Economics of Ransomware: Risk Interdependence and Large-Scale Attacks

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Abstract

Recently, the development of ransomware strains as well as changes in the marketplace for malware have greatly reduced the entry barrier for attackers to conduct large-scale ransomware attacks. In this paper, we examine how this mode of cyberattack impacts software vendors and consumer behavior. When victims face an added option to mitigate losses via a ransom payment, both the equilibrium market size and the vendor's profit under optimal pricing can actually increase in the ransom demand. Profit can also increase in the scale of residual losses following a ransom payment (which reflect the trustworthiness of the ransomware operator). We show that for intermediate levels of risk, the vendor restricts software adoption by substantially hiking up price. This lies in stark contrast to outcomes in a benchmark case involving traditional malware (non-ransomware) where the vendor decreases price as security risk increases. Social welfare is higher under ransomware compared to the benchmark in both sufficiently low and high-risk settings. However, for intermediate risk, it is better from a social standpoint if consumers do not have an option to pay ransom. We also show that the expected ransom paid is non-monotone in risk, increasing when risk is moderate in spite of a decreasing ransom-paying population. For ransomware attacks on other vectors (beyond patchable vulnerabilities), there can still be an incentive to hike price. However, market size and profits instead weakly decrease in the ransom amount. When studying a generalized model that includes both traditional and ransomware attacks, our results remain robust to a wide range of scenarios, including threat landscapes where ransomware has only a small presence.

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1 Introduction

In recent years, *ransomware* has evolved to become a prevalent class of malware due to improved use of encryption and attack vectors as well as increased maturity of cryptocurrency-based payment systems. Ransomware is an extortion-based attack that infects a computer system and subsequently prevents either access to the system (i.e., locker ransomware) or access to files or data (i.e., crypto ransomware) (Savage et al. 2015). Victims are typically threatened with permanent loss of access unless they pay a ransom. Having an additional decision for consumers[1] (i.e., whether to pay ransom) disrupts the economics underlying software usage and patching behaviors, and therefore ransomware may necessitate management strategies and policies that conflict with what served prior environmental characteristics well.

Over the past decade, ransomware experienced tremendous growth and even held the crown as the fastest growing cybersecurity threat (Cybersecurity Insiders 2017). The rate of ransomware attacks on businesses has been accelerating from one attack every 40 seconds in 2016 to one attack every 14 seconds in 2019 (Kaspersky 2016, Morgan 2019). Moreover, ransomware has continued to evolve to cause increased downtime, now averaging over 16 days (Palmer 2020). The overall damage that businesses incur from ransomware attacks (including payments, remediation, and downtime) is estimated to have exceeded $74 billion in 2020 (Emsisoft Malware Lab 2021).

Preventative actions are the best defense against ransomware (FBI 2016, U.S. Department of Justice 2017, No More Ransomware Project 2017). In fact, the U.S. Department of Health and Human Services delineates what healthcare providers are required to do to prevent ransomware infection in order to be HIPAA compliant (U.S. Department of Health and Human Services 2016). Timely patching of systems, as well as responsible access and communication management are considered among best practices, but many consumers regrettably do not comply. Sadly, this state of affairs has been the defining characteristic of security vulnerabilities for decades, and ransomware similarly exploits the same poor risk management practices.

Consider the example of WannaCry, one of the most prevalent ransomware attacks observed in the last few years which leveraged NSA-leaked infiltration and exploit tools (Sanders 2019). Microsoft had released a patch on March 14, 2017, yet two months later a sizeable number of unpatched systems enabled WannaCry to spread laterally fast, indiscriminately affecting over 230,000 computers across 150 countries in a day (Microsoft 2017, Cooper 2018). Even in 2019, WannaCry attacks still accounted for over six times as many detections com-

[1] Throughout the paper, we use "consumers" to refer to both businesses and individuals who employ the software in question.

pared to attacks from all other ransomware variants combined (Sanders 2019, Trend Micro Research 2019). These incidents highlight how large populations of unpatched consumers encourage the development of ransomware, facilitate its spread, and also keep the threat current. As software vendors and government agencies grapple with the significant losses being incurred, they have sought to understand how to respond to and operate in this new environment where affected consumers now face a ransom demand that can possibly mitigate losses.

Large-scale ransomware campaigns can also spread via unpatchable vectors such as phishing attacks and zero-day vulnerabilities. Phishing attacks bait consumers into action (e.g., opening attachments laced with malware, clicking on fake banner ads or malicious URLs). As an example, in August and September of 2017, Locky ransomware was pushed via multiple massive phishing campaigns to millions of consumers (Cabuhat et al. 2017; Palmer 2017), exploiting a long-known Microsoft Office vulnerability that Microsoft only permanently disabled in December 2017. In another example in 2019, attackers exploited a zero-day vulnerability in the widely used Oracle WebLogic server to install Sodinokibi and GandCrab ransomware on vulnerable machines, which necessitated no user interaction at all (Goddin 2019); this is among the first known cases where bad actors used a single attack to distribute two ransomware payloads (Splinters 2019).

Consumers facing large-scale cyberattacks, including ransomware, are also exposed to *interdependent* security risks. A larger at-risk population increases the risk for all individuals within the population. This can happen through a variety of mechanisms. Ransomware worms, such as ZCryptor, WannaCry, or Bad Rabbit, have the ability to self-replicate and travel laterally to other unprotected systems on the same computer network without any additional interaction or hacker intervention (Barkly 2017). In other instances, hackers install scanners on compromised systems to harvest credentials that enable them to more broadly infiltrate the corporation and possibly its partners and clients (Barak 2020). Lastly, having more consumers at risk can attract increased attention from malicious hackers. For these reasons, the risk of ransomware infection is characterized by network externalities.

Hacker motivations span human curiosity, a desire for fame, an anti-establishment agenda, economic objectives, hacktivism, and even cyberwarfare (Thomas and Stoddard 2012). Both WannaCry and NotPetya, the recent and largest ransomware attacks in history, were attributed to state actors, i.e., Russia and North Korea, respectively (Chappell and Neuman 2017; Marsh 2018). On the other hand, attackers using SamSam, Sodinokibi, Dharma or Ryuk ransomware tools appear to be more economically motivated, collecting millions of dollars in ransom payments according to FBI (Abrams 2020). In the case of Ryuk ransomware alone, attackers are estimated to have received over $150 million in ransom crypto payments

(Kremez and Carter 2021).

In that state actors' motivations are typically political in nature, those responsible for NotPetya and WannaCry did not bother to properly set up and configure effective processes to receive payments and return decryption keys to those who paid (Greenberg 2018).[2] Nevertheless, they clearly proved the feasibility of launching large-scale and disruptive ransomware attacks. An interesting question is whether these attacks could have caused greater economic damages had the ransom payment and decryption key delivery process actually been functional. For Dharma or Ryuk, which were clearly motivated by revenue generation, an open question is how would the scaling of such attacks to harness risk interdependence impact revenues, as consumers adjust their patching and usage strategies to risk expectations. This question has become more salient as some ransomware tools (e.g., Ryuk) initially designed for highly targeted attacks have evolved to include self-replicating capabilities (Arntz 2021). It is easy to see that the actual potential of ransomware has yet to be observed, and the economic models we develop in this paper help provide insight into what lies on the horizon.

Most prior work on how a software firm and its consumers react to security risk tends to model both patching costs and security losses (August and Tunca 2006; Cavusoglu et al. 2008; Dey et al. 2015). Ransomware presents a potential efficiency gain by offering a loss-mitigating payment opportunity, whereas in models of traditional attacks, victims typically do not have this opportunity and instead incur large valuation-dependent losses. To explore the impact of this cybersecurity threat, we construct a series of models that include the primitive elements that uniquely define ransomware. We then examine how the threat of ransomware affects consumers' choices as they face trade-offs between ex-ante security protection efforts like patching and ex-post ransom payments to agents with unlawful motives. The option to pay ransom shifts consumer strategies and modifies the network externality stemming from at-risk usage which fundamentally alters the decision problem that the vendor faces. These trade-offs become even more complex when both ransomware and traditional attacks are commingled in a single framework. In totality, we seek to understand how ransomware characteristics affect software pricing, usage and security in the presence of interdependent risk, and reflect on whether a shift in attack trends toward increased representation from the ransomware class is helpful or hurtful to the economy.

[2] As of Dec 2019, only 430 WannaCry victims paid the ransom demand (WebTitan 2019).

2 Literature Review

This work contributes to several research streams falling under the general topic of economics of information security, namely (i) economics of ransomware, (ii) network security externalities due to interdependent risks, and (iii) disaster recovery. Moreover, due to the peculiarities of ransomware attacks, this work is directly related to the research stream on (iv) economic dynamics of hostage taking and negotiation.

Ransomware attacks are perpetrated based on the concept of holding hostage a digital asset and demanding a ransom for its release (Young and Yung 1996). There exists an established research stream on hostage taking, ensuing negotiations, and outcomes in scenarios involving human victims. Several empirical studies explore the effect of deterrence policies and concession making on recurrence of hijacking events (Brandt and Sandler 2009, Brandt et al. 2016) and factors impacting the attackers' perpetration and negotiation effectiveness (Gaibulloev and Sandler 2009). Other studies take a behavioral approach to understand terrorist actions in hostage-taking events (e.g., Wilson 2000). Early game-theoretical studies on this topic focus on the dynamics of the interaction between rational terrorists and negotiators on the part of victims (governments, families, or other interested parties). Lapan and Sandler (1988) look at multi-period scenarios where the terrorists are considering an attack each period and there are potential reputation effects propagating through time, based on government concessions during negotiations in prior attacks. They abstract the number of victims and their model characterizes attack outcomes as constant regardless of how many victims are affected. Selten (1988) explores an extension with multiple attackers and victims but each instance of an attack represents a game with an isolated outcome in which the attacker will proceed with attacking each victim separately and only if he expects some benefit from the attack.

Drawing parallels to cyberattacks, such modeling approaches can be used to characterize attacks that are to some extent isolated (small-scale) and targeted. In contrast, in the case of *large-scale* attacks, the brunt of the impact is due to security interdependence as discussed in the Introduction.[3] In these attacks, perpetrators move laterally across at-risk systems, oftentimes in an untargeted way potentially accelerated through the implementation of worm-like self-propagation; in any case, the attacker need not work through a process of decision-making for every potential breach. Observationally, in several of these attacks, the ransom demanded is hardcoded a priori to a default level rather than being adjusted based on the value of the compromised digital asset to the consumer.[4] Furthermore, theoretical

[3]This can be true even if the onset of the attack is targeted.

[4]WannaCry, Bad Rabbit, and ZCryptor prompted victims to pay \$300-\$500, 0.05 BTC, and 1.2 BTC per

kidnapping models usually involve dynamics between two parties (negotiators and attackers). In contrast, many cyberattacks are enabled by vulnerabilities in an information system sold by a legitimate software vendor. The vendor is partially responsible for how secure its product is and can strategically create financial incentives for the adoption and patching of the system by consumers. Considering these dimensions, our framework accommodates large-scale attacks with interdependent security risks and endogenizes the role of the vendor in influencing the size of the consumer population that is vulnerable to the attack. Beyond the existence (Young and Yung 1996) and observation of cryptovirological attacks, our work focuses on their impact on software markets and the economic incentives that govern their efficacy.

The research agenda on the economics of information security has been extensively developed along multiple directions such as patching management and incentives (Cavusoglu et al. 2008, Ioannidis et al. 2012, Dey et al. 2015, August et al. 2019, Lelarge 2009), software liability (August and Tunca 2011, Kim et al. 2011), network security (August and Tunca 2006, Chen et al. 2011, August et al. 2014), piracy (August and Tunca 2008, Lahiri 2012, Kannan et al. 2016, Kim et al. 2018, Dey et al. 2019), vulnerability disclosure (Cavusoglu and Raghunathan 2007, Arora et al. 2008, Choi et al. 2010, Mitra and Ransbotham 2015), security investments (Grossklags et al. 2008), cyber-insurance (Böhme and Schwartz 2010, Johnson et al. 2011), and markets for information security and managed security services (Kannan and Telang 2005, Dey et al. 2012, Gupta and Zhdanov 2012, Ransbotham et al. 2012, Dey et al. 2014, Cezar et al. 2017). With regard to economic modeling of security in particular, the focus is often on the ex ante costly decisions (e.g., investments in patching, protection, reliability, insurance, etc.) that impact loss distributions.

On the other hand, the focus of cybersecurity recovery is often on planning and business continuity (IBM 2014, Bartock et al. 2016). The academic literature that explores economics of cybersecurity recovery is currently relatively sparse. We highlight how our work contributes to this nascent area. Chen et al. (2017) formalizes an ex-post recovery decision in the context of an infrastructure game where the designer can create redundant links for protection or add links back to the network post-attack as a means to recovery. In their model, whether and to what extent to heal the network is a recovery decision that must be made. Yang et al. (2019) consider a model with an advanced persistent threat (APT) where organizations attempt to mitigate the impact of APT via a dynamic quarantine and recovery (QAR) scheme. In APT settings, the timing of an attack event is necessarily more opaque, hence security actions tend to be governed by an optimal control problem specifying

affected system, respectively. Certain version of Locky prompted consumers to pay 0.25 BTC.

both a quarantine cost function and a recovery cost function. In a recent work that is the closest to ours, Cartwright et al. (2019) formally study the tradeoff between exerting ex ante costly effort to avoid an attack versus exerting ex post costly effort to recover. Their experiment assesses the impact of framing effects on this security trade-off. Notably, their study is motivated by ransomware where paying ransom is a means of recovery. The contribution of our paper is similarly more general being the first to examine a downstream endogenous recovery decision that influences an upstream security decision (i.e., patching) where these behaviors fundamentally alter the risk all agents face due to network externalities.

