Proofs of Propositions for Let the Pirates Patch?
An Economic Analysis of Software Security Patch Restrictions
Terrence August and Tunay I. Tunca
GSB, Stanford University

Before we proceed with the proofs of the propositions, we first present the following three lemmas that will be extensively used in the proofs.

**Lemma B.1** Define

\[ p \triangleq (1 - c_p) \left( 1 - \frac{c_p}{\pi_a \alpha} \right) \text{ and } \quad (B.1) \]

\[ p_1 \triangleq 1 - \pi_a \alpha \nu + \sqrt{(1 - \pi_a \alpha \nu)^2 + 4\pi_a \alpha \nu \pi_d c_d} \quad (B.2). \]

Suppose \( p > \pi_d c_d \). If \( \pi_a \alpha < c_p \) or both \( \pi_a \alpha \geq c_p \) and \( \pi_d c_d \geq p \), where \( p \) is as defined in (B.1) then

(i) if \( p \leq p_1 \), then \( v_{sp} = v_p = 1 \),

\[ v_s = \frac{\pi_d c_d}{2\pi_a \alpha} \left( \sqrt{(1 - \pi_a \alpha)^2 + 4\pi_a \alpha (\nu \pi_d c_d + (1 - \nu)p)} - (1 - \pi_a \alpha) \right), \quad (B.3) \]

and

\[ v_b = \frac{pv_s}{\pi_d c_d} \quad (B.4). \]

(ii) if \( p_1 < p \leq 1 \), then \( v_{sp} = v_p = v_b = 1 \) and

\[ v_s = \sqrt{(1 - \pi_a \alpha \nu)^2 + 4\pi_a \alpha \nu \pi_d c_d - (1 - \pi_a \alpha \nu)} \quad (B.5). \]

**Proof of Lemma B.1:** For the sake of clarity in exposition, we will defer the proofs for the statements \( v_{sp} = v_p = 1 \) when \( \pi_a \alpha \geq c_p \) and \( \pi_d c_d \geq p \) to Lemmas B.2 and B.3. When \( \pi_a \alpha < c_p \), since \( u \leq 1 \) and \( p > \pi_d c_d \), under both policies \( l \) and \( nl \), we have \( v_{sp} = v_p = 1 \). This is because (A.3), (A.12), and (A.17) will not hold for all \( v \in \mathcal{V} \). Now by (A.2) and (A.16), the equilibrium unpatched population is in the form

\[ u(\sigma^\ast) = \nu(v_{sp} - v_s) + (1 - \nu)(v_p - v_b). \quad (B.6) \]

When \( v_{sp} = v_p = 1 \), (B.6) implies \( u(\sigma^\ast) = \nu(1 - v_s) + (1 - \nu)(1 - v_b) \) under either policy. Suppose \( v_b < 1 \). Then, \( v_b \) satisfies (A.10) which by substituting into (A.14) yields (B.4) which, in turn, by substituting into (A.10) gives

\[ v_s - \pi_a \alpha v_s \left( \nu(1 - v_s) + (1 - \nu) \left( 1 - \frac{pv_s}{\pi_d c_d} \right) \right) - \pi_d c_d = 0. \quad (B.7) \]

(B.7) has a single root greater than \( \pi_d c_d \) and this root is given in (B.3). For \( v_b \leq 1 \) to hold, by (B.4), we must have \( v_s \leq \pi_d c_d / p \). Plugging this into (B.7), we see that \( v_b = 1 \) if and only if \( p \leq p_1 \). For \( p > p_1 \), substituting \( v_b = 1 \) into (A.14) and solving the resulting quadratic equation, we obtain (B.5). \( \square \)
Lemma B.2 Suppose $p > \pi_d c_d$, $\pi_a \alpha \geq c_p$ and $\pi_d c_d < \rho$ and that software pirates are allowed to patch vulnerabilities (i.e., $\rho = 1$). Define

$$\hat{p}_1 \triangleq \sup \{ p \mid p c_p + \pi_a \alpha \nu (\pi_d c_d - p)(p + c_p)^2 = 0 \} \quad \text{and} \quad (B.8)$$

$$\bar{p}_2 \triangleq \frac{\left(1 - c_p \right) \left(1 - \frac{c_p}{\pi_a \alpha} \right) - \pi_d c_d \nu}{1 - \nu}. \quad (B.9)$$

Then

(i) if $p \leq \min(\hat{p}_1, \bar{p}_2)$, then,

$$v_b = \sup \left\{ v_b \mid (v_b - p)^2 - \pi_a \alpha v_b^2 \left( \nu \left( c_p - \frac{\pi_d c_d (v_b - p)}{p} \right) + (1 - \nu) (c_p - v_b + p) \right) = 0 \right\}, \quad (B.10)$$

$$v_p = \frac{c_p v_b}{v_b - p}, \quad (B.11)$$

$$v_s = \frac{\pi_d c_d v_b}{p}, \quad (B.12)$$

and

$$v_{sp} = \frac{c_p v_s}{v_s - \pi_d c_d}; \quad (B.13)$$

(ii) if $\bar{p}_2 \leq \hat{p}_1$ and $\bar{p}_2 < p \leq \bar{p}_1$, then $v_{sp}$, $v_p$, $v_b$, and $v_s$ are as given in part (i) of Lemma B.1;

(iii) if $\bar{p}_2 \leq \hat{p}_1$ and $\bar{p}_1 < p \leq 1$, then $v_{sp}$, $v_p$, $v_b$, and $v_s$ are as given in part (ii) of Lemma B.1;

(iv) if $\bar{p}_2 > \hat{p}_1$ and $\bar{p}_1 < p \leq 1$, then $v_{sp}$ satisfies (B.13), $v_b = v_p$,

$$v_p = \min(p + c_p, 1), \quad (B.14)$$

and

$$v_s = \sup \left\{ v_s \mid (v_s - \pi_d c_d)^2 - \nu \pi_a \alpha v_s^2 (c_p - v_s + \pi_d c_d) = 0 \right\}. \quad (B.15)$$

Proof of Lemma B.2: Suppose $v_b < v_p < 1$ and $v_{sp} < 1$. Then, by Lemma A.1, $v_b$, $v_p$, $v_s$, and $v_{sp}$ satisfy (A.10), (A.11), (A.14), (A.15), respectively. Substituting (A.11) into (A.10) yields (B.11), (A.10) into (A.14) yields (B.12), and (A.15) into (A.14) yields (B.13). Substituting (B.11), (B.6), and (B.12) into (A.10) gives (B.10). Now, for $v_p \leq 1$ to hold, where $v_p$ is given by (B.11), we must have $v_b \geq p/(1 - c_p)$. Substituting this quantity into (B.10), we obtain that $v_p \leq 1$ if and only if $p \leq \bar{p}_2$. By (A.11) and (A.15), we have $v_p = v_{sp}$. In order to satisfy $v_p \leq 1$, and $p > \pi_d c_d$, $\bar{p}_2 > \pi_d c_d$ has to hold, which is satisfied if and only if $\pi_d c_d < \rho$. Therefore, under policy $\rho = l$, if $\pi_d c_d \geq \rho$, then $v_p = v_{sp} = 1$ as also indicated in Lemma B.1. From (B.10), we define $f(v_b) \triangleq (v_b - p)^2 - \pi_a \alpha v_b^2 (\nu (c_p - \pi_d c_d (v_b - p)/p) + (1 - \nu) (c_p - v_b + p))$. Then, $f(p + c_p) = (c_p/p) (pc_p + \pi_a \alpha \nu (\pi_d c_d - p)(p + c_p)^2)$, and therefore $v_b$ as defined in (B.10) which solves $f(v_b) = 0$ falls to the left of $p + c_p$, i.e., $v_b \leq p + c_p$, if and only if

