1. Non-Stationary Time Series (Not on the test)
   (a) “Very persistent” series (prices, volume, volatility?)
   (b) Augmented Dickey-Fuller test
   (c) Dangers of spurious correlation
   (d) Cointegration
2. Test (2.5 hrs)
2 Non-Stationary Time Series

• Thus far, we have been looking at stationary time series. We have focused on \( r_t, \sigma_t^2 \), assuming that they are stationary. For some series, this assumption is more tenable than for others.

• But suppose you want to work with non-stationary time-series, i.e. prices, volume, number of investors in a particular fund, number of funds, etc. Those processes are inherently non-stationary.

• Let \( p_t \) be the log-price. We know that

\[
p_t = p_{t-1} + \varepsilon_t
\]

or \( p_t \) is an AR(1) process with an unit-root.

• This process is non-stationary. We cannot apply the CLT.

• But we are still interested in testing the null \( \phi = 1 \) versus \( \phi < 1 \).

• Problem. Under the null, the process is non-stationary.

• Under the alternative, the process is stationary.
• It turns out that (FCLT)
\[
\frac{1}{\sqrt{T}}p_t = \frac{1}{\sqrt{T}} \sum_{s=1}^{t} \varepsilon_s \\
= \frac{1}{\sqrt{T}} \sum_{s=1}^{[rT]} \varepsilon_s \Rightarrow W(r)
\]
where $W(r)$ is a Brownian motion on $[0, 1]$.

• Q: Can’t we standardize the non-stationary processes by a power of $T$ in order for them to converge.

• A: Yes.

• Let’s get a “flavor” of how things work:
• Recall that
\[ \hat{\phi} = \frac{\sum p_t p_{t-1}}{\sum p_{t-1}^2} = \frac{\sum p_{t-1} (\phi p_{t-1} + \varepsilon_t)}{\sum p_{t-1}^2} \]
\[ = \phi + \frac{\sum p_{t-1} \varepsilon_t}{\sum p_{t-1}^2} \]

• If \( \phi < 1 \), we had
\[ \hat{\phi} = \phi + \frac{1}{T} \frac{\sum \varepsilon_t p_{t-1}}{\sum p_{t-1}^2} \rightarrow^p \phi \]
\[ \sqrt{T} (\hat{\phi} - \phi) \sim N \left(0, \sigma_{\hat{\phi}}^2\right) \]

• But if \( \phi = 1 \), the results do not hold. But
\[ \hat{\phi} = \phi + \frac{\sum \varepsilon_t p_{t-1}}{\sum p_{t-1}^2} \]
\[ T (\hat{\phi} - \phi) \Rightarrow O_p(1) \]

• In other words, \( \hat{\phi} \) is super-consistent.

• Q: But since we don’t know the distribution of \( \hat{\phi} \), can we use this result for testing?

• A: Yes, if we simulate the distribution.
• Dickey-Fuller (DF) Test:
  \[ H_o : \phi = 1 \]
  \[ H_a : \phi < 1 \]

• The test is:
  \[ t = \frac{\hat{\phi} - 1}{se(\hat{\phi})} \]

• Note: The null is of non-stationarity, as opposed to previous tests.

• Suppose that \( \varepsilon_t \) follows an AR(p) process. The distribution of the DF test is influenced by those parameters. Not good.

• To get rid of those parameters, we run the following regression:
  \[ p_t = \phi p_{t-1} + \zeta_1 \Delta p_{t-1} + \zeta_2 \Delta p_{t-2} + \ldots + \zeta_k \Delta p_{t-k} + \nu_t \]

• Then, \( t = \frac{\hat{\phi} - 1}{se(\hat{\phi})} \).

• This is called the Augmented DF, or ADF test.
Summary of ADF test–Testing for a unit root:

- Regress $p_t$ on $p_{t-1}, \Delta p_{t-1}, \Delta p_{t-2}, \ldots, \Delta p_{t=k}$
- Form: $t = \frac{\hat{\phi} - 1}{se(\hat{\phi})}$
- Get the critical value from simulations under the null.
• So, is working with non-stationary variables that easy?
• NO:
• Suppose $p^1_t$ and $p^2_t$ are the prices of the same asset traded on two markets. Then, it must be the case that
  \[ p^1_t = p^2_t \]
• Empirically, this is almost true. We find
  \[ p^1_t - p^2_t = \varepsilon_t \]
  where $\varepsilon_t$ is almost iid, and $E(\varepsilon_t) = 0$.
• How do we take advantage of this?
• Regress:
  \[ p^1_t = \gamma p^2_t + \varepsilon_t \]
• If
  \[ \varepsilon_t > 0 \]
  \[ p^1_t > \gamma p^2_t \]
• The asset is “too expensive” in market 1.
• Note: Can we do the same using returns?
• Easy, right?
• Wrong!
• Suppose we have two assets, with prices $p_t$ and $q_t$. One might be tempted to look for arbitrage strategies as in

$$p_t = \gamma q_t + \varepsilon_t$$

• If $\gamma > 0$, there is a relationship, and we can trade.
• No.
• Suppose

$$p_t = p_{t-1} + u_t$$
$$q_t = q_{t-1} + v_t$$

$$\text{cov}(u_t v_t) = 0$$

• Note: The two log-prices represent two independent discrete Brownian motions.
• But:

\[
\hat{\gamma} = \frac{\sum_{t=1}^{T} p_t q_t}{\sum_{t=1}^{T} q_t^2} = \frac{\sum_{t=1}^{T} (\sum_{s=1}^{t} v_s) (\sum_{s=1}^{t} u_s)}{\sum_{t=1}^{T} (\sum_{s=1}^{t} v_s)^2}
\]

\[
= \frac{\frac{1}{T^2} \sum_{t=1}^{T} (\sum_{s=1}^{t} v_s) (\sum_{s=1}^{t} u_s)}{\frac{1}{T^2} \sum_{t=1}^{T} (\sum_{s=1}^{t} v_s)^2} = O_p(1)
\]

• Similarly:

\[
t = \frac{\hat{\gamma}}{se(\hat{\gamma})}
\]

is not consistent.

• The $R^2$ does not converge to 0, as $T \rightarrow \infty$.

• Illustration: spurious.m
Co-integration:

- Suppose that $p_t$ and $q_t$ are unit root (integrated) processes, but there is a linear combination of $p_t$ and $q_t$ that is stationary. That is, there exists a vector $\gamma = \begin{bmatrix} 1 & -\gamma_1 \end{bmatrix}$ such that

\[ p_t - \gamma q_t = \varepsilon_t \]

and $\varepsilon_t$ is a stationary process. Then, $p_t$ and $q_t$ are said to be cointegrated.

- There are formal tests for cointegration, but they have low power against the alternative. Why? (Think spurious correlation).

- Cointegration is only between contemporaneous variables. I.e. $p_t - \gamma q_t$ is a cointegrating vector. But $\Delta p_t$ is not. However, the latter is also a way of stationarizing the process.

- Cointegration occurs “naturally” in economics. It is dictated by theory.
• Examples:
  – Prices $p^1_t$ and $p^2_t$ of the same asset traded on two markets.
  – Dividend price ratio is cointegrated, or:
    $d_t - p_t$
    must be stationary
  – The long and the short rate must be cointegrated
  – Consumption and GDP must be cointegrated, etc.
  – Lettau & Ludvigson (2001), the “cay” ratio:
    $c_t - \gamma_1 a_t - \gamma_2 y_t$
Quick Illustration: