Managing Time-Based Contracts with Delayed Payments

H. Dharma Kwon[1], **Steven A. Lippman, Kevin McCardle, and Christopher S. Tang**[2]

April 23, 2008

Abstract

Some manufacturers impose a time-based contract on their suppliers under which each supplier is paid only when all of the suppliers have completed their tasks. We investigate whether or not the manufacturer ought to demand such a delayed payment contract. In our model with one manufacturer and two suppliers, we compare the impact of both a delayed payment regime and a no delayed payment regime on each supplier's effort level and on the manufacturer's net profit in equilibrium. With deterministic supplier completion times, behavior and profits are the same for both regimes. However, with uncertainty in the suppliers' completion times, the delayed payment regime is more profitable than the no delayed payment regime if the manufacturer's project is sufficiently small; the delayed payment regime is less profitable if the project is very large.

Keywords: Time-Based Supply Contracts, Delayed Payments, Stackelberg game, Nash equilibrium.

[1]Kwon was partially supported by the UCLA Dissertation Year Fellowship and by the Harold and Pauline Price Center for Entrepreneurial Studies.

[2]Direct correspondence to any of the authors. Kwon, Lippman, McCardle and Tang: UCLA Anderson School, UCLA, 110 Westwood Plaza, CA 90095, email addresses: hyok-jon.kwon.2008@anderson.ucla.edu, slippman@anderson.ucla.edu, kmccardl@anderson.ucla.edu, ctang@anderson.ucla.edu.

1 Introduction

To mitigate the risk of late delivery or completion time of a customized product or service, customers often pay the manufacturer or service provider only when the entire job is complete and the product or service is delivered. This form of time-based contract is common when dealing with customized products or services. In turn, some manufacturers impose such a contract on their suppliers, perhaps to their own disadvantage. This paper addresses the issue of whether or not it behooves the manufacturer to demand such a contract with his suppliers.

We consider the situation in which the manufacturer's end product consists of two separate and independent components, the manufacture of which can be performed in parallel. Each component is provided by a different external supplier. The manufacturer's contract with each supplier specifies both the price to be paid to the supplier and the payment terms. We consider two different payment regimes: no delayed payment and delayed payment. Under the conventional or no delayed payment regime, a supplier receives her payment immediately after she has completed her task. Under the delayed payment regime, however, each supplier receives her payment only after both suppliers have completed their tasks.

Because of the time value of money, a manufacturer prefers to pay its suppliers later rather than earlier (other things being equal); consequently, it is not uncommon for manufacturers to require its suppliers to operate under a delayed payment contract. Clearly, at a given price and a given delivery time, each supplier prefers the no delayed payment regime and the manufacturer prefers the delayed payment regime. The time-based contracts we consider, however, specify the price to be paid to each supplier and whether or not each supplier will be paid when she completes her individual task or when both suppliers have completed their individual tasks. These contracts do not specify each supplier's time of delivery. It is conceivable (or even likely) that the delayed payment regime results in a longer time for each supplier to complete her task. This increased time on task may do more harm to the manufacturer than the benefit of delayed payment.

Both no delayed payment and delayed payment regimes are common practice. We offer three examples. First, consider the supply chain operations of a translation agency in Los Angeles called Inline Translations Services (www.inlinela.com). This agency offers one-stop written translation services to customers who need to translate materials such as employee handbooks, safety manuals, and web site content from a source language (e.g., English) to multiple target languages (e.g., Spanish and Italian). Typically, customers offer time-based contracts under which Inline receives full payment upon the completion of the entire translation project. Similar to other agencies, the translation work associated with each target language is performed by an independent translator (i.e., an external supplier).[1] It is Inline's practice to offer the no delayed payment contract to its translators; many other translation agencies offer the delayed payment contract. Second, consider the supply chain operations of various custom tailors in Asia (e.g., Hong Kong, Singapore etc.). In Asia, most custom tailors are, in essence, retailers; they first take orders from customers at their shops and then outsource the cutting, sewing, and fitting to external independent contractors. In light of the efficiency of specialization of labor, it comes as no surprise that for men's suits, the jacket and the trousers are usually made by two separate tailors. In practice, we observe both types of contract between the custom tailor and its individual supplier tailors. Third, consider the supply chain operations of Boeing's 787 Dreamliner program that has been praised by practitioners as visionary (c.f., Nolan and Kotha (2005)). The 787 Dreamliner program is the first time Boeing has elected to offer a so-called "risk-sharing" contract, akin to the delayed payment regime, to some of its strategic suppliers such as Spirit AeroSystem (c.f., Greising and Johnsson (2007)).[2] Under the risk-sharing contract, Boeing's strategic suppliers will not get paid until Boeing delivers the airplanes to its customers.[3]

Because both of these payment regimes exist in the world of business, we develop a stylized model capturing the essential features of these two regimes in order to gain intuition as to which regime imparts the larger manufacturer's profit. Specifically, our model of a supply chain consists of one manufacturer and two external suppliers.[4] To begin, we suppose that the manufacturer is paid the amount $2q$ by his customer upon delivering the product or service which comes into being when the slower of the two suppliers completes her task. The manufacturer acts as the leader in a Stackelberg game. The game starts when the manufacturer specifies the time-based contract: he selects not only the price p to be paid to each of the suppliers but also the payment regime, either the no delayed payment regime N or the delayed payment regime D. Each supplier is a follower in this Stackelberg game. Given the payment p and the regime specifying when the payment will be received, each supplier selects her optimal effort level or work rate. In selecting her effort level in the delayed payment regime D, each supplier takes into account the other supplier's effort level in accord with a Nash equilibrium.

In analyzing this Stackelberg game, we answer the following questions:

[1] While the translation work is done by external translators, most agencies offer value added services including proof-reading, typesetting, etc.

[2] For the 787 program, Boeing offers the delayed payment contract to certain strategic suppliers and the no delayed payment contract to other non-strategic suppliers.

[3] The *quid pro quo* for accepting the delayed payment contract appears to be vesting intellectual property rights associated with the systems developed by the suppliers with the suppliers rather than with Boeing. Because the 787 Dreamliner is designed so that the final assembly operations will take only three days, it has been claimed that the risk-sharing contract is virtually the same as the no delayed payment contract.

[4] Although we consider the case of two separate tasks to be performed by two external suppliers, the same approach can be used to analyze $n > 2$ tasks to be performed by n suppliers.

1. Given the payment amount p and the payment regime N, what is the supplier's optimal work rate?
2. Given the payment amount p and the payment regime D, what is the supplier's optimal work rate in equilibrium? Is this equilibrium unique?
3. Given the manufacturer's revenue $2q$, what contract (*i.e.,* price p and payment regime N or D) should the manufacturer select in order to maximize his profit?
4. What are the conditions that render the no delayed payment regime more profitable for the manufacturer?

To better answer these questions and to comport with the real world, our model incorporates both deterministic and uncertain completion times for each task. The primary contributions of this paper are three-fold. First, we construct a simple model of a time-based contract with and without delayed payments and with and without uncertainty. Second, we obtain insights regarding the optimal price p to pay to the suppliers in each regime. Third, we derive conditions under which one payment regime dominates the other.

This paper is organized as follows. Section 2 provides a brief review of related literature. Section 3 explicates the underlying mathematical structure of the model when the completion time of each task is deterministic and each supplier selects one of two prespecified effort levels. There we demonstrate that the behavior of the suppliers is the same in both payment regimes; consequently, the profit of each market participant is the same under each payment regime (Proposition 1). Section 4 deals with the case when the completion time of each task is stochastic. In particular, the completion time is exponentially distributed with parameter equal to the supplier's effort level, and the mean completion time is one of the same two prespecified effort levels available in Section 3. We determine the supplier's optimal work rate under each payment regime, and we find the Nash equilibria (Proposition 2). We show that the manufacturer tends to offer a higher price under the delayed payment regime (Proposition 3). In Proposition 4, we establish analytical conditions under which one payment regime dominates the other in equilibrium. Section 5 extends the model presented in Section 4 to the case when each supplier has a continuum of effort levels to select among. Some of the insights established in Section 4 continue to hold even when each supplier chooses her work rate from an infinite number of work rates. We conclude in Section 6. In order to enhance the presentation, all proofs are given in the Appendix.

2 Literature Review

In the lead-time and price-quotation literature, the manufacturer's contract with his customers specifies the lead time and the price. For the case when the manufacturer offers identical lead time and price for all potential customers, Palaka et al. (1998) and So and Song (1998) develop two different single server queuing models by treating the manufacturer as the server. They determine the optimal lead time and price that maximize the manufacturer's expected profit in steady-state. So (2000) extends his earlier work by considering competition among manufacturers.[5]

Our model differs from this literature in three fundamental ways. First, the focus in this paper is the manufacturer's contract with his suppliers. Upon a mutually agreed (estimated) completion time, the contract between the manufacturer and the customer specifies the price $2q$ to be paid to the manufacturer upon delivery of the product or service. The manufacturer can shorten the completion time by increasing his suppliers' incentives to finish more quickly. Naturally, the customer has no say as regards the manufacturer's contract with his suppliers. Second, in our model the supply contract has two elements: the price p paid to each supplier and the regime which determines when the supplier will be paid. This regime is either the time the supplier completes her task, or the time all suppliers complete their tasks. Third, in our model the manufacturer and the suppliers play a Stackelberg game in which the manufacturer acts as the leader and the suppliers act as (simultaneous) followers. Under the delayed payment regime, each supplier takes the other supplier's work rate into consideration: they select their work rates in accord with a Nash equilibrium.

To the best of our knowledge, a time-based contract with a delayed payment has not been examined previously in the literature.[6] In particular, there are three features of the time-based contract analyzed in this article which differ markedly from the existing supply contract literature. First, under the delayed payment regime, each supplier receives payment at the time when both suppliers have completed their tasks. Consequently, each supplier needs to take into account the other supplier's behavior when selecting her own work rate. It is through this interaction among suppliers that the several underlying supply contracts are, in effect, transformed into a single joint supply contract between the manufacturer and his multiple suppliers. This linking of the several suppliers is a fundamental and crucial departure from the traditional supply contract.

Second, in our model, the supply contract is specified by the price paid to each supplier and the time of payment; the supplier's response to these two contract terms is her work rate. These three choice variables, price offered, time of payment, and work rate, are fundamentally different from

[5]See So (2000) and the references therein for more details.

[6]The reader is referred to comprehensive reviews of supply contracts provided by Cachon (2003) and Tang (2006).

the traditional supply contracts in the Operations Management literature. There the manufacturer sets the wholesale price schedule (often a fixed cost and a constant per unit cost), and the retailer sets the order quantity (and the manufacturer has no say as regards the retail price). The reader is referred to Corbett and Tang (1998) and Cachon and Lariviere (2005) for discussions of different types of supply contracts that deal with price menus, information asymmetry, and revenue sharing.[7]

Third, our model focuses upon the (perhaps) uncertain completion time of all supplier tasks when the deliverable is a customized product or service. The number of units to be delivered is not an issue. In contrast, the primary focus in the traditional supply contract literature is uncertain customer demand. In short, the focus of our model is the time-based contract arising from delivery of the first unit of a customized product (or service) whereas the focus of the traditional supply contract literature is on the quantity-based contract arising from a non-customized product with uncertain demand.

3 The Base Model: Deterministic Completion Time

In the supply chain we consider, the manufacturer and his customer enter into a contract under which the customer will pay $2q$ to the manufacturer when the job is complete. (We implicitly assume that the estimated completion time is agreeable to both parties.) This job consists of two parallel tasks, each of which is to be performed by distinct external suppliers. The supply contract designed by the manufacturer specifies the price p and the payment regime N or D. Under the no delayed payment regime N, each supplier is paid immediately after she completes her own task. Under D, each supplier is paid only when both suppliers have completed their tasks. Because we focus on a possible delay in the time of payment, an essential feature is the time value of money, captured via the continuous time discount rate $\alpha > 0$.

For ease of exposition, we assume the tasks are of equal difficulty and the suppliers have equal capability. It then makes sense to limit our attention to the case when the contracts with the suppliers are identical. In many instances, this assumption is reasonable and innocuous. For

[7] Revenue-sharing stipulates that a supplier will reduce her supply cost in exchange for a portion of buyer's revenue. For example, as described in Mortimer (2004), Blockbuster, the buyer, shares between 30-45% of its rental revenue with various distributors (suppliers) in exchange for a reduced wholesale price (say at $8 instead of $65 for each DVD). In the economics literature, Dana and Spier (2001) showed that revenue sharing contracts can be used to coordinate the supply chain; Cachon and Lariviere (2005) proved that the revenue sharing contracts are equivalent to buy back contracts; and Tang and Deo (2008) established the conditions under which revenue sharing is beneficial to both parties. Unlike revenue sharing that, for the most part, treats the case of one buyer and one supplier, our time-based contract stipulates that the payment for each supplier depends upon the performance of all suppliers.

example, in translation services, the price for translating a document into Spanish or Italian is usually the same because the difficulty of translation is quite similar.

Without loss of generality, we assume that each task contains 1 unit of work to be performed by the supplier. Given the supply contract, the price p paid to the supplier and the payment regime N or D, each supplier selects her work rate r, where r is restricted to the values $\{1, \lambda\}$ with $\lambda > 1$. Hence, each supplier will complete her task at time 1 or time $1/\lambda$ when the work rate she selects is equal to 1 or λ, respectively. For each work rate, the associated total cost for a supplier to perform her task is comprised of two components. The first component is a fixed cost $c \geq 0$ which is independent of her work rate and is paid when the supplier is paid. The fixed cost includes the cost of materials and equipment, R&D, and other fixed costs of making the product or delivering the service. The second component is the operating cost per unit time: it depends upon the work rate and includes overtime costs and other costs associated with speedier delivery. Without loss of generality, we set the operating cost per unit time associated with work rate 1 equal to 0, and we set the operating cost per unit time associated with work rate λ equal to $k > 0$. Therefore, when operating at rate λ, the total discounted operating cost is equal to $\int_0^{1/\lambda} k \cdot e^{-\alpha t} dt = \frac{k}{\alpha}(1 - e^{\alpha/\lambda})$.

To ensure that the supplier's net profit is non-negative so that she will in fact participate, the contract price p must satisfy $p \geq c$. Similarly, $p \leq q$ is necessary to ensure that the manufacturer's net profit is non-negative.

Throughout this paper, it is the manufacturer who determines the terms of the contract with the two suppliers. Our analysis begins with the manufacturer's revenue of $2q$ already having been specified. Given the supply contract between the manufacturer and the supplier, each supplier determines her work rate (or effort level) so as to maximize her own profit. The manufacturer is playing a Stackelberg game in which he has the first move and the two suppliers simultaneously move second. The manufacturer starts by selecting the regime. As in a backward recursion, the manufacturer considers a price p and computes the work rate that the suppliers will select given the value of p and his associated profit. In view of his associated profit, the manufacturer selects that value of p, denoted by p^*, which maximizes his profit. Thus, the manufacturer informs the suppliers of the regime and the price p^*; in response, the suppliers select their optimal work rates – the work rates that the manufacturer anticipated.

3.1 N: The No Delayed Payment Regime

Under N, each supplier receives her payment p immediately after she completes her own task at time $1/r$; $\Pi_s^N(r)$, each supplier's net profit associated with work rate r, is given by

$$\Pi_s^N(r) \quad = \quad \begin{cases} e^{-\alpha}(p-c) & \text{if } r = 1 \\ e^{-\frac{\alpha}{\lambda}}(p-c) - \frac{k}{\alpha}(1-e^{-\frac{\alpha}{\lambda}}) & \text{if } r = \lambda, \end{cases} \tag{3.1}$$

where the term $\frac{k}{\alpha}(1-e^{-\frac{\alpha}{\lambda}})$ is the total discounted operating cost associated with rate λ. Setting $\theta = k(1-e^{-\alpha/\lambda})/[\alpha(e^{-\alpha/\lambda} - e^{-\alpha})]$, (3.1) reveals that $\Pi_s^N(\lambda) - \Pi_s^N(1)$ is negative when $p < c+\theta$ and positive otherwise. Therefore, the supplier's optimal work rate $r^N(p)$ satisfies

$$r^N(p) \quad = \quad \begin{cases} 1 & \text{if } c \leq p < c+\theta \\ \lambda & \text{if } p \geq c+\theta. \end{cases} \tag{3.2}$$

As evidenced in (3.2), a large payment p induces the supplier to select the faster work rate. Because the suppliers are identical in preferences and abilities, they select the same work rate for a given price and complete their tasks at the same time. Hence, (3.2) reveals that $\Pi_m^N(p,q)$, the manufacturer's net profit, is given by

$$\Pi_m^N(p,q) \quad = \quad \begin{cases} 2(q-p)e^{-\alpha} & \text{if } c \leq p < c+\theta \\ 2(q-p)e^{-\alpha/\lambda} & \text{if } p \geq c+\theta. \end{cases} \tag{3.3}$$

Let $p^N(q)$ be the price p that maximizes $\Pi_m^N(p,q):$ $p^N(q)$ is the optimal price when the customer pays $2q$. Define $\Pi_m^N(q) \equiv \Pi_m^N(p^N(q),q)$ to be the manufacturer's maximal profit. Usually, we abbreviate the optimal price and write p^N in place of $p^N(q)$ when dependence upon q is clear.

Lemma 1 *The manufacturer's optimal price p^N and optimal net profit $\Pi_m^N(q)$ satisfy*

$$p^N \quad = \quad \begin{cases} c & \text{if } c \leq q < \gamma, \\ c+\theta & \text{if } q \geq \gamma; \end{cases} \tag{3.4}$$

$$\Pi_m^N(q) \quad = \quad \begin{cases} 2(q-c)e^{-\alpha} & \text{if } c \leq q < \gamma, \\ 2(q-(c+\theta))e^{-\frac{\alpha}{\lambda}} & \text{if } q \geq \gamma, \end{cases} \tag{3.5}$$

where $\gamma = c + \theta \cdot \frac{e^{-\frac{\alpha}{\lambda}}}{(e^{-\frac{\alpha}{\lambda}} - e^{-\alpha})} = c + \frac{k(1-e^{-\frac{\alpha}{\lambda}})}{\alpha(e^{-\frac{\alpha}{\lambda}} - e^{-\alpha})} \cdot \frac{e^{-\frac{\alpha}{\lambda}}}{(e^{-\frac{\alpha}{\lambda}} - e^{-\alpha})}.$

The manufacturer's pricing strategy is particularly simple: set the lowest price consistent with supplier participation when q is small ($q < \gamma$) and set the lowest price sufficient to induce the suppliers to work at the faster rate when q is high ($q \geq \gamma$). As q increases, the manufacturer becomes more sensitive to the time the job is completed; consequently, the manufacturer's willingness to pay the suppliers a premium (equal to θ) to work faster increases with q, and this willingness crosses a threshold at $q = \gamma$.

3.2 D: The Delayed Payment Regime

Under D, each supplier receives her payment p when both suppliers have completed their tasks. When r_1 and r_2 are the work rates of suppliers 1 and 2, respectively, $\Pi_s^D(r_1, r_2)(p)$, the net profit of supplier 1, takes the form [8]

$$\Pi_s^D(r_1, r_2)(p) \quad = \quad \begin{cases} e^{-\alpha}(p-c) & \text{if } (r_1, r_2) = (1,1) \\ e^{-\alpha}(p-c) & \text{if } (r_1, r_2) = (1,\lambda) \\ e^{-\alpha}(p-c) - \frac{k}{\alpha}(1-e^{-\alpha/\lambda}) & \text{if } (r_1, r_2) = (\lambda,1) \\ e^{-\alpha/\lambda}(p-c) - \frac{k}{\alpha}(1-e^{-\alpha/\lambda}) & \text{if } (r_1, r_2) = (\lambda,\lambda) \end{cases} \tag{3.6}$$

By noting that $e^{-\alpha}(p-c) < e^{-\frac{\alpha}{\lambda}}(p-c) - \frac{k}{\alpha}(1-e^{-\frac{\alpha}{\lambda}})$ if and only if $p \geq c + \theta$, it is easy to check that $r^D(p)$, each supplier's optimal work rate in equilibrium, satisfies

$$r^D(p) \quad = \quad \begin{cases} 1 & \text{if } c \leq p < c+\theta \\ \lambda & \text{if } p \geq c+\theta. \end{cases} \tag{3.7}$$

Comparing (3.2) to (3.7), we see that $r^D(p) = r^N(p)$. Let $\Pi_m^D(p,q)$ denote the manufacturer's profit under regime D when his revenue is $2q$ and he pays each supplier p. Also, let $p^D(q)$ denote the price p which maximizes $\Pi_m^D(p,q)$ so that $\Pi_m^D(q) \equiv \Pi_m^D(p^D(q), q)$ is the manufacturer's optimal profit when his revenue is $2q$. With deterministic completion times, the manufacturer is indifferent between the two payment regimes because $r^D(p) = r^N(p)$. Hence, the following Proposition holds:

Proposition 1 *We have $p^D(q) = p^N(q)$, $\Pi_s^D(r_1, r_2) = \Pi_s^N(r_1, r_2)$, and $\Pi_m^D(q) = \Pi_m^N(q)$.*

4 Exponential Completion Time

In this section we investigate whether or not and how the choice of payment regime impacts the three market participants' behavior when the completion time is stochastic. For tractability and

[8] For brevity, we display only the profit function of supplier 1 whose work rate is r_1. The profit function of supplier 2 is easily obtained through symmetry.

to retain the same expected completion time as in the base model, we assume that the completion time is exponentially distributed with parameter r, where $r \in \{1, \lambda\}$.[9] Working at rate λ, the supplier's expected discounted operating cost equals $\int_0^\infty [\int_0^x \; k \cdot e^{-\alpha t} dt] \; \lambda e^{-\lambda x} dx = k/(\alpha + \lambda)$.

Let the supplier's completion time X be an exponential random variable with parameter r. An important quantity in our analysis is $\beta(r) = E(e^{-\alpha X})$: $\beta(r)$ represents the expected "discounted time" until the supplier completes her task. If the suppliers work at rates r_1 and r_2 and each takes an amount of time X_i to complete her task, then both tasks will be completed at time $T = \max\{X_1, X_2\}$. In this case, $\beta(r_1, r_2) = E(e^{-\alpha T})$ represents the expected discounted time until both suppliers complete their tasks.

Lemma 2 *The expected discounted times* $\beta(r)$ *and* $\beta(r_1, r_2)$ *satisfy*

$$\beta(r) \equiv E(e^{-\alpha X}) = r/(\alpha + r). \tag{4.1}$$

$$\beta(r_1, r_2) \equiv E(e^{-\alpha T}) = \frac{r_1}{\alpha + r_1} + \frac{r_2}{\alpha + r_2} - \frac{r_1 + r_2}{\alpha + r_1 + r_2}. \tag{4.2}$$

4.1 N: The No Delayed Payment Regime

Under N, each supplier receives her payment p immediately after completing her own task. Each supplier's net profit is given by [10]

$$\widetilde{\Pi}_s^N(r) = \begin{cases} (p - c)\beta(1) & \text{if } r = 1 \\ (p - c)\beta(\lambda) - \frac{k}{\alpha+\lambda} & \text{if } r = \lambda. \end{cases} \tag{4.3}$$

Comparing the net profit for these two rates, we see that $\widetilde{r}^N(p)$, the supplier's optimal work rate, is

$$\widetilde{r}^N(p) = \begin{cases} 1 & \text{if } c \le p < c + \widetilde{\theta}^N \\ \lambda & \text{if } p \ge c + \widetilde{\theta}^N, \end{cases} \tag{4.4}$$

where $\widetilde{\theta}^N = \frac{k}{\alpha+\lambda} \cdot \frac{1}{\beta(\lambda) - \beta(1)}$.

Under N, the manufacturer pays p to each supplier once she completes her own task even though the manufacturer does not receive his payment of $2q$ until both suppliers have completed

[9] Dean et al. (1969) argues that an exponential completion time is more realistic in the context of project management than the Normally distributed completion time that is commonly used (e.g., Bayiz and Corbett (2005)). In project management, it was observed that the uncertain completion time is usually caused by the occurrence of an unforeseen situation. Hence, the distribution of the completion time should be positively skewed, which is a property of the exponential distribution.

[10] We place a $\sim$ over the symbols p, r, Π, θ, and γ to distinguish their values here in section 4 in the presence of uncertainty from their values without uncertainty in section 3.

their tasks. The supplier's optimal work rate $\widetilde{r}^N(p)$ given in (4.4) and the discount factors $\beta(r)$ and $\beta(r_1, r_2)$ given in (4.1) and (4.2) enable us to determine $\widetilde{\Pi}_m^N(p,q)$, the manufacturer's expected net profit when p is the payment to each supplier and $2q$ is the customer's payment to the manufacturer:

$$\widetilde{\Pi}_m^N(p,q) \quad = \quad \begin{cases} 2q\beta(1,1) - 2p\beta(1) & \text{if } c \le p < c + \widetilde{\theta}^N \\ 2q\beta(\lambda,\lambda) - 2p\beta(\lambda) & \text{if } p \ge c + \widetilde{\theta}^N. \end{cases} \tag{4.5}$$

Let $\widetilde{p}^N(q)$, sometimes written as $\widetilde{p}^N$, denote the price p that maximizes $\widetilde{\Pi}_m^N(p,q)$ so that $\widetilde{\Pi}_m^N(q) \equiv \widetilde{\Pi}_m^N(\widetilde{p}^N, q)$. Whereas $p \ge c$ is required to ensure that the suppliers participate, $q \ge c$ is not sufficient to guarantee that the manufacturer offers a price $p \ge c$. In particular, there is a zero-profits threshold $\widetilde{z} > c$ such that $\widetilde{\Pi}_m^N(c,q) < 0$ for $q < \widetilde{z}$ and $\widetilde{\Pi}_m^N(c,q) \ge 0$ for $q \ge \widetilde{z}$.[11]

Lemma 3 *The manufacturer's optimal price* $\widetilde{p}^N$ *and optimal profit* $\widetilde{\Pi}_m^N(q)$ *satisfy*

$$\widetilde{p}^N \quad = \quad \begin{cases} c & \textit{if } \widetilde{z} \le q < \widetilde{\gamma}^N, \\ c + \widetilde{\theta}^N & \textit{if } q \ge \widetilde{\gamma}^N; \end{cases} \tag{4.6}$$

$$\widetilde{\Pi}_m^N(q) \quad = \quad \begin{cases} 2q\beta(1,1) - 2c\beta(1) & \textit{if } \widetilde{z} \le q < \widetilde{\gamma}^N, \\ 2q\beta(\lambda,\lambda) - 2(c + \widetilde{\theta}^N)\beta(\lambda) & \textit{if } q \ge \widetilde{\gamma}^N, \end{cases} \tag{4.7}$$

where $\widetilde{\gamma}^N = c\,\frac{\beta(\lambda)-\beta(1)}{\beta(\lambda,\lambda)-\beta(1,1)} + \widetilde{\theta}^N \frac{\beta(\lambda)}{\beta(\lambda,\lambda)-\beta(1,1)} = c\,\frac{\beta(\lambda)-\beta(1)}{\beta(\lambda,\lambda)-\beta(1,1)} + \frac{k}{\alpha+\lambda}\,\frac{1}{\beta(\lambda)-\beta(1)}\,\frac{\beta(\lambda)}{\beta(\lambda,\lambda)-\beta(1,1)}.$

The insight revealed by Lemma 3 is the same as that revealed by Lemma 1: the manufacturer sets the lowest price to ensure supplier participation when q is small and sets the lowest price sufficient to induce the suppliers to work at the faster rate when q is large.

Suppose $\widetilde{p}^N(q) = c$ for $c < q < c + \epsilon$ with ϵ small; then $\widetilde{\Pi}_m^N(q) = \widetilde{\Pi}_m^N(c,q) < 0$ for $q \in (c, c+\epsilon)$: the manufacturer's optimal net profit is negative when $p \ge c$ and q is small. This negative profit is caused by the time delay between paying the first supplier to finish and receiving $2q$ from the customer when both suppliers have finished. This time delay is eliminated by switching to regime D.

4.2 D: The Delayed Payment Regime

Under regime D, each supplier receives her payment p when both suppliers have completed their tasks. We use the expected discount factor $\beta(r_1, r_2)$ given in (4.2) to determine $\widetilde{\Pi}_s^D(r_1, r_2)$, the

[11] Because the manufacturer receives his payment when both suppliers have finished and one supplier receives her payment earlier, $\widetilde{\Pi}_m^N(p,p) < 0$ for all $p \ge c$. Moreover, $\widetilde{\Pi}_m^N(c,q)$ is continuous and strictly increasing in q. Hence, $\widetilde{z} > c$.

expected net profit of supplier 1:[12]

$$\widetilde{\Pi}_s^D(r_1, r_2) = \begin{cases} (p-c)\beta(1,1) & \text{if } (r_1,r_2) = (1,1) \\ (p-c)\beta(1,\lambda) & \text{if } (r_1,r_2) = (1,\lambda) \\ (p-c)\beta(\lambda,1) - \frac{k}{\alpha+\lambda} & \text{if } (r_1,r_2) = (\lambda,1) \\ (p-c)\beta(\lambda,\lambda) - \frac{k}{\alpha+\lambda} & \text{if } (r_1,r_2) = (\lambda,\lambda) \end{cases} \tag{4.8}$$

Examining the 2×2 payoff matrix of suppliers' profits associated with the possibly different work rates selected by each supplier, we obtain the following result.

Proposition 2 *There are two possible Nash equilibria:* $(1,1)$ *and* (λ,λ)*. For* $c \le p < c + \widetilde{\theta}^D$ $(1,1)$ *is the unique equilibrium, where* $\widetilde{\theta}^D = \frac{k}{\alpha+\lambda} \cdot \frac{1}{\beta(\lambda,\lambda)-\beta(\lambda,1)}$ *. For* $p > c + \frac{k}{\alpha+\lambda} \cdot \frac{1}{\beta(\lambda,1)-\beta(1,1)}$*,* (λ,λ) *is the unique equilibrium. For* $c + \widetilde{\theta}^D \le p \le c + \frac{k}{\alpha+\lambda} \cdot \frac{1}{\beta(\lambda,1)-\beta(1,1)}$*, both* $(1,1)$ *and* (λ,λ) *are Nash equilibria, and* (λ,λ) *Pareto dominates* $(1,1)$*.*

As per Proposition 2, the suppliers earn higher profits by operating at rate λ instead of rate 1 when p is in the intermediate range. Because it is Pareto optimal for the suppliers to select the equilibrium that yields higher profits, we proceed as if they do so. Hence, the equilibrium work rates, denoted by $(\widetilde{r}^D(p), \widetilde{r}^D(p))$, satisfy

$$(\widetilde{r}^D(p), \widetilde{r}^D(p)) = \begin{cases} (1,1) & \text{if } c \le p < c + \widetilde{\theta}^D, \\ (\lambda,\lambda) & \text{if } p \ge c + \widetilde{\theta}^D. \end{cases} \tag{4.9}$$

Given the manufacturer's price p, the supplier's equilibrium work rate $\widetilde{r}^D(p)$ is specified in (4.9) so that the manufacturer's net profit is

$$\widetilde{\Pi}_m^D(p,q) = \begin{cases} 2(q-p)\beta(1,1) & \text{if } c \le p < c + \widetilde{\theta}^D \\ 2(q-p)\beta(\lambda,\lambda) & \text{if } p \ge c + \widetilde{\theta}^D. \end{cases} \tag{4.10}$$

Lemma 4 *The manufacturer's optimal price* $\widetilde{p}^D$ *and optimal profit* $\widetilde{\Pi}_m^D(q)$ *satisfy*

$$\widetilde{p}^D = \begin{cases} c & \text{if } c \le q < \widetilde{\gamma}^D, \\ c + \widetilde{\theta}^D & \text{if } q \ge \widetilde{\gamma}^D; \end{cases} \tag{4.11}$$

$$\widetilde{\Pi}_m^D(q) \equiv \widetilde{\Pi}_m^D(\widetilde{p}^D, q) = \begin{cases} 2(q-c)\beta(1,1) & \text{if } c \le q < \widetilde{\gamma}^D, \\ 2(q-(c+\widetilde{\theta}^D))\beta(\lambda,\lambda) & \text{if } q \ge \widetilde{\gamma}^D, \end{cases} \tag{4.12}$$

where $\widetilde{\gamma}^D = c + \widetilde{\theta}^D \cdot \frac{\beta(\lambda,\lambda)}{\beta(\lambda,\lambda)-\beta(1,1)} = c + \frac{k}{\alpha+\lambda} \cdot \frac{1}{\beta(\lambda,\lambda)-\beta(\lambda,1)} \cdot \frac{\beta(\lambda,\lambda)}{\beta(\lambda,\lambda)-\beta(1,1)}$

[12] For brevity, we display only the profit function of supplier 1 whose work rate is r_1. The profit function of supplier 2 is easily obtained through symmetry.

4.3 Choosing the Payment Regime

It comes as no surprise that the random completion time induces the suppliers to select a slower work rate under D than under N. The reason for this is that the uncertain completion time greatly impedes the suppliers' ability to coordinate and prevents them from completing the tasks at the same time. This result, presented formally in the next lemma, follows from comparing the suppliers' optimal work rates given in (4.4) and (4.9).

Before presenting this lemma, we hasten to mention that we should not be cavalier as regards what is and what is not obvious. Consider, for example, the following:

Remark: Comparing $\widetilde{\Pi}_s$ and Π_s. A simple application of Jensen's inequality reveals that both the discount factor $E(e^{-\alpha X})$ and the expected contribution $(-kE\int_0^X e^{-\alpha t}dt)$ to profits of the variable cost are larger when X is an exponential random variable with parameter r than when X is deterministic with mean $1/r$. By considering these two facts along with (3.1) and (4.3), it is easy to check that

$$\widetilde{\Pi}_s^N(r) \geq \Pi_s^N(r), \quad \text{for } r \in \{1, \lambda\}.$$

Consequently, given the manufacturer's price p and payment regime N, the supplier's profit is larger when the completion times are uncertain (exponential) rather than deterministic.

Next, consider regime D. While Proposition 2 informs us that the Pareto Optimal Nash equilibrium is symmetric, it is easy to check from (3.6) and (4.8) that

$$\widetilde{\Pi}_s^D(1,1) > \Pi_s^D(1,1) \quad \text{if and only if } \alpha > \alpha^*,$$

and $\widetilde{\Pi}_s^D(\lambda,\lambda) > \Pi_s^D(\lambda,\lambda)$ if $\alpha/\lambda > \alpha^*$, where α^* solves $2 = (\alpha+1)(\alpha+2)\cdot e^{-\alpha}$. In addition, it can shown that there is a number $\alpha^{**} < \alpha^*$ such that

$$\widetilde{\Pi}_s^D(\lambda,\lambda) > \Pi_s^D(\lambda,\lambda) \quad \text{if and only if } \alpha/\lambda > \alpha^{**}.$$

In summary, under N, we find that the supplier's profit is greater when completion times are exponential. Under D, however, we require $\alpha > \lambda\alpha^*$ in order to ensure that the supplier's profit is greater when completion times are exponential. ■

Lemma 5 *For all* $p \geq c$, $\widetilde{\theta}^D \geq \widetilde{\theta}^N$ *so that* $\tilde{r}^D(p) \leq \tilde{r}^N(p)$.

Lemma 5 informs us that under regime D the manufacturer must offer a "premium" to induce the supplier to work as fast as she works under regime N. This is because, under D, the coupling of the random times to complete the task with receipt of payment delayed until both suppliers have completed their tasks lessens each supplier's incentive to work quickly.

Because $\tilde{r}^D(p) \leq \tilde{r}^N(p)$, the completion time of the entire job is (stochastically) larger under D. Whereas our model entails exponential completion times, a more general intuition informed by Lemma 5 suggests that each supplier will work less quickly under D than under N when the uncertain completion times are other than exponential.

In order to induce each supplier to work faster, the manufacturer must offer a higher price. This raises an interesting question: Is it optimal for the manufacturer to offer a higher price under D? Comparing the manufacturer's optimal price $\tilde{p}^N$ and $\tilde{p}^D$ given in (4.6) and (4.11), we find that the answer hinges on whether or not the following condition holds:

$$\frac{c}{k} > \frac{\lambda(\alpha+1)^2(\alpha+2\lambda)(\alpha+2)}{\alpha^2(\lambda-1)^2(2\alpha+2\lambda+1)(\alpha^2-2\lambda)}. \tag{4.13}$$

Proposition 3 *If condition (4.13) holds, then $\tilde{\gamma}^D < \tilde{\gamma}^N$ so that $\tilde{p}^D > \tilde{p}^N$. However, if condition (4.13) is violated, then $\tilde{\gamma}^D \geq \tilde{\gamma}^N$ and*

$$\tilde{p}^D = \begin{cases} c = \tilde{p}^N & \text{if } c \leq q < \tilde{\gamma}^N \\ c < c + \tilde{\theta}^N = \tilde{p}^N & \text{if } \tilde{\gamma}^N \leq q < \tilde{\gamma}^D \\ c + \tilde{\theta}^D > c + \tilde{\theta}^N = \tilde{p}^N & \text{if } q \geq \tilde{\gamma}^D. \end{cases} \tag{4.14}$$

Proposition 3 establishes conditions under which the answer to our question is affirmative: if the fixed cost c is sufficiently large or if the operating cost per unit time k is sufficiently small, then it is optimal for the manufacturer to offer a higher price under D.

Let us compare the manufacturer's expected profits $\tilde{\Pi}_m^N(q)$ and $\tilde{\Pi}_m^D(q)$ given in (4.7) and (4.12), respectively. As depicted in Figure 1, the slope of $\tilde{\Pi}_m^N(q)$ and $\tilde{\Pi}_m^D(q)$ is $\beta(1,1)$ when q is small and $\beta(\lambda,\lambda)$ when q is large. Accordingly, it is easy to observe from Figure 1 that these two profit functions will cross if and only if $\tilde{\Pi}_m^N(\tilde{\gamma}^D) > \tilde{\Pi}_m^D(\tilde{\gamma}^D)$. This condition is equivalent to (4.15) holding:

$$\frac{c}{k} < \frac{(\alpha+1)(\alpha+2\lambda)}{\alpha^2(\lambda-1)(2\alpha+2\lambda+1)}. \tag{4.15}$$

Proposition 4 *If (4.15) holds, then there exists a threshold η such that $\tilde{\Pi}_m^D(q) > \tilde{\Pi}_m^N(q)$ if and only if $q < \eta$. If (4.15) is violated, then $\tilde{\Pi}_m^D(q) \geq \tilde{\Pi}_m^N(q)$ for all q.*

Insert Figure 1 about here.

Because the right-hand-side of (4.15) is a strictly decreasing function of λ, (4.15) holds when λ is small; hence, the manufacturer should offer contract D when $q < \eta$ and contract N when $q \geq \eta$. When λ is large [so (4.15) does not hold], Proposition 4 states that the manufacturer's contract should always require D.

Boeing's problems with late deliver of the 787 Dreamliner have been widely reported. Proposition 4 suggests one plausible explanation for the Boeing 787's delay besides those reported in the press (i.e., suppliers' capabilities, Boeing's supply management skills, and unproven technologies). As Boeing and various strategic suppliers participate in supply contracts with delayed payments D, these contracts could have been optimal if the suppliers' actual work rates λ were high enough. However, due to overconfidence in the use of unproven technologies, it is plausible that the suppliers' actual work rates λ turned out to be lower than what Boeing and the suppliers had anticipated. When λ is lower than expected, condition (4.15) is likely to hold. Combine this observation with the fact that the size q of the job is high, say, $q \geq \eta$, Proposition 4 suggests that had the unproven technologies performed as anticipated then Boeing would have been correct to operate under regime D. Because these technologies did not perform well, as is evident in hindsight, Boeing should have operated under regime N. Therefore, it is plausible that the delays Boeing has experienced is partially due to operating under regime D.

5 Variable Work Rate

In this section, we extend our model: each supplier can select her work rate r to be any non-negative number. Thus, the completion time is an exponential random variable with parameter $r \in [0, \infty)$. Associated with effort level r is an operating cost $\kappa(r)$ per unit time. We suppose that $\kappa(\cdot)$ is a convex increasing function. To simplify our analysis while retaining the spirit of the model, assume the operating cost per unit time increases with the square of the work rate: $\kappa(r) = kr^2$ with $k > 0$. Hence, the supplier's expected discounted operating cost equals $\int_0^\infty \left[\int_0^x \kappa(r) \cdot e^{-rt} dt\right] \lambda e^{-rx} dx = \kappa(r)/(r + \alpha)$.

To guarantee the supplier's participation, we assume the payment p to be received satisfies $p \geq c$, where c corresponds to the fixed cost c for performing the task. As before, each supplier selects her work rate r so as to maximize her expected discounted profit. While this extension is more realistic and more complex, some but not all of the results in Section 4 continue to hold.

5.1 N: The No Delayed Payment Regime

Under regime N, each supplier receives p when her task is completed. The expected discounted profit for a supplier with work rate r is given by[13]

$$\widehat{\Pi}_s^N(r) = (p-c)\beta(r) - \frac{\kappa(r)}{r+\alpha} \ ,$$

where $\beta(r)$ is the expected discounted time given in (4.1).

Proposition 5 *The supplier's profit function $\widehat{\Pi}_s^N(r)$ is concave in r, and $\widehat{r}^N(p)$, the supplier's optimal effort level, is given by*

$$\widehat{r}^N(p) \ = \ \alpha(\sqrt{1+\frac{p-c}{\alpha k}} - 1) \ , \tag{5.1}$$

so $\widehat{r}^N(p)$ is strictly increasing on (c,∞) and unbounded in p; moreover, for each $p > c$, $\widehat{r}^N(p)$ is strictly increasing and bounded in α.

Under payment regime N, the manufacturer's expected discounted profit satisfies

$$\widehat{\Pi}_m^N(p,q) = 2q\beta(\widehat{r}^N(p),\widehat{r}^N(p)) - 2p\beta(\widehat{r}^N(p)) \ .$$

Notice that p is the manufacturer's decision variable, and the manufacturer's optimal price $\widehat{p}^N(q)$ satisfies $\widehat{\Pi}_m^N(\widehat{p}^N(q),q) \geq \widehat{\Pi}_m^N(p,q)$ for all p. Sometimes we suppress the dependence on q and simply write $\widehat{p}^N$. When convenient, we write $\widehat{\Pi}_m^N(q) \equiv \widehat{\Pi}_m^N(\widehat{p}^N(q),q)$.

The manufacturer's participation requires that q exceed a zero profits threshold $\widehat{z} > c+\alpha k$: $\widehat{\Pi}_m^N(q) = 0$ for $q < \widehat{z}$ and $\widehat{\Pi}_m^N(q) > 0$ for $q > \widehat{z}$, where $\widehat{p}^N(q) = c$ for $q \leq \widehat{z}$. This will be shown in the proof of Proposition 9.

5.2 D : The Delayed Payment Regime

Employing regime D in this subsection, we show that (a) there exists a Nash equilibrium with positive work rates, (b) the equilibrium is unique and symmetric, and (c) the associated work rate is less than that under regime N. Also, (d) we obtain a closed form expression for the equilibrium work rates (see (5.2) and (5.3)), and (e) we show that the equilibrium work rates are slower than those under regime N for any given price.

[13] We place a $\widehat{\ }$ over the symbols p, r, Π, etc. for the case when the supplier can choose any non-negative work rate.

The expected discounted profit $\widehat{\Pi}_s^D(r,\mu)$ for a supplier whose work rate is r when the other supplier is working at rate μ is given by

$$\widehat{\Pi}_s^D(r,\mu) \;=\; (p-c)\beta(r,\mu) - \frac{\kappa(r)}{\alpha+r}\,.$$

The supplier's *best response* $R(\mu)$ when the other supplier works at rate μ is that work rate r which maximizes $\widehat{\Pi}_s^D(r,\mu)$: $\widehat{\Pi}_s^D(R(\mu),\mu) \;\geq\; \widehat{\Pi}_s^D(r,\mu)$ for all $r \geq 0$.

Lemma 6 *The supplier's best response $R(\mu)$ is unique and strictly increasing in μ.*

Lemma 7 *There are no asymmetric Nash equilibria.*

By Lemma 7, all Nash equilibria are symmetric. Let $\widehat{r}^D$ denote an equilibrium work rate so $\widehat{r}^D \;=\; R(\widehat{r}^D)$. Of course, $R(0) \;=\; 0$ so $(0,0)$ is always a Nash equilibrium: if $r_1 = 0$, then supplier 2 will never receive a payment so it is optimal for supplier 2 to select $r_2 = 0$.

Proposition 6 *If $p-c \;\leq\; \alpha k$, there are no Nash equilibria other than $(0,0)$. If $p-c > \alpha k$, then $(0,0)$ is an equilibrium, and $(\widehat{r}^D,\widehat{r}^D)$ is the unique Nash equilibrium with a positive work rate; furthermore,*

$$\widehat{r}^D(p) = \alpha\left[\sqrt{1+\frac{p-c}{k\alpha}}\cos(\phi/3) - 1\right]\,, \tag{5.2}$$

and the parameter ϕ is defined by

$$\phi \equiv \pi - \arctan\sqrt{\frac{p-c}{k\alpha}}\,. \tag{5.3}$$

Moreover, $\widehat{r}^D(p)$ is strictly increasing in p for $p \geq c+\alpha k$, unbounded in p, and unimodal in α with $\widehat{r}^D(p) \to 0$ as $\alpha \to \infty$.

Corollary 1 *For each $p > c+\alpha k$, $\;\widehat{r}^D(p) \;<\; \widehat{r}^N(p)$.*

For any given p, Corollary 1 establishes that in equilibrium the delayed payment of regime D leads the suppliers to reduce their work rate as compared to the work rate under regime N. This result is essentially that of Lemma 5 in Section 4.3.

We note four additional implications of Proposition 6. First, when there is a Nash equilibrium with positive work rate, the suppliers earn positive profits; of course, the supplier's profit is 0 when her work rate is 0. Consequently, it is Pareto optimal for the suppliers to select the equilibrium associated with (5.2) when $p-c > \alpha k$.

Second, even with our simple framework and tractable cost function $\kappa(\cdot)$, the supplier's optimal work rate has a complex-looking, albeit closed-form, solution.

Third, the α-dependence of $\widehat{r}^D(p)$ is non-monotonic. In fact, $\widehat{r}^D(p)$ increases in α for small α and decreases in α for large α. The small-α behavior is similar to that of $\widehat{r}^N$. In the large-α limit, the suppliers' reward from completing the tasks is so heavily discounted that the suppliers are discouraged from exerting much effort: the optimal work rate decreases to 0 as α approaches infinity.

Fourth, the fact that the supplier's optimal work rate is unbounded in the payment p received suggests that when q is large the manufacturer might set p quite large so as to induce the suppliers to select a large work rate. Indeed, this conjecture is true. Proposition 7, presented in the next subsection, reveals that $\widehat{p}^D(q)$ and $\widehat{p}^N(q)$ both increase without bound (at a rate proportional to $q^{2/3}$) as q increases without bound.

Of course the magnitude of q influences the manufacturer's choice of p. The manufacturer's expected discounted profit is

$$\widehat{\Pi}_m^D(p,q) = 2(q-p)\beta(\widehat{r}^D(p), \widehat{r}^D(p)) ,$$

and $\widehat{p}^D(q)$, the manufacturer's optimal price, satisfies $\widehat{\Pi}_m^D(\widehat{p}^D(q),q) \geq \widehat{\Pi}_m^D(p,q)$ for all p. When convenient, we suppress the explicit dependence upon q of the optimal price and write $\widehat{p}^D$ in place of $\widehat{p}^D(q)$. We also write $\widehat{\Pi}_m^D(q) \equiv \widehat{\Pi}_m^D(\widehat{p}^D(q),q)$.

5.3 Choosing the Payment Regime

Before comparing the manufacturer's optimal price and his associated profit under the two regimes, we pause to consider the suppliers' participation. The bottom line is that supplier participation requires $q > c + \alpha k$ under both regimes.

To understand why supplier participation requires q in excess of $c+\alpha k$, first consider regime D. Obviously, $\widehat{p}^D(q) \leq q$; otherwise $\widehat{\Pi}_m^D(q) < 0$. By Proposition 6, $\widehat{r}^D(p) = 0$ for $p \leq c + \alpha k$. Hence, $\widehat{r}^D(\widehat{p}^D(q)) = 0$ and $\widehat{\Pi}_m^D(q) = 0$ for $q \leq c + \alpha k$.

Under regime N, the length of time between payment to the first supplier to finish and the time when the other supplier finishes and, simultaneously, the customer pays the manufacturer, is exponential. This gap between paying $\widehat{p}^N(q)$ and receiving $2q$ from the customer reduces the manufacturer's profit. As can be seen from (7.9) in the Appendix, this gap cannot be bridged when $q \leq c + \alpha k$. Consequently, $\widehat{p}^N(q) = c$ result in $\widehat{r}^N(c) = 0$ when $q \leq c + \alpha k$. In fact, as pointed out at the end of subsection 5.1, $\widehat{p}^N(q) = c$ for $q \le \widehat{z}$ where the manufacturers zero profits threshold $\widehat{z}$ strictly exceeds $c + \alpha k$.

Just as in Lemmas 3 and 4, Proposition 7 states that in both regimes the optimal price is strictly increasing in q.

Proposition 7 *The optimal price $\widehat{p}^D(q)$ is strictly increasing in q for $q \ge c + \alpha k$. The optimal price $\widehat{p}^N(q)$ is strictly increasing in q for $q \ge \widehat{z}$ where $\widehat{z} > c + \alpha k$.*

Lemma 8 *The profit functions $\widehat{\Pi}_m^N(q)$ and $\widehat{\Pi}_m^D(q)$ are convex and non-decreasing in q. Moreover, $\widehat{\Pi}_m^N(q)$ is strictly increasing in q for $q > \widehat{z}$, and $\widehat{\Pi}_m^D(q)$ is strictly in q for $q > c + \alpha k$.*

Lemma 8 is consistent with the shape of the profit functions as shown in Figure 1 of Section 4.3. It appears plausible that for all $q > c + \alpha k$, the manufacturer's optimal price is larger under D than under N because the delay in payment the suppliers experience requires a larger price to induce them to work at any given rate. Although this may in fact be true, we are able to prove this only when q is large and when q is small. Similarly, it is plausible that given the optimal price paid, each supplier's work rate is larger under N. The next two propositions show that this is true when q is large, but the opposite holds when q is small. In the same vein, and in sharp contrast to Proposition 4, the manufacturer's optimal profit is larger under N when q is large and smaller under N when q is small. The crossover between the optimal profits under regime N and regime D takes place conditionally in the two-rate model (Proposition 4). Propositions 8 and 9 verify that $\Pi_m^D(q) - \Pi_m^N(q)$ changes sign, from positive to negative, at least once (unconditionally) in the variable-rate model.

Proposition 8 *For sufficiently large values of q, $\widehat{p}^D(q) > \widehat{p}^N(q)$, $\widehat{r}^N(\widehat{p}^N(q)) > \widehat{r}^D(\widehat{p}^D(q))$, and $\widehat{\Pi}_m^N(q) > \widehat{\Pi}_m^D(q)$. Moreover, both $\widehat{p}^N(q)/q^{2/3}$ and $\widehat{p}^D(q)/q^{2/3}$ converge to positive constants as q increases without bound.*

Proposition 9 *For sufficiently small values of q with $q > c + \alpha k$, $\widehat{p}^D(q) > \widehat{p}^N(q)$, $\widehat{r}^N(\widehat{p}^N(q)) < \widehat{r}^D(\widehat{p}^D(q))$, and $\widehat{\Pi}_m^N(q) < \widehat{\Pi}_m^D(q)$.*

There are two countervailing forces as regards the manufacturer's expected profit. Regime D is disadvantaged in that $\widehat{r}^D(p) < \widehat{r}^N(p)$ by Corollary 1. On the other hand, under D the manufacturer benefits from not having to pay the first supplier to finish until he receives his own payment of $2q$. Proposition 9 informs us that when profits are very small (i.e., when q is closed to $c + \alpha k$), the latter force dominates. Proposition 8 informs us that the former force dominates when q is large: the suppliers work faster under N even though they are paid less.

From Propositions 8 and 9, we conclude that there is at least one sign change of $\widehat{\Pi}_m^D(q) - \widehat{\Pi}_m^N(q)$. From Figure 2, which exhibits exactly one crossing for the case $c = k = \alpha = 1$, and from other numerical examples, we suspect that $\widehat{\Pi}_m^D(q) - \widehat{\Pi}_m^N(q)$ has exactly one change of sign. Again, from the example illustrated in Figure 2, we conjecture that $\widehat{p}^D(q) > \widehat{p}^N(q)$ for all $q > c + \alpha k$, which is in agreement with Propositions 8 and 9.

Insert Figure 2 about here.

6 Summary

Our initial interest in the delayed payment regime analyzed in this paper arose from a discussion of the widely reported delivery delays of Boeing's 787 Dreamliner and one particular feature of Boeing's risk-sharing contracts that makes them, in essence, the delayed payment regime. The motivation for our model and analysis was to determine whether or not it behooves a manufacturer to offer his suppliers the conventional no-delayed payment contract or a contract with delayed payment. With deterministic completion times, both suppliers and the manufacturer are indifferent between N and D: the work rates of the suppliers are the same and both the manufacturer's profit and the suppliers' profits are the same.

In the variable-rate model of Section 5 as well as in the two-rate model of Section 4, we showed that for a given price p the suppliers work more slowly under regime D than under regime N. Hence, the manufacturer must trade off the disadvantage of a slower work rate under D for the advantage of the delayed payment to his suppliers.

While some insights established in Section 4 continue to hold in Section 5, the analysis in Section 5 is more delicate and the results more nuanced as befits the more realistic model. We obtain closed-form solutions for the suppliers' equilibrium work rates in both regimes (Proposition 5 and 6). These closed-form solutions permit us to determine how the equilibrium work rates vary as functions of the manufacturer's price p and the discount rate; specifically, given the price p, the equilibrium work rate is higher in regime N (Corollary 1).

We can think of the manufacturer's revenue $2q$ as the size of the project. Proposition 7 shows that in both regimes the manufacturer's optimal price increases with q. Finally, Propositions 8 and 9 reveal that while the optimal price paid by the manufacturer is larger under regime D than under regime N, when q is large and when q is small. Both the suppliers' equilibrium work rate and the manufacturer's equilibrium profit are larger under N than under D when q is large but smaller under N than under D when q is small. In summary, the manufacturer benefits from using the delayed payment regime when his revenue is small, but is injured when his revenue is large.

References

[1] Bayiz, M., and Corbett, C.J., "Coordination and Incentive Contracts in Project Management under Asymmetric Information," working paper, UCLA Anderson School, 2005.

[2] Cachon, G., "Supply Chain Coordination with Contracts," in *Handbooks in Operations Research and Management Science*, (Eds. De Kok, A.G., and Graves, S.,), Elsevier Publisher, 2003.

[3] Cachon, G., and Lariviere, M., "Supply Chain Coordination with Revenue Sharing Contracts: Strengths and Limitations," *Management Science*, Vol. 51, pp. 30-44, 2005.

[4] Corbett, C., and Tang, C.S., "Designing Supply Contracts: Contract Type and Information Asymmetric Information," in *Quantitative Models for Supply Chain Management*, Eds. Tayur et al., Kluwer Publisher, 1998.

[5] Dana J., and Spier, K., "Revenue Sharing and Vertical Control in the Video Rental Industry," *Journal of Industrial Economics*, Vol. 59, pp. 223-245, 2001.

[6] Dean, B., Merterl Jr. S., and Roepke, L., "Research Project Cost Distribution and Budget Forecasting," *IEEE Transactions on Engineering Management,* Vol. 16, 4, pp. 176-191, 1969.

[7] Gates, D., "Another Big Delay for Boeing's 787," *Seattle Times*, January 16, 2008.

[8] Greising, D., and Johnsson, J., "Behind Boeing's 787 Delays," *Chicago Tribune*, December 8, 2007.

[9] Mortimer, J., "Vertical Contracts in the Video Rental Industry," working paper, Department of Economics, Harvard University, 2004.

[10] Nolan, R., and Kotha, S., "Boeing 787: The Dreamliner," Harvard Business School case 9-305-101, 2005.

[11] Palaka, K.,, Erlebacher, S., and Kropp, D.H., "Lead Time Setting, Capacity Utilization, and Pricing Decisions Under Lead Time Dependent Demand," *IIE Transactions*, Vol. 30, 2, pp. 151-163, 1998.

[12] So, K.C., "Price and Time Competition for Service Delivery," *Manufacturing and Service Operations Management*, Vol. 2, 4, pp. 392-409, 2000.

[13] So, K.C., and Song, J.S., "Price, Delivery Time Guarantees and Capacity Selection," *European Journal of Operational Research*, Vol. 111, 1, pp. 28-49, 1998.

[14] Ross, S., *Introduction to Stochastic Dynamic Programming*, Academic Press, 1983.

[15] Tang, C.S., "Perspectives in Supply Chain Risk Management," *International Journal of Production Economics*, Vol. 103, pp. 451-488, 2006.

[16] Tang, C.S., and Deo, S., "Rental Price and Rental Duration under Retail Competition," forthcoming, *European Journal of Operational Research*, 2008.

7 Appendix: Proofs

Proof of Lemma 1: From (3.3), $\Pi_m^N(p)$ is decreasing in p for $p \in [c, c+\theta)$ and for $p \in [c+\theta, \infty)$ so the optimal price p^N is either equal to c or $c+\theta$. The optimal net profit follows from substitution. ■

Proof of Lemma 2: The expression for $\beta(r)$ follows from the fact that $E(e^{-\alpha X}) = \int_0^\infty e^{-\alpha x}\, re^{-rx}\, dx = r/(\alpha + r)$. Next, one way to compute $\beta(r_1, r_2)$ is by brute force: write down the density of T and then integrate $E(e^{-\alpha T})$. A second way is to observe that T is the sum of two terms. The first term is the time until one of the suppliers finishes, which is exponential with parameter $r_1 + r_2$; by (4.1) this discounted time is $(r_1 + r_2)/(\alpha + r_1 + r_2)$. The second term is, with probability $r_1/(r_1 + r_2)$, the time until the second supplier finishes and with probability $r_2/(r_1 + r_2)$ the time until the first supplier finishes. These facts yield

$$E[e^{-\alpha T}] = [\frac{(r_1 + r_2)}{\alpha + r_1 + r_2}] \quad \cdot \quad [\frac{r_1}{r_1 + r_2} \cdot \frac{r_2}{\alpha + r_2} \ + \ \frac{r_2}{r_1 + r_2} \cdot \frac{r_1}{\alpha + r_1}].$$

Simple algebraic manipulation of the above equation yields (4.2). ■

The proofs of various Lemmas and Propositions involve the same comparisons related to $\beta(r)$ and $\beta(r_1, r_2)$. To avoid duplication, we establish the following lemma for ease of reference.

Lemma 9 *The expected discount factors $\beta(r)$ and $\beta(r_1, r_2)$ possess the following properties:*

1. $\beta(\lambda) > \beta(1)$ *and* $\beta(r) > \beta(r, r)$, *for all* $r > 0$.

2. $\beta(r_1, r_1) \leq \beta(r_1, r_2) \leq \beta(r_2, r_2)$, *for all* $r_1 \leq r_2$.

3. $\beta(r_1, r_2)$ *is supermodular:* $\partial\beta(r_1, r_2)/\partial r_1 \partial r_2 > 0$.

4. $\beta(\lambda) - \beta(1) > \beta(\lambda, \lambda) - \beta(\lambda, 1)$, *for any* $\lambda > 1$.

5. $\frac{\beta(\lambda)}{\beta(\lambda)-\beta(1)} < \frac{\beta(\lambda,\lambda)}{\beta(\lambda,\lambda)-\beta(\lambda,1)}$, *for any* $\lambda > 1$.

6. $\frac{\beta(\lambda)-\beta(1)}{\beta(\lambda,\lambda)-\beta(1,1)} < 1$ *if and only if* $\lambda > \alpha^2/2$, *for any* $\lambda > 1$.

7. $\frac{\beta(\lambda)-\beta(1,1)}{\beta(\lambda,\lambda)-\beta(1,1)} > 1$, *for any* $\lambda > 1$.

Proof of Lemma 9: Statement 1 follows immediately from the fact that $\beta(r) - \beta(r, r) = \frac{\alpha r}{(\alpha+r)(\alpha+2r)} > 0$. By noting that $\frac{\partial\beta(r_1,r_2)}{\partial r_1} = \alpha \cdot (\frac{1}{(\alpha+r_1)^2} - \frac{1}{(\alpha+r_1+r_2)^2}) > 0$ and by using the fact that the function is symmetric, we obtain statement 2. Statement 3 follows immediately from the fact that $\frac{\partial^2\beta(r_1,r_2)}{\partial r_2 \partial r_1} > 0$. We now prove statements 4, 5, 6, and 7. By substituting $r_2 = \lambda$ and $r_1 = 1$ into (4.1) and (4.2) and by rearranging the terms, one can show that $[\beta(\lambda) - \beta(1)] - [\beta(\lambda, \lambda) - \beta(\lambda, 1)]$ can be simplified as $\frac{\alpha(\lambda-1)[\alpha^2+\alpha(\lambda-1)+2(\lambda^2-1)+2\alpha\lambda)}{(\alpha+\lambda)(\alpha+2\lambda)(\alpha+1)(\alpha+\lambda+1)}$. By noting this term is positive because $\lambda > 1$, we have proved statement 4. Statement 5 is true if and only if $\frac{\beta(\lambda)}{\beta(\lambda)-\beta(1)} - \frac{\beta(\lambda,\lambda)}{\beta(\lambda,\lambda)-\beta(\lambda,1)} < 0$. By expanding the terms and by rearranging the terms, the left hand side can be simplified as $\frac{-\lambda(\alpha+1)}{\alpha(\lambda-1)(2\alpha+2\lambda+1)}$, which is negative. This proves statement 5. To prove Statement 6, observe that the term $\frac{\beta(\lambda)-\beta(1)}{\beta(\lambda,\lambda)-\beta(1,1)} - 1$ can be reduced to $\frac{\alpha^2-2\lambda}{2(3\lambda+\alpha\lambda+\alpha)}$. Hence, statement 6 is true if and only if $\alpha^2/2 < \lambda$. To prove statement 7, notice that the term $\frac{\beta(\lambda)-\beta(1,1)}{\beta(\lambda,\lambda)-\beta(1,1)} - 1$ can be reduced to $\frac{\lambda(\alpha+1)(\alpha+2)}{2(\lambda-1)(3\lambda+\alpha\lambda+\alpha)} > 0$. This completes the proof. ■

Proof of Lemma 3: By using the same argument as presented in the proof of Lemma 1, the value of the optimal price $\widetilde{p}^N$ hinges on the inequality $\widetilde{\Pi}_m^N(c + \widetilde{\theta}^N) \geq \widetilde{\Pi}_m^N(c)$. By substitution and by rearranging the terms, this inequality holds when $q \geq \widetilde{\gamma}^N$. By substituting the optimal price $\widetilde{p}^N$ into (4.5), we obtain the manufacturer's optimal profit. ■

Proof of Proposition 2: First consider the case when $p \in [c + \frac{k}{\alpha+\lambda} \cdot \frac{1}{\beta(\lambda,1)-\beta(1,1)}, \infty)$. Then

$$\widetilde{\Pi}_s^D(\lambda, 1) = (p - c)\beta(\lambda, 1) - \frac{k}{\alpha + \lambda} > (p - c)\beta(1, 1) = \widetilde{\Pi}_s^D(1, 1) \tag{7.1}$$

It follows from statement 3 in Lemma 9, $\frac{1}{\beta(\lambda,1)-\beta(1,1)} > \frac{1}{\beta(\lambda,\lambda)-\beta(\lambda,1)}$. This implies that, when p is in this range, p also satisfies $p > c + \frac{k}{\alpha+\lambda} \cdot \frac{1}{\beta(\lambda,\lambda)-\beta(\lambda,1)}$. Therefore, it is easy to check that:

$$\widetilde{\Pi}_s^D(\lambda, \lambda) = (p - c)\beta(\lambda, \lambda) - \frac{k}{\alpha + \lambda} > (p - c)\beta(\lambda, 1) = \widetilde{\Pi}_s^D(1, \lambda) \tag{7.2}$$

By using the above inequalities along with the fact that $\beta(\lambda, 1) = \beta(1, \lambda)$, one can show that (λ, λ) is the unique Nash equilibrium when $p \in [c + \frac{k}{\alpha+\lambda} \cdot \frac{1}{\beta(\lambda,1)-\beta(1,1)}, \infty)$.

Next, consider the case when $p \in [c + \frac{k}{\alpha+\lambda} \cdot \frac{1}{\beta(\lambda,\lambda)-\beta(\lambda,1)}, c + \frac{k}{\alpha+\lambda} \cdot \frac{1}{\beta(\lambda,1)-\beta(1,1)})$. Then inequality (7.1) is violated, while inequality (7.2) continues to hold, so both (λ, λ) and $(1, 1)$ are Nash equilibria.

From statement 2 in Lemma 9 and (7.2), $\widetilde{\Pi}_s^D(\lambda, \lambda) = (p-c)\beta(\lambda, \lambda) - \frac{k}{\alpha+\lambda} > (p-c)\beta(\lambda, 1) > (p-c)\beta(1,1) = \widetilde{\Pi}_s^D(1,1)$. Thus, (λ, λ) Pareto-dominates $(1,1)$ in this range. Finally, we can use similar arguments to show that $(1,1)$ is the unique Nash equilibrium when $p \in [c, c + \frac{k}{\alpha+\lambda} \cdot \frac{1}{\beta(\lambda,\lambda)-\beta(\lambda,1)})$. We omit the details. ■

Proof of Lemma 4: The proof follows the same argument as presented in the proof of Lemma 3. We omit the details. ■

Proof of Lemma 5: Observe from (4.4) and (4.9) that $\widetilde{r}^D(p) \le \widetilde{r}^N(p)$ if and only if $\widetilde{\theta}^N \le \widetilde{\theta}^D$. It follows from the definitions of $\widetilde{\theta}^N$ and $\widetilde{\theta}^D$ that $\widetilde{\theta}^N \le \widetilde{\theta}^D$ if and only if $\beta(\lambda)-\beta(1) > \beta(\lambda,\lambda)-\beta(\lambda,1)$, which is just statement 4 in Lemma 9. ■

Proof of Proposition 3: Recall from the proof of Lemma 4 that $\widetilde{\theta}^N \le \widetilde{\theta}^D$. Then it is easy to check from (4.6) and (4.11) that $\widetilde{p}^N < \widetilde{p}^D$ when $\widetilde{\gamma}^D < \widetilde{\gamma}^N$, and that $\widetilde{p}^N$ and $\widetilde{p}^D$ satisfy (4.14) when $\widetilde{\gamma}^D \ge \widetilde{\gamma}^N$. Hence, it remains to identify the condition(s) under which $\widetilde{\gamma}^D < \widetilde{\gamma}^N$. Let: $Z = \frac{\beta(\lambda,\lambda)}{\beta(\lambda,\lambda)-\beta(\lambda,1)}$, $Y = \frac{\beta(1)}{\beta(\lambda)-\beta(1)}$, and $X = \frac{\beta(\lambda)-\beta(1)}{\beta(\lambda,\lambda)-\beta(1,1)}$. From statement 5 in Lemma 9, $Z > Y$ and from statement 6 in Lemma 9, $(X-1) > 0$ if and only if $\lambda < \frac{\alpha^2}{2}$. These imply that $\widetilde{\gamma}^D < \widetilde{\gamma}^N$ if and only if $\frac{c}{k} > \frac{1}{(\alpha+\lambda)(\beta(\lambda,\lambda)-\beta(1,1))} \cdot \frac{(Z-Y)}{(X-1)}$. By expanding and rearranging the terms, $\widetilde{\gamma}^D < \widetilde{\gamma}^N$ if and only if (4.13) holds. ■

Proof of Proposition 4: Let us make two observations from Figure 1. First, $\widetilde{\Pi}_m^N(q) \le \widetilde{\Pi}_m^D(q)$ for all q if the two profit functions do not cross. Second, the two profit functions will cross exactly once at $q = \eta$ if and only if $\widetilde{\Pi}_m^N(\widetilde{\gamma}^D) > \widetilde{\Pi}_m^D(\widetilde{\gamma}^D)$, in which case $\widetilde{\Pi}_m^N(q) \le \widetilde{\Pi}_m^D(q)$ when $q < \eta$ and $\widetilde{\Pi}_m^N(q) > \widetilde{\Pi}_m^D(q)$ when $q > \eta$. Therefore, it suffices to determine the condition(s) under which $\widetilde{\Pi}_m^N(\widetilde{\gamma}^D) > \widetilde{\Pi}_m^D(\widetilde{\gamma}^D)$. It follows from (4.7) and (4.12) that $\widetilde{\Pi}_m^N(\widetilde{\gamma}^D) > \widetilde{\Pi}_m^D(\widetilde{\gamma}^D)$ if and only if $2\widetilde{\gamma}^D\beta(\lambda,\lambda) - 2(c+\widetilde{\theta}^D)\beta(\lambda) > 2(\widetilde{\gamma}^D - c)\beta(1,1)$. By substituting $\widetilde{\gamma}^N$ and $\widetilde{\gamma}^N$ from Lemmas 3 and 4 into $2\widetilde{\gamma}^D\beta(\lambda,\lambda) - 2(c+\widetilde{\theta}^D)\beta(\lambda)$ and $2(\widetilde{\gamma}^D - c)\beta(1,1)$, by applying statement 7 in Lemma 9, and by rearranging terms, one can show that $2\widetilde{\gamma}^D\beta(\lambda,\lambda) - 2(c+\widetilde{\theta}^D)\beta(\lambda) > 2(\widetilde{\gamma}^D - c)\beta(1,1)$ holds if and only if condition (4.15) holds. ■

Proof of Proposition 5: The first and second derivatives of $\widehat{\Pi}_s^N(r)$ are

$$\begin{aligned} \frac{d}{dr}\widehat{\Pi}_s^N(r) &= \frac{-kr^2 - 2k\alpha r + (p-c)\alpha}{(r+\alpha)^2} \\ \frac{d^2}{dr^2}\widehat{\Pi}_s^N(r) &= -\frac{2\alpha(\alpha k + p - c)}{(r+\alpha)^3} < 0\,, \end{aligned}$$

so $\widehat{\Pi}_s^N(r)$ is concave; it is unimodal because $d\widehat{\Pi}_s^N(r)/dr$ changes sign once from positive to negative. Hence, there is a unique solution to the first-order condition $d\widehat{\Pi}_s^N(r)/dr = 0$, and it is given by (5.1). Clearly, $\widehat{r}^N(p)$ is strictly increasing in p, and $\frac{d}{d\alpha}\widehat{r}^N(p) > 0$ so $\widehat{r}^N(p)$ is strictly increasing in

α. Finally, using a Taylor series expansion in α^{-1}, we can show $\lim_{\alpha\to\infty} \widehat{r}^N(p) = (p-c)/2k < \infty$, so it is bounded in α. ■

Proof of Lemma 6: It is easy to show that $\beta(r,\mu)$ is concave in r and supermodular $(\partial^2\beta(r,\mu)/\partial r\partial\mu > 0)$ and that $-\kappa(r)/(r+\alpha)$ is concave in r. Consequently, $\widehat{\Pi}_s^D(r,\mu)$ is concave and supermodular. Therefore, the best response $R(\mu)$ is the unique solution to $d\widehat{\Pi}_s^D(r,\mu)/dr = 0$. Finally, $R(\mu)$ is strictly increasing in μ because $\widehat{\Pi}_s^D(r,\mu)$ is supermodular. ■

Proof of Lemma 7: Suppose there is an asymmetric Nash equilibrium (x,y) with $y > x$. Using Lemma 5, we have $y > x = R(y) > R(x) = y$, contradiction. ■

Proof of Proposition 6: With $f(r,\mu) \equiv (p-c)\beta(r,\mu) - \kappa(r)/(\alpha+r)$, the best response r to μ is found by solving $\partial f(r,\mu)/\partial r \equiv g(r,\mu) = 0$. That is, a symmetric Nash equilibrium (r^*,r^*) is obtained by solving $g(r,r) = 0$. We have

$$g(r,r) = -\frac{r\cdot k\cdot h(r)}{(r+\alpha)^2(2r+\alpha)^2},$$

where $h(x) \equiv 4x^3 + 12\alpha x^2 + 9\alpha^2 x - 3\frac{p-c}{k}\alpha x + 2\alpha^3 - 2\frac{p-c}{k}\alpha^2$. Hence, r^* satisfies $h(r^*) = 0$.