$$pc_p + \pi_a \alpha \nu (\pi_d c_d - p)(p + c_p)^2 \geq 0. \quad (B.16)$$

By (B.8), (B.16) is satisfied whenever $p \leq \hat{p}_1$. This proves part (i). The proofs of parts (ii) and (iii) are very similar to that of Lemma B.1.
For part (iv), suppose \( \bar{p}_2 > \hat{p}_1 \) and \( \hat{p}_1 < p \leq 1 \). Then, \( v_b \leq p + c_p \) can no longer be maintained while still satisfying (B.10). Hence \( v_b = v_p \) and both satisfy (A.14). Substituting \( v_b = v_p \) and (B.13) into (A.14) we obtain (B.15). Finally, from (B.15), we obtain that for \( v_{sp} \leq 1 \), we need \( \pi d \alpha \nu > c_p \) and \( \pi d c d < (1 - c_p)(1 - c_p/(\pi d \alpha \nu)) \), which hold if and only if \( \bar{p}_2 > \hat{p}_1 \). This completes the proof. □

**Lemma B.3** Suppose \( p > \pi d c d \). If \( \pi a \alpha \geq c_p \) and \( \pi d c d < \bar{p} \) and that software pirates are not allowed to patch vulnerabilities (i.e., \( p = n l \)). Define

\[
\hat{p}_2 \triangleq \sup \left\{ p \in \mathbb{R} \mid p c_p^2 - \pi a \alpha \nu(p + c_p)^3(p - \pi d c d) = 0 \right\}, \tag{B.17}
\]

\[
\hat{p}_3 \triangleq \sup \left\{ p \mid p c_p + \pi a \alpha \nu(p + c_p)(-p + \pi d c d(p + c_p)) = 0 \right\}. \tag{B.18}
\]

\[
\hat{p}_3 \triangleq \sup \left\{ p < 1 + \pi d c d - c_p \mid \pi a \alpha (c_p^2 - c_p(1 + \pi d c d - 2p) + (\pi d c d - p)(\nu - p)) \right. \\
+ (1 - c_p + \pi d c d - p)(c_p - \pi d c d + p)^2 = 0 \left. \right\}, \tag{B.19}
\]

\[
\bar{p}_4 \triangleq 1 - 2c_p + 2\pi d c d + \pi a \alpha \nu - \sqrt{(1 - \pi a \alpha \nu)^2 + 4\pi a \alpha \nu \pi d c d}, \tag{B.20}
\]

Then,

(i) if \( p \leq \min(\hat{p}_2, \hat{p}_3) \), then \( v_b \) satisfies (B.11), \( v_s \) satisfies (B.12),

\[
v_{sp} = \frac{(p - \pi d c d + c_p)v_s}{v_s - \pi d c d}, \tag{B.21}
\]

and

\[
v_b = \sup \left\{ v_b \mid (v_b - p)^2 - \pi a \alpha \nu \nu \left( p - \pi d c d + c_p - \frac{\pi d c d(v_b - p)}{p} \right) + (1 - \nu)(c_p - v_b + p) \right\} = 0 \right\}; \tag{B.22}
\]

(ii) if \( \bar{p}_3 \leq \hat{p}_3 \) and \( \bar{p}_3 < p \leq \min(\bar{p}_2, \hat{p}_3) \), then \( v_{sp} = 1 \), \( v_p \) satisfies (B.11), \( v_s \) satisfies (B.12) and

\[
v_b = \sup \left\{ v_b \mid (v_b - p)^2 - \pi a \alpha \nu(v_b - p) \left( 1 - \frac{\pi d c d v_b}{p} \right) - \pi a \alpha \nu^2(1 - \nu)(c_p - v_b + p) = 0 \right\}; \tag{B.23}
\]

(iii) if \( \bar{p}_3 \leq \hat{p}_2, \hat{p}_3 \leq \bar{p}_2 \), and \( \hat{p}_3 < p \leq 1 \); or if \( \bar{p}_3 > \hat{p}_2, \bar{p}_4 < p \leq 1 \), then \( v_{sp} = 1, v_b = v_p, v_s \) satisfies (B.5) and \( v_p \) satisfies (B.14);

(iv) if \( \bar{p}_3 \leq \hat{p}_2, \hat{p}_3 > \bar{p}_2, \) and \( \bar{p}_3 < p \leq \bar{p}_1 \), then \( v_{sp}, v_p, v_b, \) and \( v_s \) are as given in part (i) of Lemma B.1;

(v) if \( \bar{p}_3 \leq \hat{p}_2, \hat{p}_3 > \bar{p}_2, \) and \( \bar{p}_1 < p \leq 1 \), then \( v_{sp}, v_p, v_b, \) and \( v_s \) are as given in part (ii) of Lemma B.1;

(vi) if \( \bar{p}_3 > \hat{p}_2, \hat{p}_2 < p \leq \bar{p}_4 \), then \( v_{sp} = v_p, v_s \) satisfies (B.14), \( v_{sp} \) satisfies (B.21), and

\[
v_s = \sup \left\{ v_s \mid (v_s - \pi d c d)^2 - \pi a \alpha \nu v_s^2(p + c_p - v_s) = 0 \right\}. \tag{B.24}
\]

**Proof of Lemma B.3:** Suppose \( v_b < v_p < 1 \) and \( v_{sp} < 1 \). Then, by Lemma A.2, \( v_b, v_p, v_s, \) and \( v_{sp} \) satisfy (A.10), (A.11), (A.14), (A.19), respectively. Again, by substitution, we obtain (B.11) and (B.12). Substituting (A.19) into (A.14) yields (B.21). Substituting (B.11), (B.6), (B.12), and (B.21) into (A.10) yields (B.22). Now, for \( v_{sp} \leq 1 \) to hold, where \( v_{sp} \) is given by (B.21), we must have \( v_b \geq p/(1 - c_p - p + \pi d c d) \),
which, by (B.22), holds if and only if

$$
\pi_\alpha(c_p^2 - c_p(1 + \pi d c_d - 2p) + (\pi d c_d - p)(\nu - p)) + (1 - c_p + \pi d c_d - p)(c_p - \pi d c_d + p)^2 \leq 0,
$$  (B.25)

which is satisfied only if $\pi d c_d \leq \bar{p}$ and $p \leq \bar{p}_3$. Now by (B.22), $v_b \leq p + c_p$ holds if and only if

$$
p c_p^2 - \pi_\alpha \nu(p + c_p)^3(p - \pi d c_d) \geq 0,
$$  (B.26)

which is satisfied whenever $p \leq \hat{p}_2$. This concludes the conditions for part (i).

For part (ii), making similar substitutions as in part (i) and substituting $v_{sp} = 1$ yields (B.23). For $v_p \leq 1$ to hold, where $v_p$ is given by (B.11), we must have $v_b \geq (1 - c_p)$ which, by (B.23), is satisfied if and only if $p \leq \bar{p}_2$. Similarly, $v_b \leq p + c_p$ if and only if $p \leq \hat{p}_3$. Parts (iii) through (vi) follow the same line of proof and are omitted for conciseness. Finally, both $\bar{p}_2 > \pi d c_d$ and $\bar{p}_3 > \pi d c_d$ are satisfied if and only if $\pi d c_d < \bar{p}$. If $\pi d c_d \geq \bar{p}$, then by Lemma B.3, neither $v_{sp} < 1$ nor $v_p < 1$ can be satisfied, as stated in Lemma B.1. □

**Proof of Proposition 1:** Technically, we will prove that

(i) If $\pi d c_d < (1 - c_p)/2$, then there exists a $\omega > 0$ such that if $\pi_\alpha > \omega$, then $\Pi_{nl}(p_{nl}^*) > \Pi_l(p_l^*)$.

(ii) There exists $\gamma, \tau, \omega, \lambda, \eta > 0$ such that if $\gamma < \pi d c_d < \tau, \omega < \pi_\alpha < \lambda$, and $\nu < \eta$, then $\Pi_{nl}(p_{nl}^*) > \Pi_l(p_l^*)$.

For part (i), suppose $\rho = l$. Re-arranging (B.8), we obtain $p = \pi d c_d + p c_p/(\pi_\alpha \nu(p + c_p))^2$. As a result, and by (B.9), $\hat{p}_1$ converges to $\pi d c_d$ and $\bar{p}_2$ converges to $(1 - c_p - \pi d c_d \nu)/(1 - \nu)$ as $\pi_\alpha$ gets large. When $\pi d c_d < (1 - c_p)/2$, for sufficiently large $\pi_\alpha$ the conditions of Lemma B.2 hold, and since $(1 - c_p - \pi d c_d \nu)/(1 - \nu) > \pi d c_d$, we have $\bar{p}_2 > \hat{p}_1$. Then by part (iv) of Lemma B.2, $\Pi_l(p) = p(1 - \nu)(1 - p - c_p)$ for any $p > \pi d c_d$, and hence

$$
\max_{p > \pi d c_d} \Pi_l(p) \leq \frac{(1 - \nu)(1 - c_p)^2}{4}.
$$  (B.27)

On the other hand, when $p \leq \pi d c_d$, by Lemma 1 of August and Tunca 2006, if $\pi_\alpha \geq c_p$, and $p < \bar{p}$, then

$$
v_b = \sup \{v_b | \pi_\alpha \nu v_b^2(v_b - c_p - p) = -(v_b - p)^2\},
$$  (B.28)

By (4), $\Pi_l(p) = p(1 - v_b)$, where by (B.28) $v_b = p + c_p - (v_b - p)^2/(\pi_\alpha \nu v_b^2)$, and by writing terms in orders of $1/\pi_\alpha$ we obtain

$$
v_b = p + c_p - \frac{c_p^2}{\pi_\alpha \nu(p + c_p)^2} + O\left(\frac{1}{(\pi_\alpha)^2}\right),
$$  (B.29)

where $O$ is the common order notation, implying $f = O(g(x))$ if for sufficiently large $x$, $f(x)/g(x)$ is bounded.\(^{13}\)

By differentiating $\Pi_l(\cdot)$, computing $dv_b/dp$ by the implicit function theorem using (B.28), and substituting (B.29) into the resulting expression, we obtain

$$
\frac{d\Pi_l}{dp} = 1 - c_p - 2p - \frac{c_p^2(p - c_p)}{\pi_\alpha \nu(p + c_p)^3} + O\left(\frac{1}{(\pi_\alpha)^2}\right).
$$  (B.30)

Hence, \( p(1 - v_b) \) has an unconstrained maximizer at \( \hat{p} \) which satisfies
\[
\hat{p} = \frac{1 - c_p}{2} + \frac{2c_p^2(3c_p - 1)}{\pi_a \alpha (1 + c_p)^3} + O \left( \frac{1}{(\pi_a \alpha)^2} \right).
\] (B.31)

Since \( \pi_d c_d < (1 - c_p)/2 \), it then follows that for sufficiently large \( \pi_a \alpha \), \( \hat{p} > \pi_d c_d \) is satisfied, and hence
\[
\max_{p \leq \pi_d c_d} \Pi_l(p) = \pi_d c_d (1 - v_b(\pi_d c_d)).
\] (B.32)

Substituting (B.29) into (B.32), we obtain
\[
\max_{p \leq \pi_d c_d} \Pi_l(p) = \pi_d c_d (1 - \pi_d c_d - c_p) + \frac{c_p^2 \pi_d c_d}{\pi_a \alpha (c_p + \pi_d c_d)^2} + O \left( \frac{1}{(\pi_a \alpha)^2} \right),
\] (B.33)

and therefore, by (B.27) and (B.33), it follows that
\[
\max_{p} \Pi_l(p) \leq \max \left\{ \frac{(1 - \nu)(1 - c_p)^2}{4}, \pi_d c_d (1 - \pi_d c_d - c_p) + \frac{c_p^2 \pi_d c_d}{\pi_a \alpha (c_p + \pi_d c_d)^2} + O \left( \frac{1}{(\pi_a \alpha)^2} \right) \right\}.
\] (B.34)

For \( \rho = nl \), by (B.19), we have
\[
e^2_p - c_p (1 + \pi_d c_d - 2p) + (\pi_d c_d - p)(\nu - p) = \frac{(p + c_p - 1 - \pi_d c_d)(c_p - \pi_d c_d + p)^2}{\pi_a \alpha},
\] (B.35)

and hence, \( \bar{p}_3 \) approaches \( (\pi_d c_d + \nu - 2c_p + \sqrt{(\pi_d c_d - \nu)^2 + 4c_p (1 - \nu)})/2 \) as \( \pi_a \alpha \) gets large. Now, \( (\pi_d c_d + \nu - 2c_p + \sqrt{(\pi_d c_d - \nu)^2 + 4c_p (1 - \nu)})/2 > \pi_d c_d \) is always satisfied when \( \pi_d c_d + 2c_p - \nu \leq 0 \). However, if \( \pi_d c_d + 2c_p - \nu > 0 \), then it is satisfied if and only if \( (\sqrt{(\pi_d c_d - \nu)^2 + 4c_p (1 - \nu)})^2 > (\pi_d c_d + 2c_p - \nu)^2 \) which holds whenever \( \pi_d c_d < 1 - c_p \). Hence, for sufficiently large \( \pi_a \alpha \), \( \bar{p}_3 > \pi_d c_d \). Also by (B.17), we have \( p = \pi_d c_d + p_c^2/(\pi_a \alpha \nu (p + c_p)^3) \), and hence
\[
\hat{p}_2 = \pi_d c_d + \frac{\pi_d c_d p_c^2}{\pi_a \alpha \nu (\pi_d c_d + c_p)^3} + O \left( \frac{1}{(\pi_a \alpha)^2} \right).
\] (B.36)

As a result, for sufficiently large \( \pi_a \alpha \), \( \bar{p}_3 > \hat{p}_2 \). By (B.20), \( \bar{p}_4 \) is the larger root of the quadratic equation \( p = 1 - c_p + (p - \pi_d c_d + c_p - 1)(p - \pi_d c_d + c_p)/(\pi_a \alpha \nu) \). Thus, we have
\[
\bar{p}_4 = 1 - c_p - \frac{\pi_d c_d (1 - \pi_d c_d)}{\pi_a \alpha \nu} + O \left( \frac{1}{(\pi_a \alpha)^2} \right),
\] (B.37)