The study of economic dynamics of markets affected by ransomware also remains relatively sparse. Different from other types of cyberattacks where the full loss is realized if the attack is successful, ransomware attacks present victims with a *post-attack* choice: pay ransom (and hopefully retrieve access to the locked resource) or incur the full losses associated with giving up on that digital asset. From the perspective of consumers, the game is more complex. Laszka et al. (2017) explore security investments in risk mitigation (e.g, backups) and the strategic decision of whether to pay ransom. They abstract away from preventive effort investments by consumers (patching, firewalls, etc). In their study, the attacker's effort is customized to the victim, thus matching the dynamics of targeted attacks. In our study, in the case of large-scale, untargeted attacks with risk interdependence, preventive actions effectively lessen the spreading of the attack. Cartwright et al. (2019) adapt models by Lapan and Sandler (1988) and Selten (1988) to ransomware attacks and explore bargaining and deterrence strategies. In particular, they show that the likelihood of irrational aggression in the absence of payment and credible commitment to return files upon receipt of payment plays a key role in incentivizing victims to pay the ransom. Both Laszka et al. (2017) and Cartwright et al. (2019) consider the bargaining nature of the ransom game, where the victims have the ability to propose a counter offer to the demanded ransom and engage in negotiation. Such a modeling approach is relevant to targeted attacks on a smaller scale, where the effort is minimal on the side of the attacker to customize his handling of each victim. As mentioned above, many larger scale untargeted ransomware attacks do not allow for bargaining as the ransom is fixed and possibly hardcoded prior to the attack taking place. Hence, in our study, we focus on the consumer decision of whether to pay the ransom or not in the absence of a bargaining option.

Similar to our study, Cartwright and Cartwright (2019) and Li and Liao (2020) study untargeted ransomware attacks without an option for bargaining. In the former paper, the authors consider a repeated infinite-horizon game where a malicious agent attacks a randomly chosen victim each period. In the latter, in a setting with multiple victims, the focus is on hackers potentially engaging in an additional harmful action, that of selling victims data. Both papers consider the role of attacker reputation given its impact on victim response and overall payoff. In contrast, our results are consequentially impacted by consumer usage, protection, and ransom payment decisions that define relevant consumer segments in equilibrium, and together give rise to overall market risk. We explore the strategic pricing decisions of software vendors as well as welfare implications associated with ransomware in this context.

Finally, neither the hostage-taking literature nor the extant literature on economics of ransomware capture the possibility of negative security network externalities which often characterize large-scale cyberattacks. Cartwright et al. (2019) mention potential spillover effects of deterrence when there are two customer categories, but it is important to materially tie these effects to the size of the vulnerable population. Interdependent security risks have been explored in several other papers (e.g., Kunreuther and Heal 2003, Gal-Or and Ghose 2005, Johnson et al. 2010, August and Tunca 2011, Hui et al. 2012, Zhao et al. 2013). We extend this literature by modeling risk interdependencies in the context of ransomware.

3 Ransomware Attacks on Patchable Vulnerabilities

We begin our study by focusing on classes of ransomware risk spread via *patchable* vulnerabilities (e.g., WannaCry). In Section 4, we then examine classes of ransomware attacks that spread via unpatchable vectors (e.g., phishing scams, zero-day vulnerabilities). Then, in Section 5 and Section A.3 of the Appendix, we bring other types of traditional, non-ransomware attacks into the model to demonstrate how even a limited amount of ransomware can greatly affect the strategies of software firms and social welfare.

3.1 Model Description

We study the market for a software product that exhibits security vulnerabilities exploitable by ransomware attacks. We assume a unit-mass continuum of consumers whose valuations v for the software lie uniformly on $\mathcal{V} = [0, 1]$. Each consumer makes a decision to buy, B, or not buy, NB. Consumers who purchase pay a price p set by the vendor for the product. When a security vulnerability arises, the vendor develops a security patch and makes it freely available to all consumers of the software.[5] Each purchasing consumer makes a decision to patch, P, or not patch, NP. Consumers who decide to patch do so in a timely

[5]For software that is currently within its support period, the norm in the software industry is to make security patches widely available for free to all consumers in order to reduce risk, a policy that is cognizant of the security externalities that exist.

manner thereby incurring an expected patching cost of $c_p > 0$. Consumers who either do not patch or delay patching beyond a critical window face the risk of being hit by an attack. If operating unpatched, then a consumer gets hit with aggregated probability $\pi_r u$, where $\pi_r > 0$ is the probability the vulnerability is exploited and u is the size of the unpatched population of consumers (which is endogenous to the model). Using this specification of a network externality, we capture risk interdependence associated with ransomware attacks.[6]

We focus on large-scale, ransomware attacks. If successfully attacked, the consumer can either pay the ransom, R, or choose not to pay ransom, NR. The ransomware operator demands a single ransom $R > 0$ across all victims, which is consistent with many ransomware attacks in this family, including WannaCry, ZCryptor, and Bad Rabbit (Symantec 2016, F-Secure 2016, Barkly 2017). A ransomware victim with type v who chooses not to pay ransom incurs losses of αv, where $\alpha > 0$.[7] The parameter α can capture a wide range of loss scenarios. First, there are operational and recovery related losses. For software that drives systems that can easily be backed up and re-deployed with minimal downtime and disruption, α can be small. On the other hand, for systems characterized by more intermittent back-ups or even their absence, α can be relatively large. Recovery efforts can include hiring external providers to perform a forensic analysis of the attack and attempt to recover encrypted data without negotiating with attackers. Second, there can be potential losses associated with reputation, trust, goodwill, future business, and sensitive consumer data/privacy violations.[8] As a result, losses resulting from an attack can even go beyond the direct loss of usage and attain higher levels (i.e., $\alpha > 1$).[9]

Even consumers who pay ransom face a risk that the attacker may not release a working decryption key. According to Sussman (2020), 32% of the victims who pay ransom do not

[6] It is worth noting that a single decision maker (such as a corporate IT department) who derives independent valuations from multiple systems can make separate (and possibly different) purchasing and patching decisions for each system. For example, Boeing took such a granular approach to patching for WannaCry; this led to some systems in its Commercial Airplanes division becoming infected because they were still unpatched almost a year after WannaCry had emerged (Gates 2018). In cases like these, the aggregate valuation to a corporation is simply the sum of the individual valuations. If the role of the system is specifically to support a single, corporate individual, decision rights can be delegated to that agent. The only model requirement is that the decision maker only manages a countable set of systems.

[7] Ransomware financially impacts both businesses and individuals. Huang et al. (2018) tracked victims, ransomware operators and ransomware payments using end-to-end measurement, accounting for more than \$16 million in payments by 19,750 likely victims over 2 years. They found that 74.5% of the infected IP addresses were residential whereas 3.5% were businesses at that time. The remaining share was composed of colleges, hosting and others.

[8] For example, attackers using Maze ransomware threatened those who did not pay ransom with leakage of stolen data, and even followed through on that promise (Krebs on Security 2019).

[9] Note, however, that *expected* losses are necessarily less than the consumer's valuation. The consumer only makes trade-offs between paying ransom and incurring losses at the last stage of the game, consistent with sub-game perfection.

immediately regain access to their data, and 22% never do. This can happen for multiple reasons that interact with the wide-ranging attacker motivations discussed in Section 1. For example, an economically-motivated attacker may not release the key because he aims to extract more out of the victims (Siwicki 2016). Firm survey data suggests that as many as 10% of firms that pay an initial ransom are demanded a second ransom (Sussman 2020). Or perhaps not releasing decryption keys is a result of unintentional failures in either a manual process for producing and releasing keys or in the systems that process ransom payments (Abrams 2016; Chuang 2016; Chuang 2018). Attackers with either political agendas or other motivations may not have any intention to produce or release the keys in the first place (Frenkel et al. 2017, Marsh 2018).

Because consumers face uncertain ransomware risks based on unclear attacker motivations, we parameterize the primary loss characteristics faced by consumers. This provides the ability to analyze outcomes *across* the varied motivations that underlie hacker activity. In particular, consumers who pay ransom still incur some residual valuation-dependent losses in expectation. We model them as scaled losses by a factor $\delta \in (0,1)$. For example, a high δ can represent a case where the ransomware operator has little intention of releasing working decryption keys upon payment, and a small δ represents the opposite case where the operator uses a well-functioning and automated release system so that residual losses to paying consumers are minimal.[10]

The consumer action space is $S = \{(B,P),(B,NP,R),(B,NP,NR),(NB)\}$ and for a given strategy profile $\sigma : \mathcal{V} \to S$, the expected utility function for consumer v is given by:

$$U_{RW}(v,\sigma) \triangleq \begin{cases} v - p - c_p & if \quad \sigma(v) = (B,P)\,; \\ v - p - \pi_r u(\sigma)(R + \delta\alpha v) & if \quad \sigma(v) = (B,NP,R)\,; \\ v - p - \pi_r u(\sigma)\alpha v & if \quad \sigma(v) = (B,NP,NR)\,; \\ 0 & if \quad \sigma(v) = (NB)\,, \end{cases} \tag{1}$$

where $u(\sigma) \triangleq \int_{\mathcal{V}} \mathbb{1}_{\{\sigma(v) \in \{(B,NP,R),(B,NP,NR)\}\}}\, dv$ is the size of the unpatched population in the presence of the ransomware threat. Without loss of generality, we assume that $\pi_r \in (0,1]$, $c_p \in (0,1)$, $R \in (0,\infty)$, and $\alpha \in (0,\infty)$.[11] It is worth noting that the above modeling

[10]In general, one can vary over the $(R,\, \delta)$ parameter space to map to hacker motivations and gain insights into the equilibria that unfold under ransomware consistent with each motivation. This parametric approach is preferable here because the wide-ranging and disparate motivations behind observed ransomware would make objective specification (in malicious agent modeling) untenable (see Section 6 where we discuss the limitations). Moreover, it permits broader insights into a threat landscape that is dynamic in nature; the version and intent of ransomware seen recently in WannaCry and NotPetya may look starkly different than the successful ransomware campaign of tomorrow which our model also intends to inform upon.

[11]For consumers in our model, at the point of decision making on whether to pay ransom they are only trading off ransom and residual losses, $R + \delta\alpha v$, and full valuation-dependent losses, αv; all other costs are

framework can also examine targeted attacks with valuation-dependent ransom demands via a parameter transformation.[12]

3.2 Consumption Subgame

Before examining the impact of ransomware on the vendor's decision, we first characterize how consumers behave in equilibrium (in the consumption subgame) for a given price. There are two factors that complicate their decisions. First, the level of risk upon being unpatched is endogenously determined by the actions of all consumers. Second, this risk includes the behavior of both those who would pay ransom as well as those who would not. Thus, we first focus on understanding the effect of their strategic interactions on equilibrium behavior due to the externality generated by both subpopulations. The consumer with valuation v selects an action that solves the following maximization problem: $\max_{s \in S} U_{RW}(v, \sigma)$, where the strategy profile σ is composed of σ_{-v} (which is taken as fixed) and the choice being made, i.e., $\sigma(v) = s$. We denote the optimal action that solves her problem with $s^*(v)$. Further, we denote the equilibrium strategy profile with σ^*, and it satisfies the requirement that $\sigma^*(v) = s^*(v)$ for all $v \in \mathcal{V}$. We next characterize the structure of equilibrium consumer behavior in the subgame.

Lemma 1. [*Consumption Subgame*] *Given a price p and a set of parameters π_r, α, c_p, R, and δ, there exists a unique equilibrium strategy profile σ^* that is characterized by thresholds v_{nr}, v_r, $v_p \in [0,1]$. For each $v \in \mathcal{V}$, it satisfies*

$$\sigma^*(v) = \begin{cases} (B,P) & if \quad v_p < v \leq 1\,; \\ (B,NP,R) & if \quad v_r < v \leq v_p\,; \\ (B,NP,NR) & if \quad v_{nr} < v \leq v_r\,; \\ (NB) & if \quad 0 \leq v \leq v_{nr}\,. \end{cases} \tag{2}$$

Lemma 1 establishes that the equilibrium consumer behavior in the subgame has a threshold structure. The highest-valuation consumers have the most value to lose if attacked, so they patch in equilibrium when risk is high. Those with lower valuations remain unpatched. Of those who are unpatched, those with higher valuations are the ones who pay ransom to reduce the impact of being unpatched on their valuation-dependent losses. Importantly, it

sunk at that point in time. In that α can be greater than 1 when the attacks greatly affect firms, R can analogously be larger than 1 in situations where attackers are capitalizing on the ex-post value consumers are now concerned with mitigating.