Next, we prove that there is a unique positive root of $h(\cdot)$ when $p-c > k\alpha$, and it is given in (5.2). First, we can verify that $\widehat{r}^D$ defined in (5.2) and (5.3) satisfies $h(\widehat{r}^D) = 0$ by substituting $\widehat{r}^D$ in the function $h(\cdot)$. Second, at $p-c = k\alpha$, we have $\phi = 3\pi/4$ and $\cos(\phi/3) = 1/\sqrt{2}$ so that $\widehat{r}^D(c+k\alpha) = 0$; it is straightforward to show that $\widehat{r}^D(p)$ is increasing in p by directly calculating $d\widehat{r}^D(p)/dp$, and hence, $\widehat{r}^D > 0$ if and only if $p-c > k\alpha$. It remains to show that $\widehat{r}^D$ is the only positive root of $h(x) = 0$ when $p-c > k\alpha$ and that there are no positive roots when $p-c \le k\alpha$.

The larger root of the quadratic equation $h'(x) = 12x^2 + 24\alpha x + 9\alpha^2 - 3(p-c)\alpha/k = 0$ is $\widetilde{x} = \alpha[\sqrt{1+(p-c)/(\alpha k)} - 2]/2$ which is greater than $-\alpha$. Also from $h''(x) = 24(x+\alpha) > 0$ for $x > -\alpha$, we find that $h(\cdot)$ is strictly decreasing in the interval $(-\alpha,\widetilde{x})$ and strictly increasing in $(\widetilde{x},\infty)$. We also note that

$$h(-\alpha) = \alpha^3 + (p-c)\alpha^2/k > 0\,,\ \text{and}$$
$$h(\widetilde{x}) = -(p-c+k\alpha)\alpha^2(\sqrt{1+(p-c)/k\alpha} - 1)/k < 0\,.$$

Hence, from the continuity of $h(\cdot)$ and the limits $h(-\infty) = -\infty$ and $h(\infty) = \infty$, there is exactly one root of $h(x) = 0$ in each of the intervals $(-\infty,-\alpha)$, $(-\alpha,\widetilde{x})$, and $(\widetilde{x},\infty)$.

If $p-c \le k\alpha$, then $\widetilde{x} \le \alpha(\sqrt{2}-2)/2 < 0$ and $h(0) = 2\alpha^3 - 2\frac{p-c}{k}\alpha^2 \ge 0$ so there is no positive root of $h(x) = 0$. If $p-c > k\alpha$, then $h(0) < 0$ and $h(-\alpha) > 0$ so there is exactly one root each in the intervals $(-\infty,-\alpha)$, $(-\alpha,0)$ and $(0,\infty)$. Thus, there is exactly one positive root if and only if $p-c > k\alpha$.

The optimal work rate can written as $\widehat{r}^D(p) = \alpha f(\frac{p-c}{\alpha k})$ where

$$f(x) \equiv \sqrt{1+x}\cos[(\pi - \arctan\sqrt{x})/3] - 1\,.$$

The function $f(x)$ is strictly increasing and unbounded in x. Thus, $\widehat{r}^D(p)$ is strictly increasing and unbounded in p. Moreover, $xf(x^{-1})$ is unimodal in x, so $\widehat{r}^D(p)$ is unimodal in α. ■

Proof of Corollary 1: Notice that $d[\arctan(x)]/dx = 1/[1+x^2] > 0$ so $\arctan\sqrt{(p-c)/(k\alpha)}$ is increasing in p. By (5.3), this reveals that ϕ is decreasing in p. Moreover, because $0 < \arctan\sqrt{\frac{p-c}{k\alpha}} < \pi/2$, we have $\pi/6 < \phi/3 < \pi/3$. Because cosine is decreasing on $(0, \pi/2)$, $\cos(\phi/3)$ is increasing in p. Thus, the closed form expression given in (5.2) makes clear that $\widehat{r}^D(p)$ increases in p without bound. We also have $0 < 1/2 = \cos(\pi/3) < \cos(\phi/3) < \cos(\pi/6) = \sqrt{3}/2 < 1$. Coupling $\cos(\phi/3) < 1$ with inspection of (5.1) and (5.2) proves Corollary 1. ■

Proof of Proposition 7: Note that $\widehat{\Pi}^N_m(p, q)$ is continuously differentiable in p so that $\widehat{\Pi}^N_m(\cdot, q)$ is strictly positive on the bounded interval $(c, \overline{q})$ where $\overline{q} < q$. The point of this discussion is that $\widehat{p}^N(q)$ satisfies the first-order condition $\frac{\partial}{\partial p}\widehat{\Pi}^N_m(p, q) = 0$. Similarly, $\widehat{\Pi}^D_m(p, q)$ is continuously differentiable in p, and the set where it is strictly positive is a bounded open set. Consequently, $\widehat{p}^D$ also satisfies the first-order condition $\frac{\partial}{\partial p}\widehat{\Pi}^D_m(p, q) = 0$. From the first derivatives

$$\frac{\partial}{\partial p}\widehat{\Pi}^N_m(p, q) = 2q\frac{d\widehat{r}^N}{dp}\frac{d}{d\widehat{r}^N}\beta(\widehat{r}^N, \widehat{r}^N) - 2p\frac{d\widehat{r}^N}{dp}\frac{d}{d\widehat{r}^N}\beta(\widehat{r}^N) - 2\beta(\widehat{r}^N) \tag{7.3}$$

$$\frac{\partial}{\partial p}\widehat{\Pi}^D_m(p, q) = 2(q-p)\frac{d\widehat{r}^D}{dp}\frac{d}{d\widehat{r}^D}\beta(\widehat{r}^D, \widehat{r}^D) - 2\beta(\widehat{r}^D, \widehat{r}^D)\,, \tag{7.4}$$

we find that

$$\frac{\partial}{\partial q}\frac{\partial}{\partial p}\widehat{\Pi}^j_m(p, q) = 2q\frac{d\widehat{r}^j}{dp}\frac{d}{d\widehat{r}^j}\beta(\widehat{r}^j, \widehat{r}^j)$$

which is non-negative in general because $d\widehat{r}^j/dp > 0$ by Proposition 5. Moreover, $\frac{\partial}{\partial q}\frac{\partial}{\partial p}\widehat{\Pi}^j_m(p, q)$ is positive if $\widehat{r}^j > 0$ because $d\beta(r, r)/dr > 0$ if and only if $r > 0$. In particular, $\widehat{p}^D(q) > 0$ for $q > c + \alpha k$ and $\widehat{p}^N(q) > 0$ for $q > \widehat{z}$ as shown in the proof of Proposition 9. Thus, $\widehat{\Pi}^N_m(p, q)$ is supermodular whence the statement of the proposition follows (see Ross (1983), p. 6). ■

Proof of Lemma 8: Given any pair $q_1 < q_2$, let $p_1 = \widehat{p}^N(q_1)$ and $p_2 = \widehat{p}^N(q_2)$ so that

$$\begin{aligned}\widehat{\Pi}^N_m(p_1, q_1) &= 2q_1\beta(\widehat{r}^N(p_1), \widehat{r}^N(p_1)) - 2p_1\beta(\widehat{r}^N(p_1)) \\ &\leq 2q_2\beta(\widehat{r}^N(p_1), \widehat{r}^N(p_1)) - 2p_1\beta(\widehat{r}^N(p_1)) \leq \widehat{\Pi}^N_m(p_2, q_2)\,.\end{aligned}$$

Hence, $\widehat{\Pi}^N_m(q)$ is non-decreasing in q. Moreover, if $\widehat{r}^N(p_1) > 0$, then $\beta(\widehat{r}^N(p_1) > 0$, so $\widehat{\Pi}^N_m(p_1, q_1) < \widehat{\Pi}^N_m(p_2, q_2)$. Therefore, $\widehat{\Pi}^N_m(q)$ is strictly increasing in q for $q > \widehat{z}$. Next, let $q_x = xq_1 + (1-x)q_2$

where $x \in (0,1)$ and define $p_x = \widehat{p}^N(q_x)$. Then

$$\begin{aligned}
\widehat{\Pi}^N_m(q_x) &= 2q_x\beta(\widehat{r}^N(p_x),\widehat{r}^N(p_x)) - 2p_x\beta(\widehat{r}^N(p_x)) \\
&= x[2q_1\beta(\widehat{r}^N(p_x),\widehat{r}^N(p_x)) - 2p_x\beta(\widehat{r}^N(p_x))] \\
&\quad +(1-x)[2q_2\beta(\widehat{r}^N(p_x),\widehat{r}^N(p_x)) - 2p_x\beta(\widehat{r}^N(p_x))] \\
&\leq x\widehat{\Pi}^N_m(q_1) + (1-x)\widehat{\Pi}^N_m(q_2) \quad .
\end{aligned}$$

These arguments apply to prove that $\widehat{\Pi}^D_m(q)$ is convex and non-decreasing in q and that it is strictly increasing q for $q > c + \alpha k$. ■

Proof of Proposition 8: We first show that $\widehat{p}^j(q)$ is unbounded in q, where $j = N$ or $j = D$. Suppose on the contrary that $\lim_{q\to\infty}\widehat{p}^j = \bar{p}^j < \infty$. Because $\bar{p}^j > c$, $\widehat{p}^j$ satisfies the first-order conditions, so (7.3) and (7.4) vanish. Also, the derivatives of the discount factors $\beta(r)$ and $\beta(r,r)$ approach positive constants in the limit when $q \to \infty$. Hence, $\frac{\partial}{\partial p}\widehat{\Pi}^j_m(p,q)$ at $p = \widehat{p}^j$ for large q is a very large positive number, which contradicts the necessary first-order conditions.

In order to distinguish leading order terms from subleading order terms in both limits, we use the following notation for subleading order terms:

$$\begin{aligned}
f(x) &= O(x) \text{ if there is } K > 0 \text{ such that } f(x) \leq Kx \text{ for large } x \;, \\
f(x) &= o(x) \text{ if } \frac{f(x)}{x} \to 0 \text{ as } x \to \infty \;.
\end{aligned}$$

For large values of p,

$$\begin{aligned}
\widehat{r}^N(p) &= \alpha(\sqrt{1+p/(\alpha k)} - 1) = \alpha(\sqrt{p/(\alpha k)} - 1) + O(\sqrt{(\alpha k/p)}) && (7.5)\\
\widehat{r}^D(p) &= \alpha\left[\sqrt{1+p/(k\alpha)}\,\cos(\phi/3) - 1\right] = \alpha(\sqrt{3p/(4k\alpha)} - \frac{7}{6}) + O(\sqrt{\alpha k/p}) && (7.6)
\end{aligned}$$

The first-order conditions for $\widehat{p}^j$ in the large-q and large-p limits are

$$\begin{aligned}
2[\frac{q}{(\widehat{p}^N)^{3/2}}\cdot\frac{3\sqrt{\alpha k}}{4} - 1] + O((\widehat{p}^N)^{-2}) &= 0 \\
2[\frac{q}{(\widehat{p}^D)^{3/2}}\cdot\frac{\sqrt{3\alpha k}}{2} - 1] + O((\widehat{p}^D)^{-2}) &= 0 \,.
\end{aligned}$$

Hence,

$$\begin{aligned}
\widehat{p}^N(q) &= (\frac{3\sqrt{\alpha k}}{4}q)^{2/3}(1+o(1)) \,, && (7.7)\\
\widehat{p}^D(q) &= (\sqrt{\frac{3\alpha k}{4}}q)^{2/3}(1+o(1)) \,. && (7.8)
\end{aligned}$$

Thus, $\widehat{p}^D(q) > \widehat{p}^N(q)$ for large values of q. It follows from (7.5) and (7.6) that $\widehat{r}^N(\widehat{p}^N(q)) > \widehat{r}^D(\widehat{p}^D(q))$ for large q.