and hence, \( \max_{p \leq \bar{p}_4} \Pi_{nl}(p) \) is given by part (vi) of Lemma B.3. Rearranging (B.24) and writing terms in orders of \( 1/\pi_a \alpha \), we obtain
\[
v_a = p + c_p - \frac{(p + c_p - \pi_d c_d)^2}{\pi_a \alpha \nu(p + c_p)^2} + O \left( \frac{1}{(\pi_a \alpha)^2} \right),
\] (B.38)
and by differentiating (5) and substituting (B.14), (B.21), and (B.38) into the resulting expression for the derivative, we obtain
\[
\frac{d\Pi_{nl}}{dp} = 1 - c_p - 2p + \frac{c_p \pi d_{cd}(\pi d_{cd} - c_p) - p \pi d_{cd}(\pi d_{cd} + c_p)}{\pi a \alpha (p + c_p)^3} + O \left( \frac{1}{(\pi a \alpha)^2} \right). \tag{B.39}
\]
Equating (B.39) to zero and solving for \(p\), the unconstrained maximizer \(\tilde{p}\) satisfies
\[
\tilde{p} = \frac{1 - c_p}{2} - \frac{2s d_{cd}(c_p(1 + c_p) + \pi d_{cd}(1 - 3c_p))}{\pi a \alpha (1 + c_p)^3} + O \left( \frac{1}{(\pi a \alpha)^2} \right). \tag{B.40}
\]
By (B.36) and (B.37), for sufficiently large \(\pi a \alpha\), \(\hat{\nu} < \tilde{p} < \bar{p}_4\), and hence, \(\tilde{p} = \arg\max_{\hat{\nu}} \hat{\nu} \leq \bar{p}_4\) \(\Pi_{nl}(p)\). Therefore,
\[
\max_p \Pi_{nl}(p) \geq \max_{\hat{\nu} < \tilde{p} < \bar{p}_4} \Pi_{nl}(p) = \left( \frac{1 - c_p}{4} \right) - \frac{\pi d_{cd}(1 - c_p)(1 + c_p - 2\pi d_{cd})}{\pi a \alpha (1 + c_p)^2} + O \left( \frac{1}{(\pi a \alpha)^2} \right). \tag{B.41}
\]
By (B.33), (B.41), and since \(\pi d_{cd} < (1 - c_p)/2\), there exists \(\varpi\) such that for all \(\pi a \alpha > \varpi\), we have \(\max_p \Pi_{nl}(p) > \max_p \Pi_{l}(p)\).

For part \((ii)\), define
\[
\overline{\kappa} \triangleq \min \left( \frac{c_p}{(1 - \pi d_{cd})(1 + c_p)^3}, \frac{c_p}{(1 - \pi d_{cd}(1 + c_p))(1 + c_p)} \right), \tag{B.42}
\]
and \(\overline{\varpi} \triangleq \overline{\kappa}/\nu\). By (B.17) and (B.18), \(\pi a \alpha < \overline{\varpi}\) implies that \(\tilde{p}_2 > 1\) and \(\tilde{p}_3 > 1\). Let \(v_b\) satisfy (B.22) and \(\pi a \alpha = k/\sqrt{\nu}\) for \(0 < k < \overline{\kappa}/\sqrt{\nu}\). Then we have
\[
v_b = p + c_p - \frac{c_p^2 \sqrt{\nu}}{k(p + c_p)^2} + \left( p + c_p - \pi d_{cd}(1 + c_p/p) + \frac{2c_p^3 p}{k^2(p + c_p)^2} \right) \nu + O \left( \nu^{3/2} \right). \tag{B.43}
\]
Now, applying the implicit function theorem to (B.22) to obtain \(dv_b/dv_p\), we have
\[
\frac{dv_b}{dp} = \frac{2p^3 - \pi a \alpha \pi d_{cd} v_b^3 - p^2 v_b(2 + \pi a \alpha v_b)}{p \left( 2p^2 + 2v_b(\pi a \alpha (p + c_p) - 1) - 3\pi a \alpha v_b^2 (\nu + \pi d_{cd} + (1 - \nu)p) \right)}. \tag{B.44}
\]
By taking the derivative with respect to \(p\) in (5), and substituting (B.43), (B.44), and \(\pi a \alpha = k/\sqrt{\nu}\) into the resulting expression, we have
\[
\frac{d\Pi_{nl}}{dp} = 1 - c_p - 2p - \frac{c_p^2(p - c_p) \sqrt{\nu}}{k(p + c_p)^3} + \kappa_1 \nu + O(\nu^{3/2}), \tag{B.45}
\]
where \(\kappa_1 \in \mathbb{R}\) is a constant. For \(p > \bar{p}_2\), parts \((iv)\) and \((v)\) of Lemma B.3 apply, and the vendor’s profit approaches zero when \(\nu\) is sufficiently small. Equating (B.45) to zero, solving for \(p\) and writing the terms in orders of \(\nu\), we obtain
\[
p_{nl}^* = \frac{1 - c_p}{2} + \frac{2c_p^2(3c_p - 1) \sqrt{\nu}}{k(1 + c_p)^3} + O(\nu). \tag{B.46}
\]
Hence, by (B.9) and (B.46), \(p_{nl}^* < \bar{p}_2\). As \(\nu\) becomes small, by (B.19), \(\bar{p}_3\) converges to the larger root of the equation \(c_p^2 - c_p(1 + \pi d_{cd} - 2p) - p(\pi d_{cd} - p) = 0\), and hence, \(\bar{p}_3 > (1 - c_p)/2\) if and only if \(\pi d_{cd} > (1 -
c_p^2/(2(1-c_p)). Since (1-c_p^2)/(2(1-c_p)) < (1-c_p)/2 is always satisfied, whenever \( \pi_d c_d \in ((1-c_p^2)/(2(1-c_p)), (1-c_p)/2) \), we have \( p_{nl}^* < \min(\bar{\rho}_l, \bar{\pi}_l) \). Hence, substituting (B.43) and (B.46) into (5), and by part (i) of Lemma B.3, we obtain

\[
\Pi_{nl}(p_{nl}^*) = \frac{(1-c_p^2)}{4} + \frac{2c_p^2(1-c_p)\sqrt{\nu}}{k(1+c_p)^2} + z_1 + O(\nu^{3/2}),
\]

(B.47)

where \( z_1 \in \mathbb{R} \) is a constant satisfying

\[
z_1 = \frac{4c_p^3(5c_p^2 - 2) + 5c_p - 4}{k^2(1+c_p)^6} + \frac{(1+c_p)^2(c_p + 2\pi_d c_d - 1)}{8c_p}.
\]

(B.48)

Similarly, under policy \( \rho = l \), it can shown that \( p_l^* \) satisfies part (i) of Lemma B.2. Then, by (B.10),

\[
v_b = p + c_p - \frac{c_p^2\sqrt{\nu}}{k(p+c_p)^2} + \left( \frac{c_p}{p+c_p} - \frac{\pi_d c_d c_p}{p} - \frac{2c_p^4}{k^2(p+c_p)^5} + \frac{2c_p^3}{k^2(p+c_p)^4} \right) \nu + O(\nu^{3/2}).
\]

(B.49)

By the implicit function theorem and (B.10),

\[
\frac{dv_b}{dp} = \frac{2p(v_b - p) + \pi_d \alpha v_b \left( \frac{\nu \pi_d c_d v_b}{p} + (1-\nu)p \right)}{2p(v_b - p) + \pi_d \alpha v_b ((\nu \pi_d c_d + (1-\nu)p)(v_b + 2(v_b - p)) - 2c_p)}.
\]

(B.50)

By (4) and substituting \( \pi_d \alpha = k/\sqrt{\nu} \), (B.49), and (B.50) we have

\[
\frac{d\Pi_l}{dp} = 1 - c_p - 2p - \frac{c_p^2(p - c_p)\sqrt{\nu}}{k(p+c_p)^3} + \kappa_2 \nu + O(\nu^{3/2}),
\]

(B.51)

and hence,

\[
\Pi_l(p_l^*) = \frac{(1-c_p^2)}{4} + \frac{2c_p^2(1-c_p)\sqrt{\nu}}{k(1+c_p)^2} + z_2 + O(\nu^{3/2}),
\]

(B.52)

where \( \kappa_2, z_2 \in \mathbb{R} \) are again constants with \( z_2 \) satisfying

\[
z_2 = \frac{4c_p^3(5c_p^2 - 2) + 5c_p - 4}{k^2(1+c_p)^6} + \frac{c_p(c_p + 4\pi_d c_d - 1)}{4}.
\]

(B.53)

Comparing (B.47) and (B.52), \( \Pi_{nl}(p_{nl}^*) > \Pi_l(p_l^*) \) if and only if \( z_1 > z_2 \) which, by comparing (B.48) and (B.53) and carrying out the algebra, is satisfied if and only if \( \pi_d c_d > (1-c_p^2)/(2(1+3c_p)) \). Since

\[
\frac{(1-c_p^2)}{2(1+c_p)} < \frac{1-c_p^2}{2(1+3c_p)} < \frac{1-c_p}{2},
\]

(B.54)

there exist \( \gamma < \pi_d c_d < \bar{\gamma} \), where \( \Pi_{nl}(p_{nl}^*) > \Pi_l(p_l^*) \). ■

**Proof of Proposition 2:** We will show that

(i) There exists \( \tilde{\gamma}, \tilde{\omega}, \bar{\omega}, \bar{\pi} > 0 \) such that if \( \pi_d c_d < \tilde{\gamma}, \omega < \pi_a \alpha < \bar{\omega}, \) and \( \nu < \bar{\pi} \), then \( \Pi_l(p_l^*) > \Pi_{nl}(p_{nl}^*) \).

(ii) There exist \( \tilde{\theta}, \tilde{\gamma}, \omega, \bar{\omega} > 0 \) such that if \( \pi_d c_d < \tilde{\gamma}, \omega < \tilde{\theta}, \) and \( \omega \leq \pi_a \alpha \leq \bar{\omega} \), then \( \Pi_l(p_l^*) > \Pi_{nl}(p_{nl}^*) \).
For part (i), for sufficiently small \( \nu \), by the proof of part (ii) of Proposition 1, \( p_{nl}^* < p_2 \), and by (B.19), \( p_2 < (1 - c_p)/2 \) if and only if \( \pi_c c_d < (1 - c_p)^2/(2(1 - c_p)) \). Hence, \( v_b \) satisfies (B.23), and when \( \pi_c \alpha = k/\sqrt{\nu} \) for \( 0 < k < \overline{k}/\sqrt{\nu} \), where \( \overline{k} \) is given by (B.42), we have

\[
v_b = p + c_p - \frac{\sqrt{\nu}}{k(p + c_p)^2} + \left( \frac{c_p}{p + c_p} + \frac{2c_p^3}{k^2(p + c_p)^4} + \frac{2c_p^4}{k^2(p + c_p)^5} - \frac{\pi c_d c_p}{p} \right) \nu + O \left( \nu^{3/2} \right). \tag{B.55}
\]

By (B.23) and the implicit function theorem,

\[
\frac{dv_b}{dp} = \frac{\pi_c \alpha \pi_d c_d \nu v_b^3 - 2p^3 + p^2 v_b(2 - \pi_c \alpha \nu + \pi_c \alpha v_b(1 - \nu))}{p^3(\pi_c \alpha \nu - 2) + 3p \pi_c \alpha v_b^2(\pi_c c_d \nu + p - \nu v) + 2p^2 v_b(1 - \pi_c \alpha(c_p(1 - \nu) + \nu + \nu \pi_c c_d + p - \nu v))}. \tag{B.56}
\]

By part (ii) of Lemma B.3, for all \( \delta > 0 \), by (B.4), (B.10) and (B.14) with \( b \in (p, \infty) \), we see that \( \Pi_1(p_1^*) > \Pi_1(p_{nl}^*) \) if and only if \( z_2 > z_3 \), which is always satisfied.

To see part (ii), let \( 1/\nu < k < 4/\nu \) and suppose

\[
0 < \pi_c c_d < \min \left( 1, \frac{1}{k\nu} \right). \tag{B.59}
\]

Since \( k > 1 \) and by (B.59), for \( \pi_c \alpha = k c_p \), there exists \( \varepsilon > 0 \) such that when \( c_p < \varepsilon, \pi_c \alpha \geq c_p \) and \( \pi_c c_d < (1 - c_p)(1 - c_p/(\pi_c \alpha)) \), are satisfied. Then by (B.8),

\[
pc_p + \pi_c \alpha \nu(\pi_c c_d - p)(p + c_p)^2 = (p + \pi_c c_d k
\nu p^2 - k\nu p^3)c_p + O(c_p^2), \tag{B.60}
\]

and hence, \( \hat{p}_1 \) approaches \((\pi c_d + \sqrt{1/(k\nu) + (\pi c_d)^2})/2, \) and by (B.9), \( \overline{p}_2 \) approaches \((1-k+\pi c_d k \nu)/(1-k) \) for sufficiently small \( c_p \). Since \( 1 - 1/(k\nu) < 1/2 + (k - 2)/(2k\nu) \), by (B.59), \( \pi c_d c < 1/2 + (k - 2)/(2k\nu) \), which is satisfied if and only if \( \overline{p}_2 > 1/2 \). Further, \( \hat{p}_1 > 1/2 \) is satisfied if and only if \( \pi c_d c > 1/2 - 2/(k\nu) \), which is always satisfied since \( k\nu < 4 \). By (B.2), \( \hat{p}_1 \) approaches 1 as \( c_p \) gets small. Thus, by part (iii) of Lemma B.2, for all \( \delta > 0 \), there exists an \( \varepsilon > 0 \) such that when \( c_p < \varepsilon \), and \( 0 < \nu < 1 - \delta \), \( v_b < 1 \). Then by (B.4), (B.10) and (B.14) \( v_b \) approaches \( p \) for sufficiently small \( c_p \). It follows that \( p_1^* = \arg\max_{0 \leq p \leq 1} \Pi(p) \) approaches 1. Then, since \( \min(\overline{p}_2, \hat{p}_1) > 1/2 \), by part (i) of Lemma B.2, when \( p = p_1^* \), \( v_b \) satisfies (B.10).

By (4),

\[
\frac{d\Pi}{dp} = 1 - v_b - p \cdot \frac{dv_b}{dp}. \tag{B.61}
\]

Substituting (B.50) into (B.61), and by (B.10), \( v_b = p + z_1 c_p + O(c_p^2) \), and hence

\[
p_1^* = 1/2 + \frac{(-8z_2^2 - 2k + k z_1(1 - \nu))c_p}{2(k + k\nu(2\pi c_d - 1) + 8z_1)} + O(c_p^2), \tag{B.62}
\]

B.8
By (4), (B.62), and again since $v_b = p + z_1 c_p + O(c_p^2)$, we have

$$\Pi_l(p_i^*) = \frac{1 - \nu}{4} - \frac{(1 - \nu)z_1 c_p}{2} + O(c_p^2).$$

(B.64)

Finally, $\pi_{d,c_d}(1 - \pi_{d,c_d}) < (1 - \nu)/4$ is satisfied if and only if $\pi_{d,c_d} < (1 - \sqrt{\nu})/2$, which holds by (B.59), and hence the vendor will not set $p \leq \pi_{d,c_d}$, which verifies the optimality of (B.62).

