Forecasting the Correlation Between Stock and Bond Returns: A Variance-Choleski Model (VACFA or Var-Chol)

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Goal:

Model the covariance matrix (and the correlation) between asset returns (stock and bond) as a function of observable state variables. The model should:

- Help us understand the economic factors behind variations in the correlation
- Be parsimonious
- Be simple to estimate and interpret (KISS)

Why:

- There is evidence (US and international) that the correlation between asset returns is time-varying (Ibbotson).
- GARCH literature is not doing a good job on several accounts:
  - Exclusively autoregressive, no economic factors (Why is there time-variation?)
  - Not parsimonious and difficult to estimate
  - Correlation between assets is often assumed constant for tractability (Bollerslev (1990))
- There is an extensive literature on forecasting mean returns using state variables. (2% for 1-month ahead). There is no work done on forecasting the correlation as a function of state variables.
- It is important to know how the correlation changes as a function of economic variables.
  - If a recession is coming, the default spread widens. What is the effect of changes in the default spread on the correlation?
  - What is the effect of (contractionary) monetary policy shocks on the correlation?
- The unrestricted estimation procedure will “let the data speak” without imposing too many theoretical assumptions. The results can serve as guide for future theoretical and empirical work.
Framework:

To keep it simple, we focus on 2 assets ($R_{1,t}$ and $R_{2,t}$) and 1 exogenous variable, $X_{t-1}$ (think D/P ratio). First, we model the conditional means as:

\[
R_{1,t+1} = \mu_1 + k_1 X_t + Y_1,_{t+1} \\
R_{2,t+1} = \mu_2 + k_2 X_t + Y_2,_{t+1}
\]

Let $Y_{t+1} = (Y_{1,t+1} Y_{2,t+1})$, and $\Sigma_t = E \left( Y_{t+1}' Y_{t+1} | X_t \right) = E \left( \begin{array}{cc} Y_{1,t+1}^2 & Y_{1,t+1} Y_{2,t+1} \\
Y_{1,t+1} Y_{2,t+1} & Y_{2,t+1}^2 \end{array} \right) | X_t$.

- **NOTE:** Usually, we handle conditional expectations with projections (regressions), but we cannot regress $Y_{1,t+1}, Y_{1,t+1} Y_{2,t+1},$ and $Y_{2,t+1}^2$ on $X_t$ because $\Sigma_t$ must be positive semi-definite.

**LET:**

\[
\Sigma_t = \hat{\Sigma}_t + \begin{pmatrix} \varepsilon_{11,t+1} & \varepsilon_{12,t+1} \\
\varepsilon_{21,t+1} & \varepsilon_{22,t+1} \end{pmatrix}
\]

Then, we can use any triangular decomposition (say Cholesky) to write:

\[
\hat{\Sigma}_t = U_t' U_t
\]

where

\[
U_t = \begin{pmatrix} U_{11,t} & U_{12,t} \\
0 & U_{22,t} \end{pmatrix} = \begin{pmatrix} \alpha_{11} + \beta_{11} X_t + \gamma_{11} U_{11,t-1} & \alpha_{12} + \beta_{12} X_t + \gamma_{12} U_{12,t-1} \\
0 & \alpha_{22} + \beta_{22} X_t + \gamma_{22} U_{22,t-1} \end{pmatrix}
\]

Then, we can write

\[
Y_{1,t+1}^2 = \hat{\Sigma}_{11,t} + \varepsilon_{11,t+1} = U_{11,t}^2 + \varepsilon_{11,t+1} \\
Y_{1,t+1} Y_{2,t+1} = \hat{\Sigma}_{12,t} + \varepsilon_{12,t+1} = U_{11} U_{12} + \varepsilon_{12,t+1} \\
Y_{2,t+1}^2 = \hat{\Sigma}_{22,t} + \varepsilon_{22,t+1} = U_{22,t}^2 + U_{22,t}^2 + \varepsilon_{22,t+1}
\]

- **NOTE:** The positive definiteness restrictions are insured by the Cholesky decomposition.

- **NOTE:** Estimating