[12] Specifically, suppose that the valuation-dependent ransom demand is expressed as $R(v) = R_0 + R_1 v$, satisfying $R_0 > 0$ and $R_1 + \delta\alpha < \alpha$. Then, we obtain the same model as in (1) with the transformations $R^\dagger = R_0$ and $\delta^\dagger = \frac{R_1 + \delta\alpha}{\alpha}$.

can be the case that no unpatched consumer (if R or δ is sufficiently high) pays ransom or even all unpatched consumers pay ransom (if R or δ is sufficiently low).

3.3 Pricing Subgame

Next, we turn our attention to the optimal pricing problem that the vendor faces in the first stage. We characterize the equilibrium in the pricing subgame which is the game being posed in its entirety. We denote the vendor's profit function by

$$\Pi(p) = p \int_{\mathcal{V}} \mathbb{1}_{\{\sigma^*(v|p) \in \{(B,NP,NR),(B,NP,R)(B,P)\}\}} dv \,, \tag{3}$$

noting that marginal costs are negligible. In the pricing subgame, the vendor sets a price p for the software by solving the following problem: $\max\limits_{p \in [0,1]} \Pi(p)$, such that (v_{nr}, v_r, v_p) are given by $\sigma^*(\cdot \,|\, p)$. With the optimal price p^* that solves the vendor's problem, we denote the associated profits by $\Pi^* \triangleq \Pi(p^*)$.

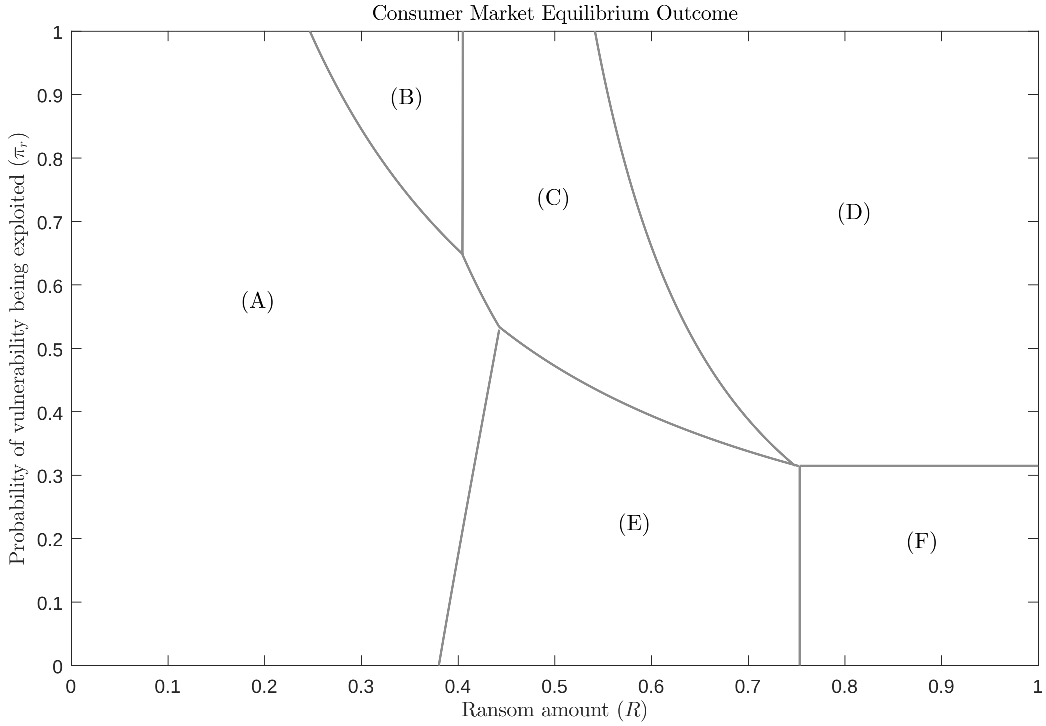
Because our model parameter space induces many different equilibria including those not commonly found in practical settings, we focus our analysis on the more relevant subspaces. As an example, it is natural that, in equilibrium, if patching costs (c_p) are too high, then no consumer patches; however, this outcome is not characteristic of settings that are commonly observed. To better focus on regions where key trade-offs are active, we assume the following:

Assumption 1. $0 < c_p < 2 - \sqrt{3}$*, and*

Assumption 2. $\frac{2}{(1-c_p)^2} - 2 < \alpha < 2(2-c_p)^2$.

While costs of patching include downtime during the patching process, patch distribution and installation processes have been greatly streamlined and sometimes even partially automated. When patching is done properly and in tandem with fail-safe measures (e.g., restoration capabilities / backups) the associated business costs are usually within reasonable ranges. Similarly, such fail-safe measures can reduce the extent of damage of a ransomware (or other type of malware) attack. For simplicity, we assume that patching is effective at preventing the exploitation of the vulnerability. Moreover, the assumptions on c_p and α above are sufficient conditions to obtain the findings in our paper, which can extend well beyond this focal region.

A software producer like Microsoft is keenly aware of the value of security that its products offer consumers (see, e.g., Microsoft's (2021) value of security calculator). This affects the total value proposition to consumers which is a fundamental factor in pricing. Over the years, Microsoft has also shown an understanding of how a lack of patching can impact



Consumer Market Segments Represented

Region (A)	[Not Using / Paying Ransom]
Region (B)	[Not Using / Paying Ransom / Patching]
Region (C)	[Not Using / Not Paying Ransom / Paying Ransom / Patching]
Region (D)	[Not Using / Not Paying Ransom / Patching]
Region (E)	[Not Using / Not Paying Ransom / Paying Ransom]
Region (F)	[Not Using / Not Paying Ransom]

Figure 1: Characterization of the consumer market equilibrium structure under the vendor's optimal pricing across regions in the ransom demanded (R) and security loss factor (π_r). Region labels describe the consumer segments that arise in each region in order of increasing consumer valuations (from left to right). Patching costs ($c_p = 0.12$), security loss factor ($\alpha = 0.8$), and residual loss factor ($\delta = 0.05$) are selected to ensure all consumer patching and ransom-paying behaviors are present for some sub-region.

security for its entire consumer base; it has even taken stances to force updates in certain cases (Newman 2015, Khalili 2020). Given this backdrop, it is useful to understand how the vendor influences consumers via pricing dependent upon characteristics of the ransomware setting, e.g. size of the ransom demand R and security risk factor level π_r. Figure 1 provides a helpful illustration of how the equilibrium in the consumer market is affected. It depicts an instance of the focal region we study and serves as a reference for the reader to keep in mind which structures are in play.[13] In the figure, we can see that when R and π_r are

[13]Some subsequent figures study vertical and horizontal slices across Figure 1 which can also be visualized here. For consistency, the capital letter labels in these cases refer back to the region labels of Figure 1.

sufficiently low, then prices are set such that all consumers remain unpatched and pay the ransom if hit. Overall, the expected losses are sufficiently low in Region (A) that consumers do not find it worthwhile to incur the cost to protect themselves by patching, and, if hit, they also prefer to pay the ransom because R is relatively low. At the other extreme, if R and π_r are sufficiently high, unpatched consumers are no longer willing to pay ransom. This is seen in Region (D), in which the equilibrium outcome is $0 < v_{nr} < v_p < 1$. In a setting with a high security risk factor π_r, higher-valuation consumers have a strong incentive to protect themselves from risk by patching. Those with lower valuations prefer to remain unpatched and do not pay high ransom demands; this results in the described market outcome. In the middle ground between these two scenarios, we see that the equilibrium outcome aligns well with observations of today's world. In particular, in Region (C) the consumer market is characterized by $0 < v_{nr} < v_r < v_p < 1$, in which all consumer segments emerge. If the risk factor is not as high as in Region (C), then one might expect $0 < v_{nr} < v_r < 1$ to arise in equilibrium, in which no consumer patches and higher-valuation consumers pay ransom if hit. Such a region does arise, and it is depicted as Region (E). Similarly, with a high risk factor but a smaller expected ransom demand, the outcome $0 < v_r < v_p < 1$ in which the highest-valuation consumers patch and all unpatched consumers pay the ransom if hit also arises, depicted as Region (B). Lastly, when the risk factor is low but the ransom amount is high, consumers no longer have an incentive to patch and, if struck by an attack, they will not pay the high ransom, inducing an equilibrium $0 < v_{nr} < 1$, as seen in Region (F).

3.4 Impact of Ransomware Characteristics

In this paper, we discuss the central role R and δ play in shaping the vendor's pricing strategy and, ultimately, consumer decisions. In the main body, we focus on the impact of the ransom demand under low residual losses. We then broaden our scope to explore higher residual losses (which can stem from diverse hacker motivations) - due to length limitations, this complete analysis is provided in Section A.1 of the Appendix.

3.4.1 Role of Ransom Amount and Risk

With the newfound understanding of how the equilibrium outcome unfolds across different regions of the parameter space, we next investigate regions of interest in more depth. In the rest of this section, we describe and illustrate several insights into ransomware economics. For example, one might expect that a higher ransom demand would negatively impact the vendor and reduce the market share of the affected product. However, that is not always the case, as is shown in the first proposition. For the majority of results and discussions in this paper, we focus on residual losses for ransom-paying consumers being reasonably low (i.e., δ satisfying an upper bound) such that paying ransom can be incentive compatible. This scenario matches more recent ransomware trends (Disparte 2018).[14] To gain a broader view into diverse motivations and provide an overall more complete analysis, we discuss scenarios of intermediate and high residual losses in Appendix A.1.

Proposition 1. *There exists $\tilde{\delta} > 0$ such that if $\delta < \tilde{\delta}$ and $\pi_r > \bar{\pi}_r$, then:*

(a) if $0 < R < R_1$, then equilibrium consumption satisfies $0 < v_r < 1$. As R increases, so does the vendor's price but market size and profits decrease;

(b) if $R_1 \leq R < R_2$, then equilibrium consumption satisfies $0 < v_r < v_p < 1$. As R increases, the vendor's price, market size, and profits all decrease;

(c) if $R_2 \leq R < R_3$, then equilibrium consumption satisfies $0 < v_{nr} < v_r < v_p < 1$. As R increases, the vendor's price, market size, and profits all increase;

(d) if $R \geq R_3$, then equilibrium consumption satisfies $0 < v_{nr} < v_p < 1$, and there exists $\omega > R_3$ such that, as R increases,

(i) the vendor's price and profits increase while the market size decreases on $R < \omega$;

(ii) the vendor's price, market size, and profits are constant on $R \geq \omega$.[15]

Proposition 1 is illustrated in Figure 2 which depicts how the consumer choices, vendor price and profit react to changes in the ransom amount, R.[16] For the parameters used in Figure 2, the thresholds identified in Proposition 1 are computed to be approximately $R_1 = 0.34$, $R_2 = 0.41$, $R_3 = 0.58$, and $\omega = 0.59$. When the ransom demand is not too low and the potential losses from the attack are sufficiently high, then high-valuation consumers elect to patch ex ante. This patching behavior can be seen entering into panel (a) of Figure 2 for $R \geq 0.34$. The trade-off here centers on c_p versus $\pi_r u(\sigma)(R + \delta\alpha v)$ (i.e., the *expected* costs

[14] An economically-motivated hacker would generally deploy ransomware with characteristics satisfying such conditions because the hacker's goal is to generate ransom payments which would be negatively impacted by post-payment malicious behavior. Despite ransom-paying consumers requiring some belief about "honor among thieves", for certain classes of hacker motivations, maintaining this honor would be in everyone's best interest (Fleishman 2016).

[15] The existence, characterization, and relative ordering (e.g., $0 < R_1 < R_2 < R_3$) of the presented bounds are formally established in the proof under the focal region (see Assumptions 1 and 2), noting that $R_1 \to \frac{(2-c_p)c_p}{(1-c_p)^2\pi_r}$, $R_2 \to \frac{\alpha}{2-c_p}$, $R_3 \to \frac{\alpha\sqrt{\pi_r}+\sqrt{\alpha(16c_p+\alpha\pi_r)}}{4\sqrt{\pi_r}}$, and $\bar{\pi}_r \to \frac{c_p(2-c_p)^2}{\alpha(1-c_p)^2}$ as δ becomes small.

[16] The impact to the consumer market structure itself can also be viewed as a cross-section of Figure 1 horizontally at $\pi_r = 0.75$.

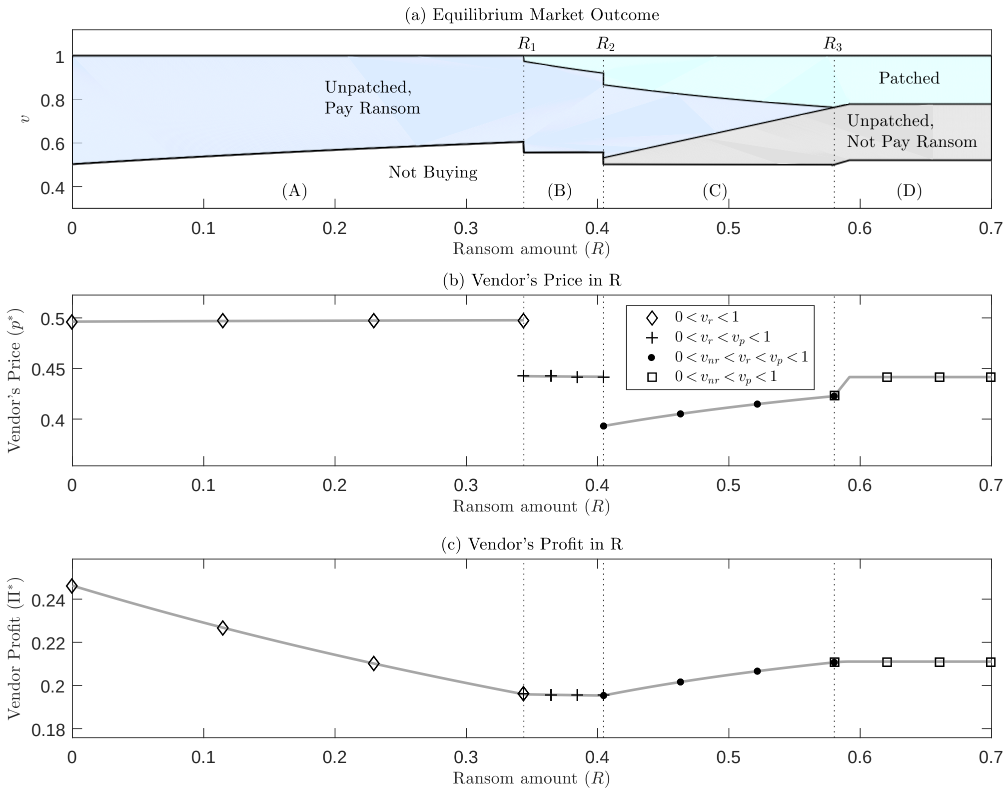


Figure 2: Impact of ransom demand (R) on equilibrium consumption, vendor's price, and the vendor's profit. The parameter values are $c_p = 0.12$, $\alpha = 0.8$, $\delta = 0.05$, and $\pi_r = 0.75$. The capitalized letter region labels correspond to region labels in Figure 1. The legend in panel (b) also applies to panel (c).

under a ransom-paying strategy), with a patching population emerging only when R is large relative to c_p.

When ransom demands are small, part (a) of Proposition 1 establishes that a patching strategy does not emerge. Consumers remain unpatched and all pay ransom if an attack arises, as depicted in panel (a) of Figure 2 for $R < 0.34$, which falls within Region (A) of Figure 1. Hence, in equilibrium, all purchasing consumers are directly and negatively impacted by a higher R. In this situation, the vendor elevates price in order to reduce the size of the unpatched population and help mitigate the risk of an attack. Throughout this region, increases in R will hurt vendor profits.

Part (b) of Proposition 1 covers a region in which the ransom demand is still low enough that all unpatched consumers simply pay the ransom if hit but not so low that nobody patches. This can be observed for $0.34 \leq R < 0.41$ in Figure 2, which falls under Region (B) of Figure 1. Higher valuation consumers now patch, which provides risk relief to unpatched consumers. The vendor has less incentive to use a high price to contain risk via a reduced consumer population, therefore as R moves from Region (A) to Region (B), it chooses to drop price and expand the market. Nevertheless, within Region (B), as R increases further, to reduce the additional burden on lower valuation consumers it gradually decreases price without completely compensating for the increase in expected losses. The net effect of a decreasing price and market size ultimately hurt profitability as is depicted in panel (c) of Figure 2.

As R moves relatively higher (part (c) of Proposition 1), all three adopter segments emerge, with the unpatched population splitting into sub-populations of ransom payers and non-payers. This is captured in Figure 2 for $0.41 \leq R < 0.58$ and falls under Region (C) of Figure 1. An increase in R incentivizes some unpatched consumers who would have paid ransom to strictly prefer patching over risking being hit with ransomware. On the other hand, unpatched consumers with lower valuations are not directly impacted by an increase in R because they do not pay ransom anyway. However, these consumers are still indirectly impacted by R because the negative externality that they endure upon remaining unpatched is reduced by the increased patching behavior of high valuation consumers. As a result, low valuation unpatched consumers are now better off under a higher ransom demand being charged. The market size therefore expands in R, as non-adopters now find it beneficial to adopt the product. In turn, the vendor is able to profitably extract additional surplus by charging a higher price.

Finally, when the ransom demand is sufficiently high (i.e., $R \geq 0.58$ in Figure 2 which falls under Region (D) of Figure 1), then no unpatched consumer pays ransom if hit. There are two cases. At the highest level of R, it becomes *naturally* cost prohibitive from the consumer perspective, hence market size and vendor price and profit do not change in R. However, prior to reaching this level and corresponding to $R < \omega$ in part (d.i) of Proposition 1, price and vendor profit are increasing in R despite the absence of ransom-paying consumers. Very close to the transition point, the ransom-paying option is still viable for consumers; hence, setting too high of a price would instead encourage some would-be patching consumers to pay ransom, leading to a suboptimal risk level in the market. Therefore, the vendor prices at the highest point that *just* prevents consumers from paying ransom. This price hike compensates for the loss in market size, thereby increasing profits.

By offering victims a chance to reduce their losses, ransomware attackers can sometimes segment the unpatched population into two interdependent tiers. The expansion or reduction of either tier indirectly impacts both tiers simultaneously because all unpatched hosts are potential vectors for the spread of ransomware. But now, in contrast to traditional modes of attack, an increase in the ransom demand may *directly* affect only a single tier which helps the vendor to discriminate. Because of these characteristics, ransomware can give rise to some unique pricing strategies. Next, we explore how the market changes as the inherent risk (π_r) of the software being breached increases.

Proposition 2. *There exist bounds $\tilde{\delta} > 0$ and $\hat{\omega} > \frac{\alpha}{2-c_p}$ such that if $\delta < \tilde{\delta}$, then:*

(a) if $0 < R \leq \hat{R}_1$, then the vendor's price is continuously decreasing in π_r;

(b) if $\hat{R}_1 < R \leq \hat{R}_2$, then the vendor's price is piecewise decreasing in π_r on adjacent intervals $(0, \pi_1)$ and $(\pi_1, 1)$ while jumping downward at π_1;

(c) if $\hat{R}_2 < R \leq \hat{\omega}$, then there exists $\hat{\pi}$ such that the vendor's price is piecewise decreasing in π_r on adjacent intervals $(0, \hat{\pi})$, $(\hat{\pi}, \tilde{\pi})$, and $(\tilde{\pi}, 1)$. Its strategy is discontinuous in π_r: the price should be jumped up at $\hat{\pi}$ and significantly jumped down at $\tilde{\pi}$.[17]

When the level of ransom demand is low ($R \leq \hat{R}_1$), all consumers prefer to remain unpatched and pay ransom if hit. Therefore, as risk increases, all purchasing consumers are directly affected by the associated increase in expected ransom payments. In order to throttle consumers from discontinuing use while also ensuring the risk level stays in check, the vendor gradually decreases price in a controlled way.

At slightly higher ransom levels ($\hat{R}_1 < R \leq \hat{R}_2$), once risk passes a certain threshold, π_1, high valuation consumers now find it incentive compatible to patch, leading to a consumer market characterized by $0 < v_r < v_p < 1$. As risk transitions into this region, the vendor discontinuously drops its price to expand market coverage at the lower end. From there, the vendor manages further increases in risk through gradual downward price adjustments.

Once ransom demands increase to a range characterized by richer trade-offs ($\hat{R}_2 \leq R \leq \hat{\omega}$), a more complex pricing strategy unfolds. Figure 3 illustrates this particular scenario, with panel (b) explicitly capturing the price sensitivity with respect to risk. The corresponding cutoff points are $\hat{\pi} \approx 0.424$ and $\tilde{\pi} \approx 0.568$. When the inherent risk factor is low, i.e., $\pi_r < \hat{\pi}$, consumers do not patch. However, the higher ransom demand in this region induces consumers to segment: the unpatched population separates into those who pay and those who do not pay ransom if hit leading to a consumer market outcome characterized by $0 < v_{nr} < v_r < 1$. This outcome corresponds to Region (E) in panel (a) of Figure 3. As the inherent risk factor increases through this range, the vendor mitigates the impact of increased risk on its consumers by lowering price.

[17] The existence, characterization, and relative ordering of the presented bounds are formally established in the proofs under the focal region, noting that $\hat{R}_1 \to \frac{1}{(1-c_p)^2} - 1$, $\hat{R}_2 \to \frac{\alpha}{2}$, $\pi_1 \to \frac{(2-c_p)c_p}{(1-c_p)^2 R}$, and $\pi_2 \to \frac{c_p \alpha}{R^2 - c_p R \alpha}$ as δ becomes small, and $\tilde{\pi} = \min(\pi_1, \pi_2)$. Implicit bounds $\hat{\omega}$ and $\hat{\pi}$ are characterized as such in Lemma B.4 of Section B.1.2 of the Appendix.

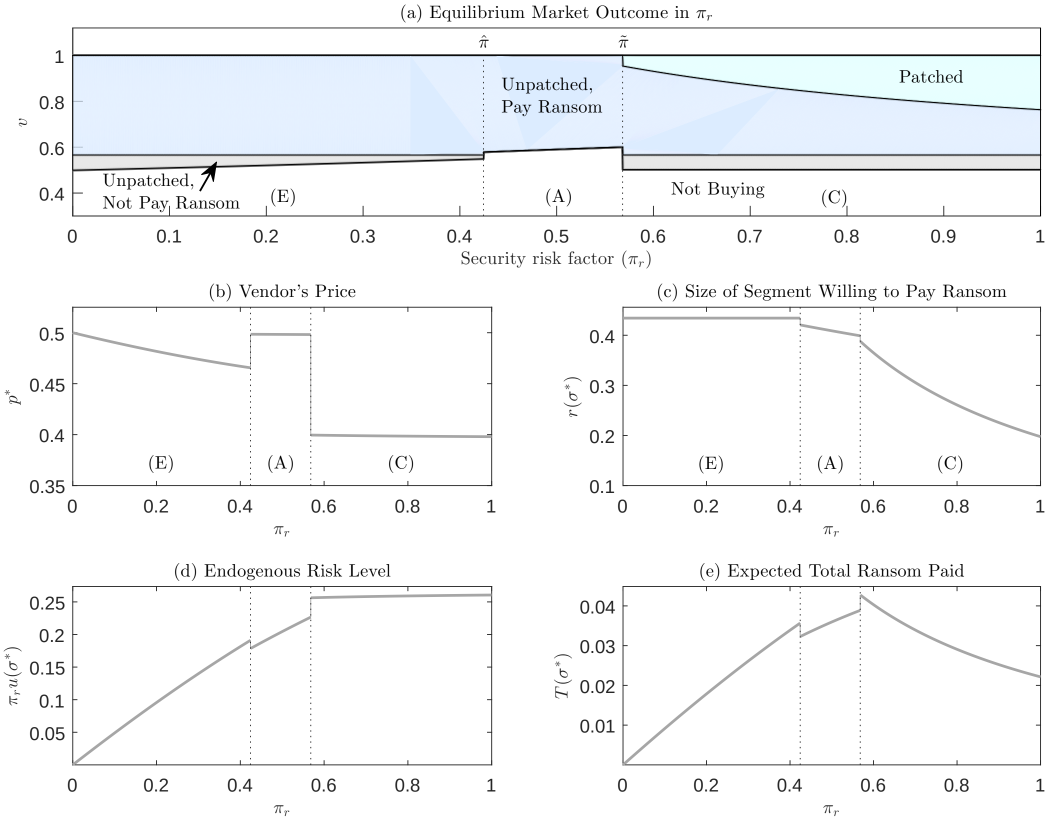


Figure 3: Impact of the risk factor (π_r) on the equilibrium market outcome, vendor's price, size of the market segment willing to pay ransom if hit, endogenous risk level, and expected total ransom paid. The parameter values are $c_p = 0.12$, $\alpha = 0.8$, $\delta = 0.05$, and $R = 0.43$.

However, when the risk factor increases further and crosses the threshold at $\hat{\pi}$, there is a significant and strategic change in the equilibrium pricing behavior of the vendor. This is the primary message contained in part (c) of Proposition 2: when the risk factor lies in a middle range (i.e., $\pi_r \in (\hat{\pi}, \tilde{\pi})$), the vendor implements a strategically higher price to focus only on higher-valuation consumers. This pricing strategy changes the consumer market characterization to $0 < v_r < 1$ in which all purchasing consumers remain unpatched and pay ransom if hit, illustrated by Region (A) in panel (a) of Figure 3. Lower-valuation consumers drop out of the market due to the increased risk and higher price. Notably, as discussed in Section 5 and Section A.2 of the Appendix, such a strategic discontinuous increase in pricing is not observed in a comparative setting where ransomware is not present in the market.

Finally, as the inherent risk factor increases further to exceed $\tilde{\pi}$, consumers naturally have a stronger incentive to patch. To facilitate consumer patching, reduce the security externality, and substantially increase usage of its software, the vendor discontinuously lowers price at $\tilde{\pi}$, and employs gradual price reductions as risk increases thereafter.

When examining the impact of the risk factor on the software market, one question of interest is how the expected, total ransom paid by victims is affected. Although the size of the consumer population willing to pay ransom if hit always shrinks as the risk factor increases (as seen in panel (c) of Figure 3), the total, expected ransom paid is non-monotone and exhibits greater complexity (as seen in panel (e)).

Proposition 3. *Under the conditions of Proposition 2,*

(a) the vendor's profit and market size are piecewise decreasing in π_r;

(b) if $0 < R < R_2$, then the total, expected ransom paid is piecewise increasing in π_r. Moreover, the size of the ransom-paying population decreases in π_r if and only if $\pi_r \in (\hat{\pi}, \pi_1)$;

(c) if $R_2 \leq R < \hat{\omega}$, then the total, expected ransom paid is piecewise increasing in π_r on $(0, \pi_2)$ and decreasing in π_r on $(\pi_2, 1)$. The size of the ransom-paying population is piecewise weakly decreasing in π_r everywhere.[18]

First, it is natural that the vendor's profit and market size decrease in π_r because every unpatched segment (whether paying ransom or not) is directly affected by the increased prospect of an attack. In that overall risk is endogenous, there will always be a segment of unpatched consumers in the market who are sensitive to ransomware attacks.

The more material takeaway lies in parts (b) and (c) of Proposition 3 where we establish a risk region in which the total, expected ransom paid increases in π_r while the ransom-paying population size decreases. We focus our discussion on the scenario presented in part (c) which is illustrated in Figure 3. Panel (c) plots the size of the consumer segment that pays ransom if hit, $r(\sigma^*)$.[19] Panel (d) illustrates the endogenous risk level, $\pi_r u(\sigma^*)$.[20] Panel (e) illustrates the total, expected ransom paid, i.e., $T(\sigma^*) \triangleq \pi_r u(\sigma^*) r(\sigma^*) R$. The key point to note here is that the externality $u(\sigma^*)$ depends on *all* unpatched consumers, including those who are not willing to pay the ransom. When the ransom demand R is moderate, the unpatched population splits into the two tiers under low risk (π_r), resulting in a market structure $0 < v_{nr} < v_r < 1$. This is depicted in Region (E) of panel (a) of Figure 3. The valuation of the indifferent consumer, between paying and not paying ransom ($v = \frac{R}{\alpha(1-\delta)}$), is independent of the risk factor because the tradeoff is between incurring a loss of αv and incurring a loss of $R + \delta\alpha v$. Since consumers who remain unpatched but do not pay ransom

[18]See footnote 17 for further specification of the bounds. A more comprehensive statement of Proposition 3 can be found in Section B.5 of the Appendix.

[19]$r(\sigma^*)$ measures the mass of consumers whose equilibrium strategy is (B, NP, R).

[20]$u(\sigma^*)$ measures the total mass of consumers who remain unpatched in equilibrium, choosing either (B, NP, NR) or (B, NP, R).

have valuations even lower than this threshold, they are the ones to drop out of the market first as π_r increases. Consequently, the size of the consumer population willing to pay ransom if hit remains constant in π_r at first (see Region (E) of panel (c)). Although $u(\sigma^*)$ necessarily shrinks, the overall risk $\pi_r u(\sigma^*)$ increases as π_r increases (same region of panel (d)) such that the expected total paid by victims is also increasing in π_r (same region of panel (e)).

When the vendor strategically increases price (at the boundary between Regions (E) and (A) in Figure 3), these same low-valuation consumers drop out of the market. Additionally, the size of the ransom-paying group shrinks which now becomes the only segment present in the market (i.e., the market structure is given by $0 < v_r < 1$). As the market structure changes, the overall risk externality $\pi_r u(\sigma^*) = \pi_r r(\sigma^*)$ also suddenly drops. The net impact of these two effects is a drop in the size of the expected total ransom paid by victims, which can be seen at $\pi_r = \hat{\pi}$ in Figure 3. Nevertheless, as risk further increases within this range of the risk factor (i.e., $\pi_r \in (\hat{\pi}, \pi_2)$), the ransom-paying population does not decrease steeply which is depicted in Region (A) of panel (c). Hence, the expected total ransom paid by victims remains monotonically increasing in π_r, albeit initially trailing behind the levels just prior to the change in market structure.

At the junction between Regions (A) and (C), as described in part (c) of Proposition 2, the vendor finds it profitable to significantly cut price so that all three market segments emerge in equilibrium (i.e., $0 < v_{nr} < v_r < v_p < 1$). This shift in pricing strategy invites additional consumers to enter at the low end of the market, who naturally remain unpatched. The increased risk they create induces high valuation consumers to shield themselves from risk by patching. In aggregate, the ransom-paying population $r(\sigma^*)$ shrinks because more consumers switch from paying ransom to patching. The sudden jump in the unpatched population $u(\sigma^*)$ compensates for the drop in $r(\sigma^*)$, and overall the total, expected ransom paid momentarily jumps upward. Notably, a change in monotonicity occurs. As the risk increases even further, higher-valuation consumers who were willing to just pay ransom if hit switch to patching at a much steeper rate such that those low-valuation consumers who do not pay ransom are only marginally impacted by the increased risk. Moreover, as discussed before, the marginal consumer indifferent between paying and not paying ransom is not affected by the overall risk level. Thus, both the ransom-paying population $r(\sigma^*)$ and the overall unpatched population $u(\sigma^*)$ keep shrinking. Nevertheless, unlike in regions of lower risk (i.e., Regions (E) and (A) in Figure 3), this dual shrinking effect dominates the increase in the risk factor (π_r) and the expected overall ransom paid decreases.

While the option to pay ransom offers a recourse to mitigate value-dependent losses, it also involves a secondary risk. When considering the ransom payment, victims in general are not sure a priori whether the attacker will deliver the promised decryption keys. As mentioned in Section 3.1, this secondary risk is captured by the parameter δ to account for these residual losses. We study how δ impacts the vendor's profit, expected aggregate losses incurred by the unpatched population, and aggregate consumer surplus. When all market segments are observed in equilibrium, the vendor benefits from a higher δ because it can then charge a premium for a safer product that is patched by a greater percentage of consumers. On the other hand, once δ becomes too high, a shift away from paying ransom reduces the impact that ransomware-specific characteristics (R, δ) have on outcome measures. To see this analysis, we direct the reader to Section A.1 of the Appendix. Our formal findings on the role of residual losses have been carried there in an effort to keep the main body concise.

4 Other Ransomware Attack Vectors

In the prior section, we studied threats on patchable vulnerabilities. Ransomware can also operate via other infection vectors such as zero-day vulnerabilities and social engineering campaigns including phishing (Gendre 2019a; Goodin 2019). One vector employed in recent years is Microsoft's collaboration platform SharePoint (Guida 2018; Gatlan 2019; Gendre 2019b). Attackers send potential victims links to SharePoint documents, which upon being clicked lead these victims to spoofed Office 365 login pages. By stealing credentials, attackers can then send additional phishing emails or SharePoint documents with ransomware attached from within the victim's organization to other organizations (such as suppliers or clients). Because SharePoint documents are passed, Microsoft cannot blacklist links to these documents without negatively impacting consumers' ability to collaborate. Consequently, all consumers are faced with this risk, and as more consumers use the service, the more attractive the attack vector becomes. We also discussed in the Introduction other ransomware attacks on Microsoft Office and Oracle WebLogic server software that were not patchable. In all of these instances, only adopters of a particular software or service are vulnerable to such an attack, which means interdependent risk is being induced by the entire consumer population.[21]

We construct a modified model to capture the unique aspects of other common ransomware attack vectors and refer to it as *RW-OV* (*ransomware, other vectors*):

[21] There are other types of unpatchable attacks that are not at the application level (e.g., phishing campaigns designed to syphon credentials by rerouting to a copy-cat web page replicating that of an official service provider). Such attacks are not captured through the lens of an economic decision to adopt a single software package in isolation, as they in general also impact non-users of that particular package.

$$U_{RW-OV}(v,\sigma) \triangleq \begin{cases} v - p - \pi_r n(\sigma)(R + \delta\alpha v) & if \quad \sigma(v) = (B,R)\,; \\ v - p - \pi_r n(\sigma)\alpha v & if \quad \sigma(v) = (B,NR)\,; \\ 0 & if \quad \sigma(v) = (NB)\,, \end{cases} \tag{4}$$

where $n(\sigma) \triangleq \int_{\mathcal{V}} \mathbb{1}_{\{\sigma(v) \in \{(B,R),(B,NR)\}\}}\, dv$ is the total size of the adopting population in the presence of this class of ransomware threat. Unlike in the case of an attack on a patchable vulnerability, *all* consumers are exposed to this risk; hence, risk interdependence is tied directly to usage.

One of the primary goals of this section is to make meaningful comparisons of outcomes under *RW-OV* with those established in Section 3 under *RW*. Analogous to Assumption 2 in Section 3, we similarly focus on a sub-range of the loss factor (specifically, $\alpha \in (\sqrt{3}, 6)$) to place the models on similar footing.[22]

Lemma 2. *Under RW-OV, given a price p and a set of parameters $\pi_r, \alpha, R,$ and δ, there exists a unique equilibrium consumer strategy profile σ^* that is characterized by thresholds v_{nr} and v_r $\in [0,1]$. For each $v \in \mathcal{V}$, it satisfies:*

$$\sigma^*(v) = \begin{cases} (B,R) & if \quad v_r < v \leq 1\,; \\ (B,NR) & if \quad v_{nr} < v \leq v_r\,; \\ (NB) & if \quad 0 \leq v \leq v_{nr}\,. \end{cases} \tag{5}$$

For this structure, some of the consumer segments may not appear as thresholds collapse into each other; however, if they appear, the segments will be ordered as presented in equation (5). Figure 4 illustrates the market outcome (under equilibrium pricing) in an analogous manner to Figure 1, using the same region labels - (A), (E), (F). The market structures that emerge under *RW-OV* are relatively less complex and are formally presented in the following proposition.

Proposition 4. *Under RW-OV, there exist bounds $\tilde{\delta} > 0$ and $\tilde{R}_1 \leq \alpha(1-\delta)$ such that if $\delta < \tilde{\delta}$, then:*

(a) if $R \leq \tilde{R}_1$, then equilibrium consumption satisfies $0 < v_r < 1$. As R increases, so does the vendor's price but market size and profits decrease;[23]

[22] In Assumption 2, the lower bound on α is increasing in c_p, while the upper bound on α is decreasing in c_p. Thus, taking the upper bound on c_p in Assumption 1, i.e., $2-\sqrt{3}$, and replacing in Assumption 2, we obtain the interval $(\sqrt{3}, 6)$ which is nested inside $(\frac{2}{(1-c_p)^2} - 2, 2(2-c_p)^2)$ for any c_p satisfying Assumption 1. Therefore, for any such c_p, we can compare and contrast the outcomes under *RW* and *RW-OV* directly provided that $\alpha \in (\sqrt{3}, 6)$.

[23] The characterization of $\tilde{R}_1$ is provided in Lemma B.9 of Section B.1.2 of the Appendix.

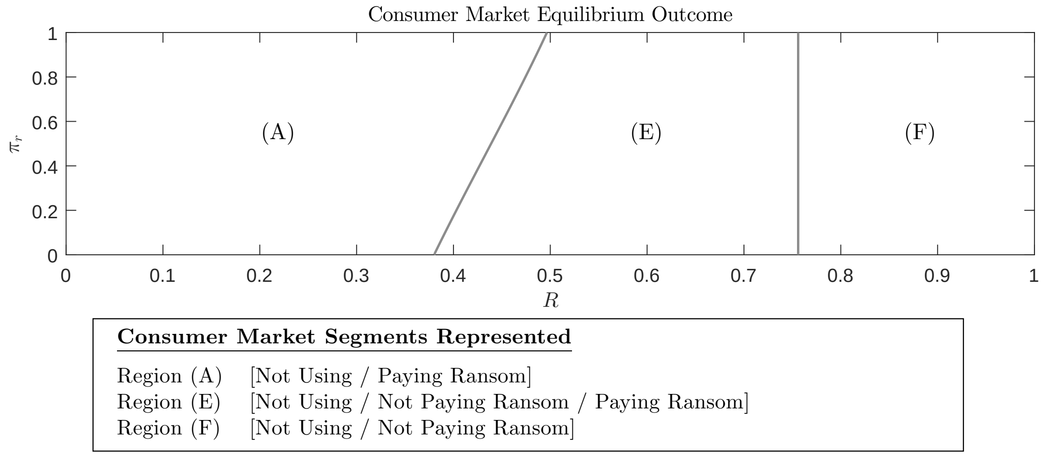


Figure 4: Characterization of equilibrium structures across regions in the ransom demanded (R) and security loss factor (π_r). Region labels describe the consumer segments that arise in each region in order of increasing consumer valuations (from left to right). Security loss factor ($\alpha = 0.8$) and residual loss factor ($\delta = 0.05$) are selected to ensure all strategies are present for some sub-region.

(b) If $R > \tilde{R}_1$, then equilibrium consumption satisfies $0 < v_{nr} < v_r < 1$ when $R < \alpha(1-\delta)$ and $0 < v_{nr} < 1$ when $R \geq \alpha(1-\delta)$. As R increases, the vendor's price, market size, and profits are constant in R.

In part (c) of Proposition 1 (depicted by Region (C) in Figure 2), we learned that with ransomware attacks on patchable vulnerabilities, it is possible for the vendor's price, market size and profits to simultaneously increase in R. Under *RW-OV*, such a comparative static can no longer arise, as we further explain below. This highlights the importance that the patching option has on the vendor's strategy when facing a ransomware threat. Specifically, the market can expand and the vendor can be better off with increasing ransoms *only* in a context of ransomware attacks on patchable vulnerabilities.

In Figure 5, we illustrate how market structure, price, and vendor profit change in R under *RW-OV*. For the parameter set being used, $\tilde{R}_1$ is approximately 0.47. In Region (A), when the ransom amount is low, all adopters would prefer to pay the ransom if successfully attacked. Within this region, as the ransom level increases, the firm controls risk in the market by maintaining a high price and gradually increasing it. This, together with the increasing ransom level, effectively reduces the market size, and hence interdependent risk. However, profits are declining in R because the limited increase in price does not compensate for the large reduction in market size. Beyond $\tilde{R}_1$, this strategy is no longer tenable. Instead, the firm lowers price in an effort to expand the market to consumers who balk at the size of the ransom request. The marginal consumer's valuation satisfies $v - \pi_r(1-v)\alpha v = p$,

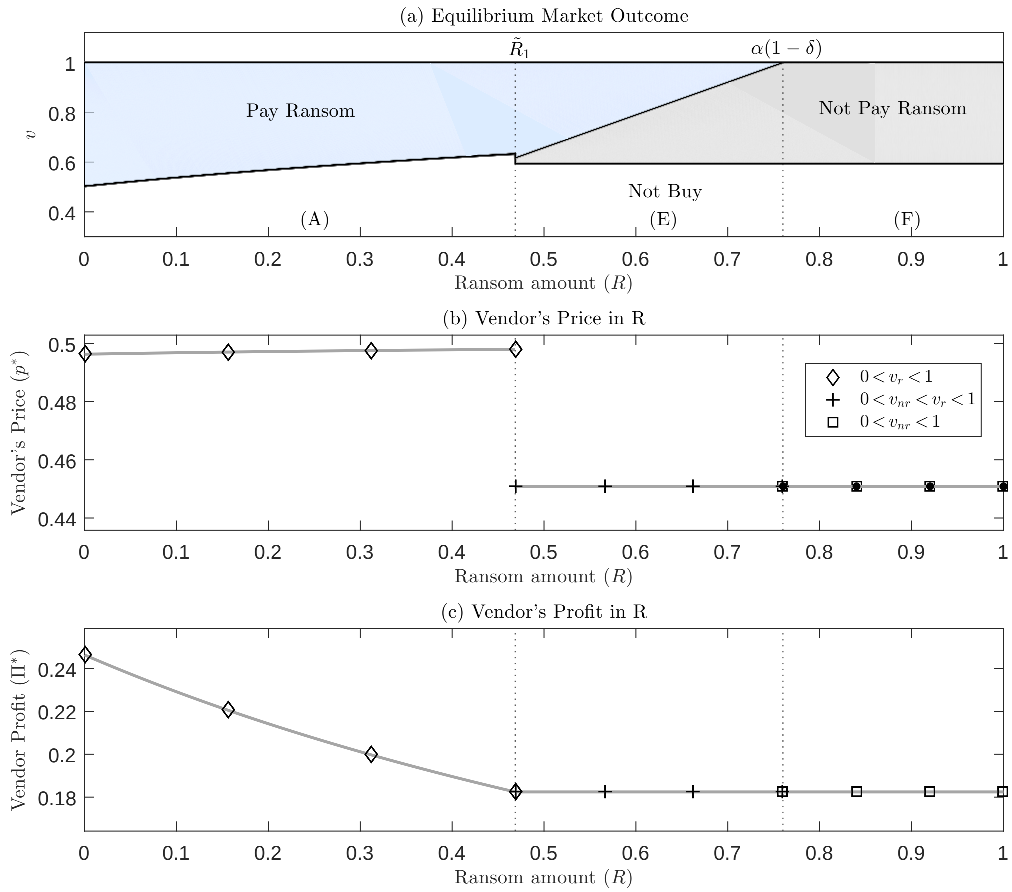


Figure 5: Impact of ransom demand (R) on the equilibrium market outcome, vendor's price, and profit. The parameter values are $\alpha = 0.8$, $\delta = 0.05$, and $\pi_r = 0.75$. The capitalized letter region labels correspond to region labels in Figure 4. The legend in panel (b) also applies to panel (c).

which is no longer impacted by the ransom amount. As such, market size, price and profits become insensitive to R in Regions (E) and (F).

In the next two propositions, we establish that some primary comparative statics we characterize using our main model (*RW*) are robust to the class of ransomware attacks being modeled under *RW-OV*. First, we examine how the inherent risk factor impacts the vendor's price for ransomware in the presence of alternative vectors to patchable vulnerabilities.

Proposition 5. *Under RW-OV, there exists* $\tilde{\delta} > 0$ *such that if* $\delta < \tilde{\delta}$*, then:*

(a) if $0 < R \le \tilde{R}_2$*, then equilibrium consumption satisfies* $0 < v_r < 1$*. Price is continuous in* π_r*, and if* $\alpha > 2$ *and* $R > 1$*, then it decreases in* π_r *on* $(0, \frac{1}{R})$ *and increases in* π_r *on* $[\frac{1}{R}, 1)$*. Otherwise, it only decreases in* π_r*.*

(b) if $\tilde{R}_2 < R < \tilde{R}_3$, then:

(i) if $0 < \pi_r < \pi'$, then equilibrium consumption satisfies $0 < v_{nr} < v_r < 1$. Price decreases in π_r.[24]

(ii) if $\pi' \leq \pi_r < 1$, then equilibrium consumption satisfies $0 < v_r < 1$. If $R > 1$, then price decreases in π_r on $(\pi', \frac{1}{R})$ and increases in π_r on $[\frac{1}{R}, 1)$. Otherwise, it decreases in π_r.

(iii) the vendor discontinuously hikes price at $\pi_r = \pi'$.[25]

We visually present the essence of Proposition 5 in panels (a) and (b) of Figure 6 where we have chosen a ransom level to satisfy part (b) of the proposition statement. Under that parameter set, π' is approximately 0.42. As can be seen when moving from Region (E) to (A), the vendor still discontinuously raises price which is consistent with our earlier finding in part (c) of Proposition 2. Therefore, this inherent incentive for the vendor to strategically raise price at a higher level of ransom demand is quite robust to the class of ransomware attacks being studied. Moreover, it is this price hike that sets vendor strategies in the presence of a ransomware threat apart from strategies employed in its absence, as we will further see in Section 5.

It is also interesting to contrast *RW* and *RW-OV*. In a context of patchable vulnerabilities, as the risk factor increases, consumers have greater incentives to patch their systems as a means of protection. This leads to a second discontinuous and strategic price shift by the vendor (this time downward) as it can expand the market when more systems are being patched. We illustrated this point earlier in Region (C) of panels (a) and (b) of Figure 3. On the other hand, for ransomware vectors that do not target patchable vulnerabilities as in *RW-OV*, this strategic behavior cannot arise and is absent from Proposition 5.

From panels (c) and (e) of Figure 6, it can be observed that the expected ransom paid increases piecewise in π_r even as the segment of consumers willing to pay ransom shrinks with the risk factor. In contrast, as mentioned in part (c) of Proposition 3 and illustrated in panel (e) of Figure 3, under patchable vulnerabilities, the expected ransom paid would decrease in the risk factor over region (C) which corresponds to high risk. The following result formalizes this argument:

Proposition 6. *Under RW-OV and the same conditions as Proposition 5,*

(a) if $0 < R \leq \tilde{R}_2$, then:

[24]The characterization of π' provided in Lemma B.10 of Section B.1.2 of the Appendix.

[25]The characterizations of $\tilde{R}_2$ and $\tilde{R}_3$ are provided in Lemma B.10 of Section B.1.2 of the Appendix.

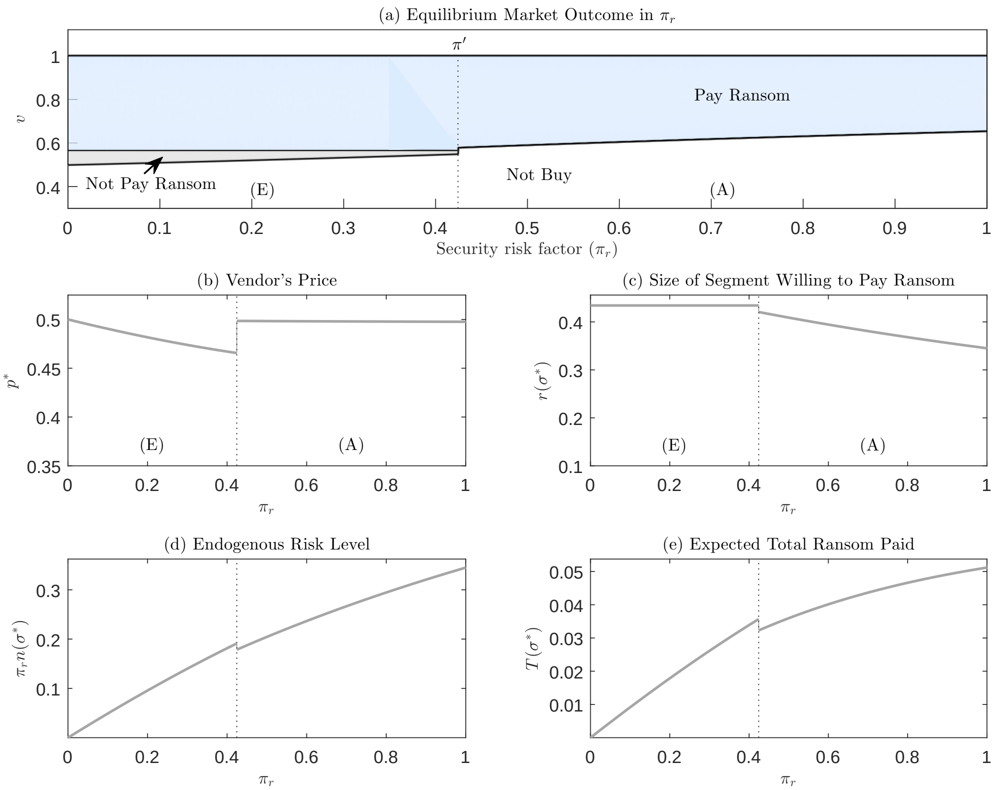


Figure 6: Impact of risk factor (π_r) on the equilibrium market outcome, vendor's price, size of the market segment willing to pay ransom if hit, endogenous risk level, and total expected ransom paid. The parameter values are $\alpha = 0.8$, $\delta = 0.05$, and $R = 0.43$.

(i) if $\alpha \leq 2$, or $\alpha > 2$ and $R \leq 1$, then the total, expected ransom paid is increasing in π_r. Otherwise, the total, expected ransom paid increases in π_r on $(0, \frac{1}{R})$ and decreases in π_r on $[\frac{1}{R}, 1)$.

(b) if $\tilde{R}_2 < R < \tilde{R}_3$, then:

(i) if $0 < \pi_r < \pi'$, then the size of the population willing to pay ransom is constant in π_r while the total, expected ransom paid increases in π_r;

(ii) if $\pi' \leq \pi_r < 1$ and if $R \leq 1$, then the size of the population willing to pay ransom shrinks in π_r while the total, expected ransom paid increases. On the other hand, if $R > 1$, then the size of the population willing to pay ransom shrinks in π_r while the total, expected ransom paid increases in π_r for $\pi_r < \frac{1}{R}$ and decreases for $\pi_r \geq \frac{1}{R}$.

(iii) the vendor's price hike at $\pi_r = \pi'$ reduces usage risk to the extent that the total, expected ransom paid decreases at the discontinuity as well.

The vendor's profit and market size are decreasing in π_r on each of the specified intervals above, regardless of R.

The primary differences between the equilibrium outcomes under patchable vulnerabilities (RW) and unpatchable vulnerabilities (RW-OV) are due to the fact that Region (C) arises under RW. This particular region corresponds to all segments being present in the market, which is commonly observed in practice. Thus, increases in either ransom value or risk shift consumers who were previously willing to pay ransom toward either patching or no longer paying the ransom demanded while remaining unpatched. The opportunity to patch significantly alters the economics of the situation.

5 Multiple Classes of Threats on Patchable Vulnerabilities

In this section, we extend our main model to account for the possibility of other attacks in addition to ransomware. We examine attacks whose exploits share a common vulnerability. For example, EternalRocks and WannaCry are both worms that rely on NSA-leaked tools EternalBlue (for lateral propagation) and DoublePulsar (for backdoor implantation) to exploit a Windows Server Message Block vulnerability. However, WannaCry delivers a ransomware payload whereas EternalRocks does not (Ng 2017). In fact, there are over a dozen known large-scale malware campaigns that weaponized EternalBlue to facilitate lateral spread (Keshet 2020).

We explore how the presence of ransomware alters software firm strategies and market outcomes even when it is among other threats. For a consumer of type v, we fix the payload impact from the attack to be the same, i.e., αv, for both ransomware and non-ransomware attacks. This enables us to tease out the effect of having a ransom-paying option associated with one of the threats without creating a confounding effect due to differences in payload.[26] Given that the same vulnerability is exploited, we assume interdependent security risks under all threats, parameterizing the risk factor associated with a non-ransomware payload uniquely as π_n such that a successful non-ransomware attack occurs with probability $\pi_n u(\sigma)$. By permitting π_r and π_n to vary freely, we can examine different attack profiles.

We denote this model of *multiple threats* with MT and express the consumer's utility

[26] Different payload magnitudes can be accommodated via a transformation. If the payload impact from the non-ransomware attack is $\alpha_n v$ with $\alpha_n \neq \alpha$, then we can define $\bar{\pi}_n = \pi_n \alpha_n / \alpha$, ensuring that $\pi_n \alpha_n = \bar{\pi}_n \alpha$ and we transform the model into one with equivalent payloads from both types of attacks and risk factor $\bar{\pi}_n$ for the non-ransomware attack.

function as follows:

$$U_{MT}(v,\sigma) \triangleq \begin{cases} v-p-c_p & if \quad \sigma(v)=(B,P)\,; \\ v-p-\pi_r u(\sigma)(R+\delta\alpha v)-\pi_n u(\sigma)\alpha v & if \quad \sigma(v)=(B,NP,R)\,; \\ v-p-(\pi_r+\pi_n)u(\sigma)\alpha v & if \quad \sigma(v)=(B,NP,NR)\,; \\ 0 & if \quad \sigma(v)=(NB)\,, \end{cases} \tag{6}$$

where $u(\sigma) \triangleq \int_{\mathcal{V}} \mathbb{1}_{\{\sigma(v)\,\in\,\{(B,NP,R),(B,NP,NR)\}\}}\,dv$ is the size of the unpatched population in the presence of the security threat which facilitates both attacks. Similar to before, we assume that $\delta \in (0,1)$, $\pi_r, \pi_n \in [0,1]$, $c_p \in (0,1)$, $R \in (0,\infty)$, and $\alpha \in (0,\infty)$.

We begin by exploring the differences in market dynamics between ransomware and non-ransomware threats in the presence of negative security externalities. We study the following two scenarios which serve to highlight the impact of having ransomware present versus absent from the threat landscape:

- **Scenario 1**: only a ransomware threat is present. In this special case, $\pi_n = 0$ and MT becomes equivalent to the main model in Section 3, i.e., $U_{MT} \equiv U_{RW}$.

- **Scenario 2**: only a non-ransomware threat is present. In this special case, $\pi_r = 0$ and MT becomes equivalent to a *benchmark* model (henceforth denoted BM) specified in August and Tunca (2006).[27] That is, $U_{MT} \equiv U_{BM}$.

In this way, MT is a generalization that integrates ransomware and non-ransomware threats while preserving the integrity of the models that inform the respective components.

In order for the comparisons we make to be meaningful, we hold $\pi_r + \pi_n = \pi$ as constant with $\pi > 0$. Under scenario 1, we specify $\pi_r = \pi$ and $\pi_n = 0$, whereas under scenario 2, we specify $\pi_r = 0$ and $\pi_n = \pi$. Thus, we focus in both cases on similar attack vectors for infiltration and spread within networks. By comparing outcomes between the two scenarios given a common risk factor, any difference in the nature and magnitude of outcomes is attributable to the additional option consumers have in scenario 1 (which is to pay ransom) and the strategic behavior that results from its presence.

[27] Their model captures fundamental characteristics of software markets in the presence of malware attacks with no ransom option. Specifically, they study three potential consumer strategies, $S_{BM} = \{(B,P),(B,NP),(NB,NP)\}$ and capture the network externalities that exist. For a given strategy profile $\sigma : \mathcal{V} \to S$, the expected utility function for consumer v is given by:

$$U_{BM}(v,\sigma) \triangleq \begin{cases} v-p-c_p & if \quad \sigma(v)=(B,P)\,; \\ v-p-\pi_n u_{BM}(\sigma)\alpha v & if \quad \sigma(v)=(B,NP)\,; \\ 0 & if \quad \sigma(v)=(NB)\,, \end{cases} \tag{7}$$

where $u_{BM}(\sigma) \triangleq \int_{\mathcal{V}} \mathbf{1}_{\{\sigma(v)\,\in\,\{(B,NP)\}\}}\,dv$ is the size of the unpatched population under the benchmark case.

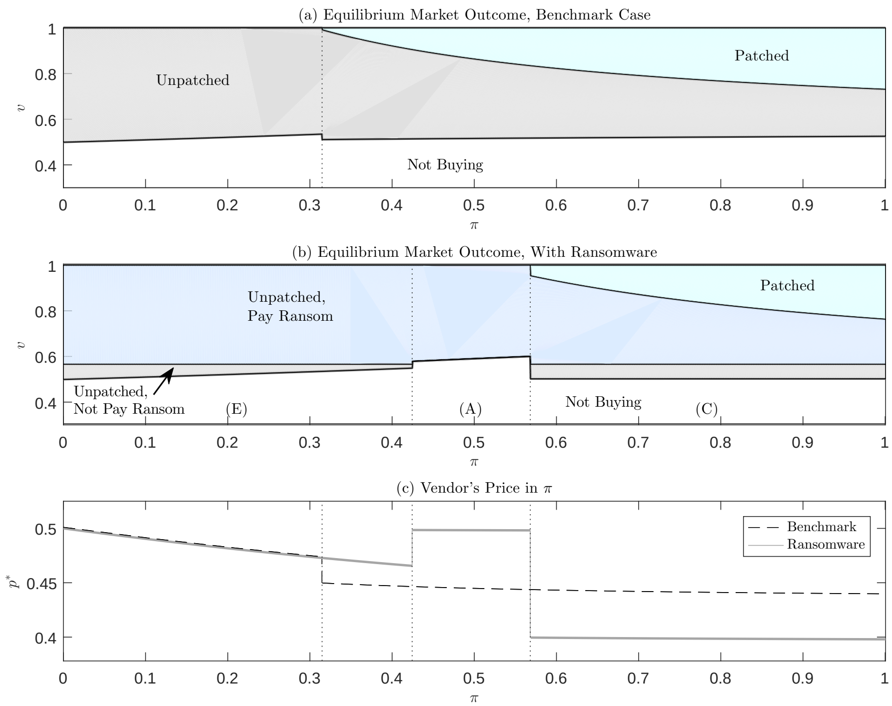


Figure 7: Sensitivity of equilibrium consumer market structure and price with respect to risk factor π under both benchmark and ransomware cases. The parameter values are $c_p = 0.12$, $\alpha = 0.8$, $\delta = 0.05$, and $R = 0.43$.

Similar to the discussion in Section 3, the more interesting comparative analysis occurs for low δ which describes ransomware designed with revenue generation in mind. First, we explore how the vendor's optimal pricing strategy is different under the two scenarios.

Proposition 7. *There exist* $\tilde{\delta}, \hat{\omega} > 0$ *and non-overlapping intervals with bounds satisfying* $0 < \tilde{\pi}_L < \underset{\sim}{\pi}_M < \tilde{\pi}_M < \underset{\sim}{\pi}_H < 1$ *such that if* $\delta \leq \tilde{\delta}$ *and* $R_2 < R < \hat{\omega}$*:*[28]

(a) if $0 < \pi < \tilde{\pi}_L$*, then* $p^*_{RW} = p^*_{BM}$*;*

(b) if $\underset{\sim}{\pi}_M < \pi < \tilde{\pi}_M$*, then* $p^*_{RW} > p^*_{BM}$*;*

(c) if $\underset{\sim}{\pi}_H < \pi < 1$*, then* $p^*_{RW} < p^*_{BM}$*.*

Proposition 7 is illustrated in Figure 7. For the ransomware scenario, Figure 7 depicts a cross-section of Figure 1 in the π-direction at $R = 0.43$. In particular, for π ranging from 0 to 1, this slice cuts through Regions (E), (A), and (C) of Figure 1; these region labels are also provided in panel (b) of Figure 7. When the risk factor (π) is sufficiently small, whether ransomware is present in the landscape or not, no consumer has any incentive to patch

[28] R_2 appears in Proposition 1 and is defined in Lemma B.3.

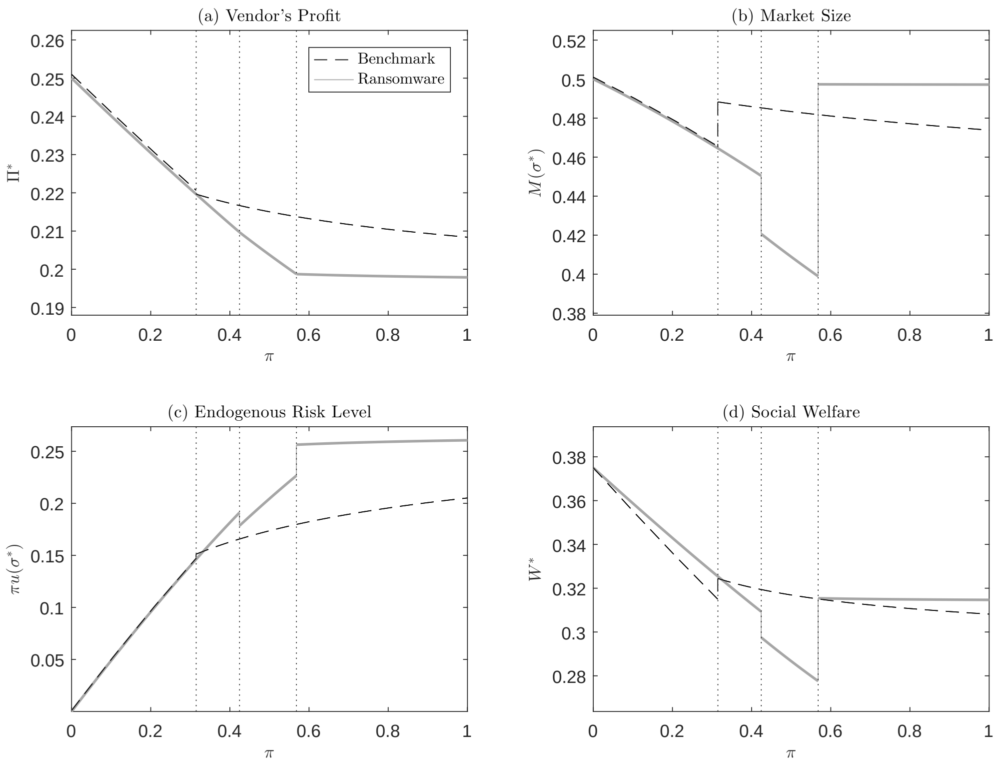


Figure 8: Sensitivity of vendor's profit, consumer market size, endogenous total risk level (πu), and social welfare in equilibrium with respect to the risk factor (π) under both benchmark and ransomware scenarios. The parameter values are $c_p = 0.12$, $\alpha = 0.8$, $\delta = 0.05$ and $R = 0.43$.

since expected losses are small relative to the cost of patching. Under RW, the unpatched consumer indifferent between paying ransom and not paying ransom (v_r) derives strictly positive utility in equilibrium. Thus, the emerging market structure is $0 < v_{nr} < v_r < 1$ (as seen in panel (b) of Figure 7 to the left of $\pi \approx 0.31$). Low-valuation consumers who adopt under the given risk circumstances face expected losses that are small enough that paying ransom is not incentive-compatible for them. Hence, the lowest-valuation adopter under RW is unaffected by small perturbations of the ransom demand, although she is affected by the overall unpatched population size. Since the vendor cares about the entire adopter population, the optimal price, profit, and total market size under both RW and BM are the same in this region. This can be seen in Figures 7 and 8. In particular, panels (a) and (b) of Figure 8 show the impact of π on the vendor's profit and equilibrium market size, $M(\sigma^*) \triangleq \int_{\mathcal{V}} \mathbb{1}_{\{\sigma^*(v) \in \{(B,NP,NR),(B,NP,R),(B,P)\}\}} \, dv$.

As π increases into moderate and high ranges, we see differences in pricing strategy (and corresponding market outcomes) between the two threat landscapes. While in both scenarios the price remains piecewise decreasing, the two pricing strategies present jumps at points of discontinuity (corresponding to changes in market structure) that highlight significant differences in the vendor's approach toward mitigating risk via pricing. Under BM, when the risk factor first reaches a moderate level ($\pi \approx 0.31$), the endogenous security risk πu in equilibrium has also become relatively high (as can be seen from panel (c) of Figure 8). At such a risk level, the vendor has an incentive to significantly drop its price (a discontinuous reduction, illustrated in panel (c) of Figure 7) in a strategic manner to profitably increase the market size. Facing a larger unpatched population, the highest-valuation consumers opt to patch, which insulates them from the added externality introduced by more lower-valuation consumers joining the market but not patching.

However, in the vicinity of $\pi \approx 0.31$, this dynamic does not occur under RW. In particular, if the vendor were to drop the price, some of the highest-valuation consumers would not patch, even when facing this increased risk. Instead, they would still opt to just pay the ransom and bear the valuation-dependent losses $\delta\alpha v$. As long as these highest-valuation consumers remain unpatched, they continue to impose a negative externality on all other unpatched consumers in the market. Because of that, the vendor cannot profitably expand the market through a significant drop in price; thus, $p^*_{RW} > p^*_{BM}$ in this region.