Now, by (B.19), $p_3 = \pi_{d,c_d} + z_2 c_p + O(c_p^2)$ where $z_2$ is the larger root of

$$z_2^2 + (2 + k(\pi_{d,c_d} - \nu))z_2 + 1 - k + \pi_{d,c_d}k = 0,$$

(B.65)

and, by (B.17), $\hat{p}_2 = \pi_{d,c_d} + c_p/(k\nu(\pi_{d,c_d})^2) + O(c_p^2)$. Substituting $z_2 = 1/(k\nu(\pi_{d,c_d})^2)$ into (B.65), it follows that $\pi_{d,c_d} \leq \hat{p}_2$ if and only if $1 + k\nu(\pi_{d,c_d})(\pi_{d,c_d} - 1) \geq 0$, which is satisfied since $k\nu \leq 4$. By (B.18), $\hat{p}_3$ approaches $1/(k\nu(1 - \pi_{d,c_d}))$ as $c_p$ gets small, and by (B.59), $\hat{p}_3 < \hat{p}_2$. Further, for any $\delta > 0$, there exists a $\varepsilon > 0$ such that when $0 < c_p < \varepsilon$, for $p > \pi_{d,c_d} + \delta$, by parts (ii) and (iii) of Lemma B.3, $v_{sp} = 1$ and $v_p < 1$.

When $v_b < v_p$ by (B.23), $v_b$ satisfies

$$(v_b - p)^2 - \pi_{a,c_d} c_p(v_b - p) \left(1 - \frac{\pi_{d,c_d} c_p}{p}\right) - \pi_{a,c_d} c_p (1 - \nu)(c_p - v_b + p) = 0,$$

(B.66)

and by the implicit function theorem, $d v_b / dp$ satisfies (B.56). By part (ii) of Lemma B.3, $v_{sp} = 1$, and hence by (5),

$$\frac{d \Pi_{nl}}{dp} = 1 - v_b - p \cdot \frac{d v_b}{dp}.$$

(B.67)

Substituting (B.56) into (B.67), and by (B.66), $v_b = p + z_3 c_p + O(c_p^2)$. Substituting again into (B.67), the unconstrained maximizer of $\Pi_{nl}$ when $v_b < v_p$ then satisfies

$$p = 1/2 + \frac{(k(1 - \nu)(z_3 - 2) - 8z_3^2)(c_p - v_b + p)}{2(k + 2k\nu(\pi_{d,c_d} - 3) + 8z_3)} + O(c_p^2),$$

(B.68)

where

$$z_3 = \frac{1}{8} \left(\sqrt{16k(1 - \nu) + (k - 3k\nu + 2k\nu\pi_{d,c_d})^2} - (k - 3k\nu + 2k\nu\pi_{d,c_d})\right),$$

(B.69)

and hence,

$$\Pi_{nl}^{iii} \leq \frac{1 - \nu}{4} - \frac{(1 - \nu)z_3 c_p}{2} + O(c_p^2),$$

(B.70)

where $\Pi_{nl}^{iii}$ is the maximum profit attained when $\bar{p}_3 \leq \bar{p}_2$ and $\bar{p}_3 < p \leq \min(\bar{p}_2, \bar{p}_3)$. By (B.64) and (B.70), $\Pi_l(p^*_i) > \Pi_{nl}^{iii}$ if $z_1 < z_3$ which holds if and only if $\pi_{d,c_d} < 1/2 + (k - 2)/(2k\nu)$, which is satisfied by (B.59).

For the case when $v_b = v_p$, by part (iii) of Lemma B.3 and (5), $\Pi_{nl}(p) = p(1 - \nu)(1 - p - c_p)$, which has an unconstrained maximizer at $p = (1 - c_p)/2$, and therefore,

$$\Pi_{nl}^{iii} \leq \frac{1 - \nu}{4} - \frac{(1 - \nu)c_p}{2} + O(c_p^2),$$

(B.71)
where where $\Pi_{nl}^\alpha$ is the maximum profit attained when $\bar{p}_3 \leq \tilde{p}_2$, $\bar{p}_3 \leq \tilde{p}$ and $\bar{p}_3 < \tilde{p} < 1$. By (B.64) and (B.71), $\Pi(p) > \Pi_{nl}^\alpha$ if $z_1 < 1$ which holds if and only if $\pi_{dc} > 1/2 - (k\nu)$, which is always satisfied since $k\nu < 4$. Therefore $\Pi_{nl}(p_{nl}^*) \leq \max (\Pi_{nl}^\alpha, \Pi_{nl}^\beta)$, and the proof is complete. ■

**Proof of Proposition 3:** We will prove that if $\pi_{dc} < (1 - \sqrt{c_p(2 - c_p)})/2$, there exist $0 < \pi_a \alpha < \pi_a \alpha$ such that if $\nu > c_p(2 - c_p)$, $0 < \omega_1 < \pi_a \alpha$ and $\omega_2 > \pi_a \alpha$ then

$$\Pi_{nl}(p_{nl}^*)|_{\pi_a=\omega_2} = \max_{\rho \in \{l, nl\}} \Pi_{\rho}(p_{\rho})|_{\pi_a=\omega_2} > \max_{\rho \in \{l, nl\}} \Pi_{\rho}(p_{\rho})|_{\pi_a=\omega_1}.$$  \hfill (B.72)

When $p \leq \pi_{dc}$, by Lemma 1 of August and Tunca 2006, if $\pi_a < c_p$ or both $\pi_a \geq c_p$ and $p \geq \overline{p}$, then $v_p = 1$ and

$$v_b = \frac{1 - \pi_a \alpha}{2\pi_a \alpha} + \frac{1}{2\pi_a \alpha} \sqrt{(1 - \pi_a \alpha)^2 + 4\pi_a \alpha p}.$$  \hfill (B.73)

By this fact and Lemma B.1, $v_{sp} = 1$ for sufficiently small $\pi_a \alpha$, and hence, by (4) and (5), $\Pi(p) = \Pi_{nl}(p)$. Further, $\Pi_{nl}(p) > 0$ if and only if $p < \overline{p}_1$. Hence, by (B.73) and (B.4), $v_b = p + O(\pi_a \alpha)$. Since $\pi_{dc} < (1 - \sqrt{c_p(2 - c_p)})/2$,

$$\max_{0 < p \leq \pi_{dc}} \Pi_{nl}(p) < \left(1 - \sqrt{c_p(2 - c_p)}\right) \left(1 + \sqrt{c_p(2 - c_p)}\right) = \frac{(1 - c_p)^2}{4},$$  \hfill (B.74)

for sufficiently small $\pi_a \alpha$. By (5) and since $v_b = p + O(\pi_a \alpha)$, we also have

$$\max_{p \leq \pi_{dc} < 1} \Pi_{nl}(p) < \frac{1 - \nu}{4}.$$  \hfill (B.75)

On the other hand, by the proof of Proposition 1, when $\pi_a \alpha$ is sufficiently large, $\max_p \Pi_{nl}(p) > \max_p \Pi(p)$, where $\max_p \Pi_{nl}(p)$ approaches $(1 - c_p)^2/4$. Then, by (B.74), (B.75), and since $(1 - c_p)^2 > (1 - \nu)$ holds if and only if $\nu > c_p(2 - c_p)$ is satisfied, the result follows. ■

**Proof of Proposition 4:** We will prove that if $c_p < 1/3$ and $\pi_a \alpha \geq \omega$,

1. $\frac{d\Pi_{nl}(p^*)}{dp_{dc}} < 0$ if $\pi_{dc} < \frac{1 + c_p}{4}$, and
2. $\frac{d\Pi_{nl}(p^*)}{dp_{dc}} > 0$ if $\frac{1 + c_p}{4} < \pi_{dc} < \frac{1 - c_p}{2}$.

Let $\xi \equiv 1/(\pi_a \alpha)$. For sufficiently large $\pi_a \alpha$, by part (vi) of Lemma B.3, the proof of part (i) of Proposition 1, and (B.24), we obtain

$$v_s = p + c_p - \frac{(p + c_p - \pi_{dc})^2 \xi}{\nu(p + c_p)^2} + O(\xi^2),$$  \hfill (B.76)

and by (B.21) and (B.76),

$$v_{sp} = p + c_p + \frac{\pi_{dc}(p + c_p - \pi_{dc}) \xi}{\nu(p + c_p)^2} + O(\xi^2).$$  \hfill (B.77)
By part (vi) of Lemma B.3 and (B.14), \( v_b = p + c_p \) for \( 0 \leq p \leq 1 - c_p \) as \( \pi_a \alpha \) gets large. Substituting \( v_b \) and (B.77) into (5) and differentiating yields

\[
\frac{d\Pi_{nl}(p)}{dp} = 1 - c_p - 2p + \frac{\pi_d c_d (c_p - \pi_d c_d) + p(c_p + \pi_d c_d)\xi}{(p + c_p)^3} + O(\xi^2). \tag{B.78}
\]

Equating (B.78) to zero yields the first order condition, solving which we obtain the optimal price as

\[
p^* = \frac{1 - c_p}{2} - \frac{2\pi_d c_d (1 + c_p)(1 + 3c_p)\xi}{(1 + c_p)^3} + O(\xi^2). \tag{B.79}
\]

Hence it follows that

\[
\Pi_{nl}(p^*) = \frac{(1 - c_p)^2}{4} - \frac{\pi_d c_d (1 + c_p)(1 + 2c_p)\xi}{(1 + c_p)^2} + O(\xi^2). \tag{B.80}
\]

By (B.80), we obtain

\[
d\Pi_{nl}(p^*)/(d(\pi_d c_d) = (1 - c_p)(1 + c_p)^{-2}(4\pi_d c_d - 1 - c_p)\xi + O(\xi^2). \]

Therefore, for sufficiently large \( \pi_d \alpha \) and by Proposition 1, \( d\Pi_{nl}(p^*)/(d(\pi_d c_d) < 0 \) for \( \pi_d c_d < (1 + c_p)/4 \) and \( d\Pi_{nl}(p^*)/(d(\pi_d c_d) > 0 \) for \( (1 + c_p)/4 < \pi_d c_d < (1 - c_p)/2 \). Since \( p^* = n_l \), by Proposition 1, the result follows. \( \blacksquare \)

**Proof of Proposition 5:** Technically, we will first show that there exist threshold values \( \omega > 0 \) and \( \gamma < (1 - c_p)/2 \) such that if \( \pi_a \alpha \geq \omega \), then

\[
\lim_{\pi_d c_d \to \gamma^-} W_{\rho^*}(p^*) < \lim_{\pi_d c_d \to \gamma^+} W_{\rho^*}(p^*). \tag{B.81}
\]

We will then prove that

\[
\lim_{\gamma \to \gamma^-} \frac{dW_{nl}(p^*)}{d(\pi_d c_d)} \bigg|_{\pi_d c_d = \gamma} > 0. \tag{B.82}
\]

Let \( \xi \triangleq 1/(\pi_a \alpha) \). Suppose that \( p = \pi_d c_d < (1 - c_p)/2 \). As \( \pi_a \alpha \) grows large, by (B.28), we have

\[
v_b = \pi_d c_d + c_p - \frac{c_p^2 \xi}{(\pi_d c_d + c_p)^2} + O(\xi^2). \tag{B.83}
\]

By (4) and (5), \( \Pi_{nl}(\pi_d c_d) = \Pi_l(\pi_d c_d) \) and by substituting (B.83), we obtain

\[
\Pi_{nl}(\pi_d c_d) = \pi_d c_d (1 - c_p - \pi_d c_d) + \frac{\pi_d c_d^2 c_p^2 \xi}{(\pi_d c_d + c_p)^2} + O(\xi^2). \tag{B.84}
\]

Suppose that \( p > \pi_d c_d \). By (B.24) and part (vi) of Lemma B.3,

\[
v_s = p + c_p - \frac{(p - \pi_d c_d + c_p)^2 \xi}{\nu(p + c_p)^2} + O(\xi^2). \tag{B.85}
\]

Substituting (B.14), (B.21), and (B.85) into (5) gives

\[
\Pi_{nl}(p) = p(1 - p - c_p) - \frac{\pi_d c_d (p - \pi_d c_d + c_p) \xi}{(p + c_p)^2} + O(\xi^2), \tag{B.86}
\]

B.11
and by taking first order conditions yields
\[
p^*_n = \frac{1 - c_p}{2} - \frac{2\pi_d c_d (1 + c_p) + \pi_d c_d (1 - 3c_p)}{(1 + c_p)^3} + O(\xi^2).
\] (B.87)

By (B.86) and (B.87), we obtain
\[
\Pi_n(p^*_n) = \frac{(1 - c_p)^2}{4} - \pi_d c_d (1 - c_p)\left(1 + c_p - 2\pi_d c_d\right)\xi + O(\xi^2),
\] (B.88)
and by equating (B.84) and (B.88), it follows that \(\hat{d}\) and therefore \(\gamma\) and further by evaluating (B.90) at \(\hat{d}\), we have
\[
\hat{d} = \frac{1 - c_p}{2} - \sqrt{\frac{c_p (1 - c_p)\xi}{1 + c_p}} + O(\xi).
\] (B.89)

By (B.84) and (B.88), we have
\[
\frac{d[\Pi_n(\pi_d c_d) - \Pi_n(p^*_n)]}{d(\pi_d c_d)} = 1 - c_p - 2\pi_d c_d + \left(-1 - \frac{2}{1 + c_p} - \frac{4\pi_d c_d (1 - c_p)}{(1 + c_p)^2} + \frac{c_p^2 (c_p - \pi_d c_d)}{(c_p + \pi_d c_d)^2}\right)\xi + O(\xi^2),
\] (B.90)
and further by evaluating (B.90) at \(\hat{d}\), we have
\[
\frac{d[\Pi_n(\pi_d c_d) - \Pi_n(p^*_n)]}{d(\pi_d c_d)} \bigg|_{\pi_d c_d = \hat{d}} = \sqrt{\frac{4c_p (1 - c_p)\xi}{1 + c_p}} + O(\xi),
\] (B.91)
and therefore \(d[\Pi_n(\pi_d c_d) - \Pi_n(p^*_n)]/d(\pi_d c_d) > 0\) for all \(0 \leq \pi_d c_d \leq \hat{d}\). This implies that at \(\pi_d c_d = \hat{d}\), the vendor switches price from \(p^*_n\) to \(\pi_d c_d\). As \(\pi_d c_d \to \hat{d}^+\), by (7), (B.85), (B.87), and (B.89), we obtain
\[
W_n(p^*_n) = \frac{3(1 - c_p)^2}{8} + \frac{\pi_d c_d \left(\pi_d c_d + c_p (3 + 4c_p + c_p^2 - \pi_d c_d (8 + c_p))\right)}{(1 + c_p)^3} + O(\xi^{3/2}).
\] (B.92)

However, as \(\pi_d c_d \to \hat{d}^+\), by (7), (B.83), and (B.89), we have
\[
W_n(\pi_d c_d) = \frac{3(1 - c_p)^2}{8} + \frac{1 - c_p}{2} \sqrt{\frac{c_p (1 - c_p)\xi}{1 + c_p}} + O(\xi),
\] (B.93)
which proves that \(\lim_{\pi_d c_d \to \hat{d}^-} W_{p^*}(p^*) < \lim_{\pi_d c_d \to \hat{d}^+} W_{p^*}(p^*)\). By (B.92), we obtain
\[
\lim_{\gamma \to \hat{d}^-} \frac{dW_n(p^*_n)}{d(\pi_d c_d)} \bigg|_{\pi_d c_d = \gamma} = \frac{1 + c_p (6 + c_p)(2c_p - 1)\xi}{(1 + c_p)^3} + O(\xi^{3/2}),
\] (B.94)
which is positive for all \(0 < c_p < 1\). This completes the proof. \(\blacksquare\)

**Proof of Proposition 6:** We will show that there exist \(\hat{\theta}, \omega, \bar{\omega} > 0\) such that if \(c_p < \hat{\theta}\), and \(\omega \leq \pi_a \alpha \leq \bar{\omega}\), then

(i) there exists \(\eta > 0\) that if \(0 < \pi_d c_d < (1 - \sqrt{\eta})/4\) and \(\nu < \eta\), then \(W_1(p^*_1) \geq W_n(p^*_n)\);

(ii) there exist \(0 < \bar{\lambda} < \bar{\lambda}\) that if \(\bar{\lambda} < \pi_d c_d < \bar{\lambda}\), then \(W_n(p^*_n) > W_1(p^*_1)\).
For convenience in exposition, define $v_{s_p, \rho}$, $v_{sp, \rho}$, $v_{b, \rho}$ and $v_{p, \rho}$ as the corresponding threshold values under policy $\rho$. Since $\pi_d c_d < (1 - \sqrt{\nu})/4$ and by part (ii) of the proof of Proposition 2, both $p_\rho^*$ and $p_{nl}^*$ approach $1/2$ for sufficiently small $c_p$. Again, by part (ii) of the proof of Proposition 2, since $\min(\pi_2, \hat{p}_l) > 1/2$, by part (i) of Lemma B.2, we have $v_{sp, l} < 1$. Now, suppose $u_l > u_{nl}$, where $u_l$ and $u_{nl}$ are the sizes of the unpatched populations under policy $l$ and $nl$ as given by (B.6), respectively. By (A.14), $v_{s, l} = \pi_d c_d / (1 - \pi_a \alpha u_l)$ and $v_{s, nl} = \pi_d c_d / (1 - \pi_a \alpha u_{nl})$, hence $v_{s, l} > v_{s, nl}$. By (A.15), $v_{sp, l} = \pi_d c_d / (\pi_a \alpha u_l)$, and by (A.19), $v_{sp, nl} = \min(1, (p + \pi_d c_d) / (\pi_a \alpha u_{nl}))$, and hence $v_{sp, nl} > v_{sp, l}$. Now, defining $u_{p}^L$ and $u_{p}^H$ as the sizes of the unpatched populations under policy $\rho$ in the Type $L$ and Type $H$ consumer populations, respectively, it follows that $u_p = u_{p}^L + u_{p}^H$. Since $v_{sp, nl} > v_{sp, l}$ and $v_{l} > v_{s, nl}$, we obtain $u_{nl}^L > u_{l}^H$, and since $u_l > u_{nl}$, we have $u_{nl}^L < u_{l}^L$. By part (i) of Lemma B.2, $v_{b, l} < v_{p, l}$, and by (A.5), (A.8), and (A.9), we have $v_{b, l} = p/(1 - \pi_a \alpha u_l)$ and $v_{p, l} = p/(\pi_a \alpha u_l)$. Since $v_{b, l} < p + c_p$ and $u_l > u_{nl}$, we obtain $v_{b, nl} = p/(1 - \pi_a \alpha u_l) < v_{b, l} < p + c_p$. By (A.5) and (A.8), $v_{p, nl} = \min(1, c_p / (\pi_a \alpha u_{nl}))$, and hence, $v_{p, nl} > v_{p, l}$. It follows that $u_{nl}^L > u_{l}^L$, which is a contradiction. Therefore, $u_l \leq u_{nl}$, and hence, $v_{s, l} \leq v_{s, nl}$.

Now, by (B.18), we know that for sufficiently small $c_p$, $\hat{p}_l > 1/2$ for $k < 2/\nu$. Then, by part (ii) of Proposition 2, $v_p < 1$ and $p_{nl}^*$ satisfies (B.68). Comparing with (B.62), it then follows that $p_{nl}^* - p_{l}^* = \nu c_p / 2 + O(\nu^2)$ and therefore, for sufficiently small $\nu$, $p_{nl}^* \geq p_{l}^*$ and hence $v_{b, l} \leq v_{b, nl}$. As a result, $(v - C(v, \theta, \sigma^*))^*$ is greater under $\rho = l$ than $\rho = nl$ for each consumer and by (7), it follows that $W_l(p_{l}^*) \geq W_{nl}(p_{nl}^*)$. This proves (i).

For (ii), first notice that $(1 - \sqrt{\nu}) / 2 < 1 - 1/(k\nu)$ if and only if $k > 2/(\nu(1 + \sqrt{\nu}))$ and that $2/(\nu(1 + \sqrt{\nu})) > 1/\nu$ for all $\nu \in (0, 1)$. Let $2/(\nu(1 + \sqrt{\nu})) < k < 4/\nu$ and $0 < \pi_d c_d < (1 - \sqrt{\nu})/2$. Then (B.59) is satisfied and since $k > 1$, for $\pi_a \alpha = k c_p$, there exists $\delta > 0$ such that when $c_p < \varepsilon$, $\pi_a \alpha \geq c_p$ and $\pi_d c_d < (1 - c_p)/(\pi_a \alpha)$. By (B.64), (B.70) and (B.71), we then have $\Pi_l(p_{l}^*) = \max_{\theta > \pi_a c_d} \Pi_l(p_{l}^*)^{\pi_{nl}}$. By (4), (5), and (B.28),

$$\Pi_l(\pi_d c_d) = \Pi_{nl}(\pi_d c_d) = \pi_d c_d (1 - \pi_d c_d) - z_0 c_p + O(c_p^2),$$

where $z_0 = (\pi_d c_d)^2 (k \pi_d c_d - \sqrt{4k + (k \pi_d c_d)^2})/2$. Carrying out the algebra, it follows that $z_0 < z_4$ as given in (B.63). Therefore, by (B.64), and (B.95), there exists $\varepsilon > 0$ such that for any $c_p < \varepsilon$, there exists $\delta > 0$ such that if $(1 - \sqrt{\nu})/2 - \delta < \pi_d c_d < (1 - \sqrt{\nu})/2$ then $\Pi_l(\pi_d c_d) = \Pi_{nl}(\pi_d c_d) > \Pi_l(p_{l}^*)$. Consequently, by (B.64) and the continuity of $\Pi_l(\pi_d c_d)$ as in (B.95), there exist $\lambda, \lambda \in (0, (1 - \sqrt{\nu})/2)$ such that when $\lambda < \pi_d c_d < \lambda$, $\max_{\theta > \pi_a c_d} \Pi_{nl}(\theta) < \Pi_l(\pi_d c_d)$. Hence $p_{nl}^* = \pi_d c_d$ and $p_{l}^*$ is characterized by (B.62). But by (7), for sufficiently small $c_p$, $W_{nl}(\pi_d c_d)$ approaches $(1 - (\pi_d c_d)^2)/2$ by (B.62). $W_l(p_{l}^*)$ approaches $(3 + \nu(1 - 4(\pi_d c_d)^2))/8$. Since $\pi_d c_d < 1/2$, it follows that $\Pi_l(p_{l}^*) > \Pi_{nl}(p_{nl}^*)$ and $W_{nl}^* > W_{l}^*$. This completes the proof. ■