Next, consider the manufacturer's optimal profit. For large q, (7.5) and (7.7) yield

$$\widehat{\Pi}_m^N(q) = 2q\beta_2(\widehat{r}^N, \widehat{r}^N) - 2\widehat{p}^N\beta_1(\widehat{r}^N) = 2[q - 3(\frac{3\sqrt{\alpha k}}{4} \cdot q)^{2/3}] + o(q^{2/3}).$$

Similarly, (7.6) and (7.8) yield

$$\widehat{\Pi}_m^D(q) = 2(q - \widehat{p}^D(q))\beta_2(\widehat{r}^D, \widehat{r}^D) = 2[q - 3(\frac{\sqrt{3\alpha k}}{2} \cdot q)^{2/3}] + o(q^{2/3}) .$$

Therefore, $\widehat{\Pi}_m^N(q) > \widehat{\Pi}_m^D(q)$ for large q. ■

Proof of Proposition 9: From (5.1), the supplier's work rate is strictly positive if and only if $p > c$ whence $p \leq c$ implies $\widehat{\Pi}_m^N(\widehat{p}^N(q), q) = 0$. Next, note that

$$\widehat{\Pi}_m^N(p, q) = 2\frac{\widehat{r}^N(p)\alpha^2 k}{(\widehat{r}^N(p) + \alpha)(2\widehat{r}^N(p) + \alpha)}[f(\frac{p-c}{\alpha k}) - \frac{c}{\alpha k} + (q - c - \alpha k)\frac{2\widehat{r}^N(p)}{\alpha k}] , \tag{7.9}$$

where $f(x) = 2(\sqrt{1+x} - 1)(1-x) - x$. It is easily verified that $\max_{x\geq 0} f(x) = 0 = f(0)$. Coupling this fact with (7.9) reveals that for all $q \leq c+\alpha k$ we have $\widehat{p}^N(q) = c$ and $\widehat{\Pi}_m^N(p, q) \leq 0$; moreover, $\widehat{\Pi}_m^N(\widehat{p}^N, c + \alpha k) = 0$.

Now let $q - c - \alpha k = \epsilon > 0$. For sufficiently small values of ϵ,

$$f(\frac{p-c}{\alpha k}) - \frac{c}{\alpha k} + \epsilon\frac{2\widehat{r}^N(p)}{\alpha k} < 0$$

for any $p \in (c, q)$, and hence $\widehat{p}^N = c$. It follows that $\widehat{\Pi}_m^N(\widehat{p}^N(q), q)$ is strictly 0 for q sufficiently close to $c + \alpha k$. Hence, there is $\widehat{z} > c + \alpha k$ such that $\widehat{\Pi}_m^N(\widehat{p}^N(q), q) = 0$ and $\widehat{p}^N(q) = c$ if and only if $q \leq \widehat{z}$ because $\widehat{\Pi}_m^N(\widehat{p}^N(q), q)$ and $\widehat{p}^N(q)$ are non-decreasing in q.

As in the proof of Proposition 8, we use the following notation for subleading order terms:

$$\begin{aligned} f(x) &= O(x) \text{ if there is } K > 0 \text{ such that } f(x) \leq Kx \text{ for small } x \ , \\ f(x) &= o(x) \text{ if } \frac{f(x)}{x} \to 0 \text{ as } x \to 0 \ . \end{aligned}$$

For the D regime, Proposition 6 informs us that if $p \leq c+\alpha k$, then $\widehat{r}^D(p) = 0$ so that $\widehat{\Pi}_m^D(p, q) = 0$. Of course, the manufacturer's profit cannot be positive if $p \geq q$. Hence, $\widehat{\Pi}_m^D(\widehat{p}^D(q), q) = 0$ for $q \leq c + \alpha k$. Defining $x \equiv (p - c - \alpha k)/\alpha k$ and $\bar{x} = (q - c - \alpha k)/\alpha k$ with the constraint $p < q$ (or $x < \bar{x}$), we find that

$$\widehat{r}^D(p) = \alpha\frac{x}{3} + O(\bar{x}^2) \text{ and}$$

$$\widehat{\Pi}_m^D(p, q) = \frac{4}{9}\alpha k(\bar{x} - x)x^2 + O(\bar{x}^4) .$$

Noting that $(\bar{x} - x)x^2$ is a unimodal function of x with the maximum value $4\bar{x}^3/27 > 0$ which occurs at $x = 2\bar{x}/3$, we find that $\widehat{\Pi}_m^D(q) > 0$ for all $q > c+\alpha k$. Therefore, $\widehat{\Pi}_m^D(q) > \widehat{\Pi}_m^N(q)$ for small enough values of q. Moreover, $\widehat{p}^D(q) > \widehat{p}^N(q) = c$ and $\widehat{r}^D(\widehat{p}^D(q)) = \alpha x/3 + O(\bar{x}^2) > \widehat{r}^N(\widehat{p}^N(q)) = 0$. ■

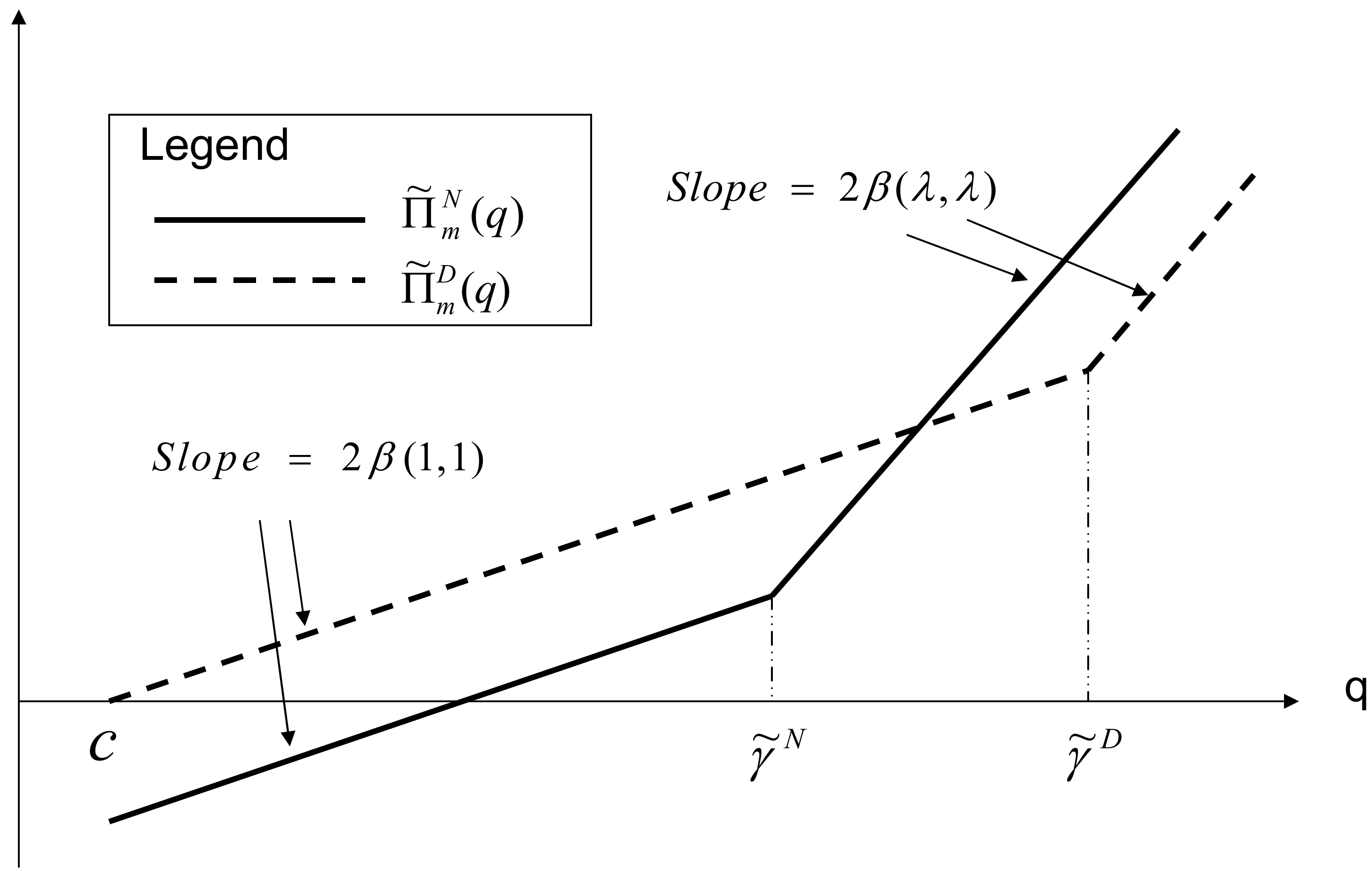

Figure 1. The manufacturer's optimal profit under payment regime N and D.

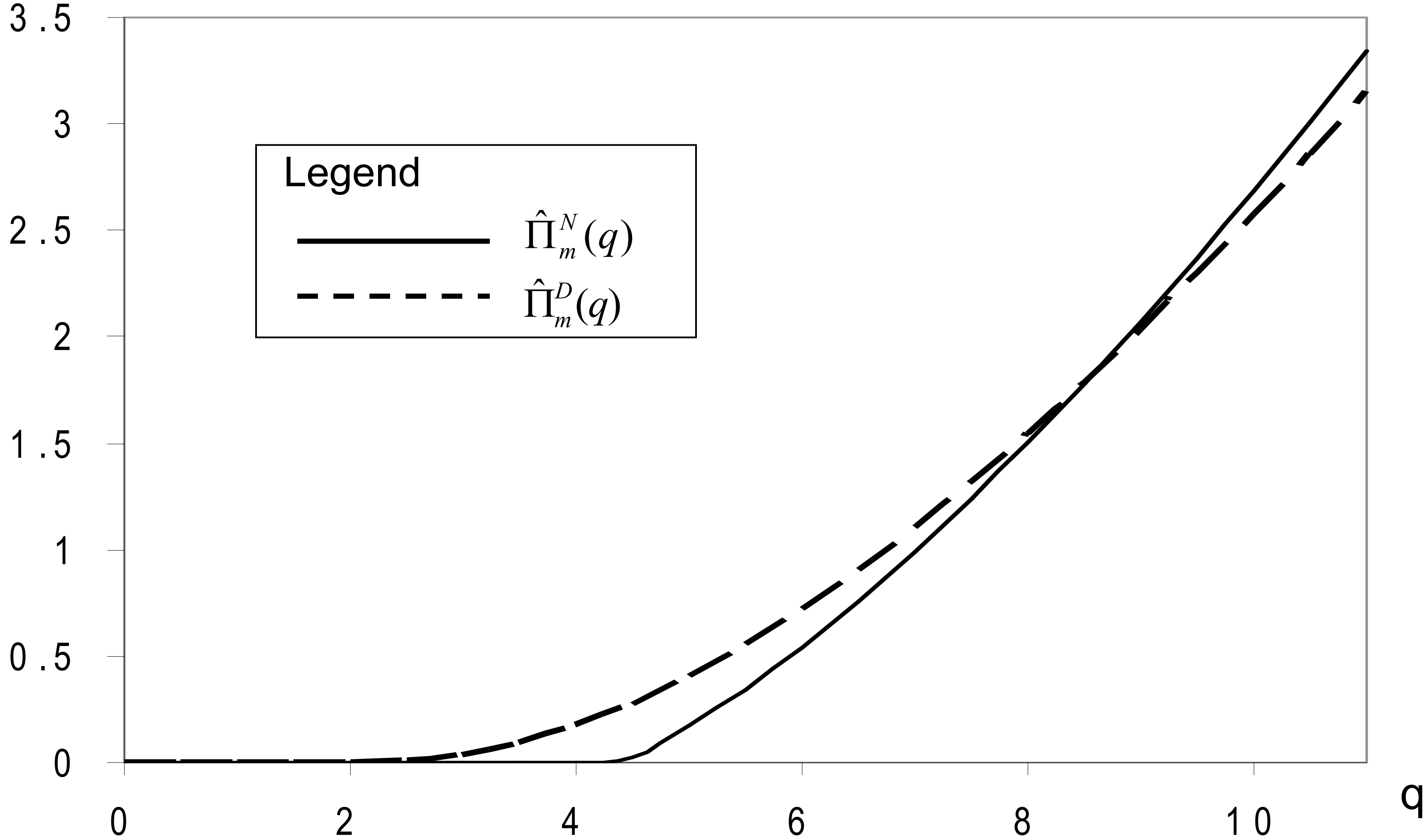

Figure 2. The manufacturer’s optimal profit under payment regime N and D.