In stark contrast to the benchmark scenario, under RW the vendor actually has an incentive to hike the price altogether to a higher range as risk increases further. This happens for π between 0.42 and 0.57, as can be seen in panel (c) of Figure 7. This outcome and the trade-offs that drive it have been analyzed in Proposition 2 and the related discussion. This is an important point of contrast specific to ransomware; notably, under BM, the vendor never discontinuously hikes price when changing the market structure from $0 < v_{nr} < 1$ to $0 < v_{nr} < v_p < 1$ as the risk factor increases. This is because lower-valuation consumers who do not patch would be strongly impacted since there is not an option to pay ransom in order to mitigate losses. As such, a price increase would hurt the unpatched population even more, leading to a significant drop in market size and lower profits.[29]

In a similar way, once the risk factor becomes sufficiently high, then the vendor drops price significantly under RW as well, even below that seen in the benchmark level. This occurs near $\pi \approx 0.57$ in panel (c) of Figure 7. This move expands the market significantly at the low end, inviting a large mass of unpatched consumers to join the market with a

[29]In Appendix A.1, we consider the robustness of our results. Specifically, in Figure A.3 we plot the vendor's equilibrium price under BM over the entire space of patching costs (c_p) and the effective security loss factor ($\pi_n \alpha$) to demonstrate that a price hike does not occur. The only price discontinuity occurs when the vendor significantly scales back price as the loss factor exceeds a threshold. At that point, the vendor's price provides the right incentives for a patching population to emerge in equilibrium.

sizable fraction of them opting to not pay the ransom if hit, as seen in panel (b) of Figure 7. This behavior introduces significant overall risk to the market which is depicted in panel (c) of Figure 8. As a result of this endogenous increase in aggregate risk, high-valuation consumers find it optimal to protect themselves by patching. Consequently, the resulting consumer market equilibrium outcome is $0 < v_{nr} < v_r < v_p < 1$, which most closely resembles today's software markets.

Interestingly, panel (a) of Figure 8 shows that, once the risk factor is high enough ($\pi > 0.31$), the vendor is strictly better off in scenario 2 (benchmark) than in scenario 1 (ransomware). As π increases from 0.31 to 0.42, under RW, the market keeps shrinking even though the price is dropping. In contrast, under BM, the big drop in price leads to a significant expansion in market size. This enables the vendor to obtain higher profits. When π ranges from 0.42 to 0.57, under RW, the vendor strategically and significantly increases price. As π increases through this region, the vendor is better off letting usage shrink rather than lowering the price. Moreover, the unpatched group that pays ransom is highly elastic to risk in this region. On the other hand, under BM, the overall population remains relatively stable (only slightly decreasing), as the presence of security risk alters the sizes of the patched and unpatched groups in opposite directions in a balanced way. With prices relatively inelastic in this region, the profit under the benchmark is superior. When the risk factor is even higher ($\pi > 0.57$), the gap between the two profit levels is shrinking but the profit under the benchmark scenario still dominates.

Lastly, we study the impact of ransomware on social welfare. Under RW, we denote the expected losses incurred by unpatched consumers who do not pay ransom with $NL_{RW} \triangleq \int_{\mathcal{V}} \mathbb{1}_{\{\sigma^*(v)=(B,NP,NR)\}} \pi u(\sigma^*) \alpha v dv$. Similarly, we denote the expected losses incurred by unpatched consumers who pay ransom with $RL_{RW} \triangleq \int_{\mathcal{V}} \mathbb{1}_{\{\sigma^*(v)=(B,NP,R)\}} \times \pi u(\sigma^*)(R + \delta\alpha v) dv$. The aggregate expected patching costs are given by $PL_{RW} \triangleq \int_{\mathcal{V}} \mathbb{1}_{\{\sigma^*(v)=(B,P)\}} \times c_p dv$. Summing these components, the expected security-related losses are $L_{RW} \triangleq NL_{RW} + RL_{RW} + PL_{RW}$. Social welfare is then given by $W_{RW} \triangleq \int_{\mathcal{V}} \mathbb{1}_{\{\sigma^*(v)\in\{(B,NP,NR),(B,NP,R),(B,P)\}\}} v dv - L_{RW}$. Similarly, for the benchmark case, we define $NL_{BM} \triangleq \int_{\mathcal{V}} \mathbb{1}_{\{\sigma^*(v)=(B,NP)\}} \pi u(\sigma^*) \alpha v dv$, $PL_{BM} \triangleq \int_{\mathcal{V}} \mathbb{1}_{\{\sigma^*(v)=(B,P)\}} c_p dv$, $L_{BM} \triangleq NL_{BM} + PL_{BM}$, $W_{BM} \triangleq \int_{\mathcal{V}} \mathbb{1}_{\{\sigma^*(v)\in\{(B,NP),(B,P)\}\}} v dv - L_{BM}$. The presence of an option to pay ransom can have a significantly impact welfare in complex ways, which we illustrate in the next result.

Proposition 8. *There exist $\tilde{\delta}, \hat{\omega} > 0$, $\tilde{c}_p \in (0, 2-\sqrt{3})$, and non-overlapping intervals with bounds satisfying $0 < \tilde{\pi}_L < \underset{\sim}{\pi}_M < \tilde{\pi}_M < \underset{\sim}{\pi}_H < 1$ such that if $\delta \le \tilde{\delta}$, $c_p > \tilde{c}_p$, $\alpha > \frac{1}{c_p}$, and $R_2 < R < \hat{\omega}$:*

(a) if $0 < \pi < \tilde{\pi}_L$, then $W_{RW} > W_{BM}$;

(b) if $\underset{\sim}{\pi}_M < \pi < \tilde{\pi}_M$, then $W_{RW} < W_{BM}$;

(c) if $\tilde{\pi}_H < \pi < 1$, then $W_{RW} > W_{BM}$.

When risk is sufficiently low, consumers do not have strong incentives to patch. Whether they have the option to pay ransom or not, consumers remain unpatched and, as discussed before, the optimal price and market size are identical under the two scenarios, as seen in panels (b) and (c) of Figure 8 for $\pi < 0.31$. The only difference is that, under RW, unpatched consumers with higher valuations counter the potentially high valuation-dependent losses by paying ransom. Therefore, overall losses are lower under RW, which leads to higher social welfare as can be seen in panel (d) of Figure 8.

Social welfare is also higher under RW when risk is high ($\pi > 0.57$) but for a different reason. In this region, high-valuation consumers patch under both scenarios. However, in the face of ransomware, the vendor employs a significantly lower price and achieves a larger market size, including a larger unpatched population. Moreover, some of the unpatched consumers are able to reduce their losses by paying ransom instead. Combining these two effects leads to higher social welfare under RW in comparison to BM. Notably, in the high-risk region, the vendor's interests are not aligned with the scenario that yields higher social welfare. Instead, the vendor actually prefers that consumers do not have the recourse of paying ransom. This in turn places additional pressure on consumers, leading to a reduced unpatched population, and ultimately enabling the vendor to charge a higher price.

When the risk is within an intermediate range (π between 0.31 and 0.57 in Figure 7), the vendor sets a higher price under RW relative to BM, resulting in a market size that is significantly lower. While most (and sometimes all) unpatched consumers pay ransom and nobody patches, the significant difference in market size under RW and BM pushes social welfare higher in the latter case, also matching the vendor's scenario preference.

In Section A.1 of the Appendix, we also study how a higher level of residual losses, δ, which may be associated with political motivations, impacts outcomes. We demonstrate that benchmark and ransomware outcomes are equivalent. Said differently, consumers cease paying ransom, hence an open question is whether a politically-motivated hacker could inflict greater damage by more appropriate tuning of residual losses. We show that the hacker can indeed minimize welfare at a moderate range of residual losses that encourages ransom payments to occur.[30]

We also perform a robust analysis of the results presented in this paper. In our discussion of Proposition 7 and Figure 7, an important contrast in the benchmark case is that the vendor only drops price in response to increasing risk (as opposed to the price hike observed with ransomware). In Section A.2, we demonstrate that this contrasted pricing behavior in the

[30]A numerical illustration of this result can be found in Figure A.2 in the Appendix.

benchmark case is robust over a wide range of risk and patching costs.

Second, it is important to show that our results extend broadly to the general scenario with multiple classes of threats (MT). We examine this formally in Section A.3 of the Appendix. A common theme that emerges from this robustness analysis is that ransomware need not be the sole (or even dominant) form of security attack in the market for our results to be applicable. In particular, some of the effects that we establish are specific to the existence of ransomware, and, moreover, these effects only require a small proportion (or likelihood) of ransomware to be in play to already take hold. Since these effects alter the strategic decisions of the software vendor, even the existence of ransomware in a given software market should warrant careful consideration.

6 Conclusion

With the rise of cryptocurrency-based payment systems, malicious hackers are finding it increasingly profitable to conduct ransomware attacks. Modern ransomware variants have exhibited the capability to spread laterally across unprotected systems leading to large scale reach and damage. In this paper, we study the impact of ransomware attacks on software markets. The presence of ransomware in a software product's threat landscape can qualitatively change the nature of the consumer market structure that obtains in equilibrium. In particular, by giving consumers an opportunity to mitigate their losses by paying ransom, ransomware operators segment the unpatched population into two interdependent tiers. Ransomware directly impacts the ransom-paying consumer segment while indirectly impacting all market segments through the negative security externality that all unpatched consumers generate. This segmentation of consumer behavior drives unexpected findings. For example, both the equilibrium market size and the vendor's profit under equilibrium pricing can increase in the ransom demand. Also, the vendor's profit can increase in the residual loss factor (related to the trustworthiness of the ransomware operator). Furthermore, we also show that the expected total ransom paid is non-monotone in the risk of success of the attack, increasing when the risk is moderate in spite of a decreasing ransom-paying population.

In order to properly assess the market changes induced by the option to pay ransom, we also compare and contrast market outcomes in the ransomware case to similar outcomes under a benchmark scenario where consumers do not have the option to mitigate the losses by paying ransom. For intermediate levels of risk, the vendor under the ransomware case restricts software adoption by hiking the price to a significantly high level. This lies in stark contrast to outcomes in the benchmark case where any jump in price as security risk increases will be downward. While in low and high-risk settings, social welfare is higher under the ransomware case compared to the benchmark, it turns out that for intermediate risks levels, it is better from a social standpoint for consumers to *not* have an option to pay ransom.

We expand our study in two dimensions by exploring (i) other variants of ransomware whose attacks are not specific to patchable vulnerabilities (e.g., phishing attacks and zero-day attacks), and (ii) other classical attacks on patchable vulnerabilities in addition to ransomware in a generalized model of multiple, concomitant classes of threats. The first expansion clarifies that while the impact of the ransom amount and risk level on equilibrium measures have a similar nature, some important findings are specific to the presence of ransomware risk related to patchable vulnerabilities. For example, the opportunity highlighted where price, market size and profits can all increase in the ransom amount hinges on the impact to patching incentives. Similarly, while strategic price hikes can be observed in the presence of ransomware whether the vulnerability is patchable or not, a strategic price drop that is observed with patchable vulnerabilities requires a higher incentive to patch that occurs in higher risk regions. Overall, the insights we obtain in our primary study are quite robust to generalized attacks in a wide range of scenarios, including threat landscapes where ransomware has only a small presence. Said differently, a little ransomware in the risk profile can be quite influential.

From our findings, we provide some practical guidance to consumers of vulnerable software and the software firms that produce it. First, for consumers, the upside of economically-motivated ransomware (in comparison to traditional attacks) is possessing another option to choose from when minimizing losses. However, when many consumers substitute away from patching toward paying ransom as a means to reduce losses, risk can increase because more consumers remain unpatched. From the interaction of these forces with properties of the ransomware environment (such as the size of ransom amount and the likelihood of an attack), we demonstrate how consumers must carefully adjust their usage and patching strategies in response to the changing environment and connected strategic price adaptations. To make this concrete, panel (a) in Figure 2 shows how a consumer with a valuation near the lower threshold of being in or out of the market would choose to be: (in market, pay ransom) $\rightarrow$ (out of market) $\rightarrow$ (in market, pay ransom) $\rightarrow$ (in market, *not* pay ransom), as the ransom amount increased across its range. Notably, in Region (C) of the figure, the vendor strategically raises price in response to larger ransoms but the steep increase in patching behavior by consumers makes the product more secure and encourages more consumers at the low end of valuations to enter the market. Our findings provide guidance to consumers on how to manage their software in a setting with security interdependence.

Second, our findings help provide guidance to software vendors on how pricing can help mitigate risk via a richer understanding of consumer incentives. While the pricing of software products is a complex affair involving many relevant factors, our model aims to inform on the directional pressure on price from relevant cybersecurity factors. We highlight the extent to which pricing can influence consumption and security behaviors, making it a powerful tool in the arsenal to shape cybersecurity defense. Specifically, in risk contexts where consumers do not have sufficient incentives to patch and instead bear risk and pay ransom, we find that vendors can effectively raise prices to better serve a smaller consumer population by providing an inherently reduced risk level. As ransomware attacks continue on, if the economic returns to hacker activity ultimately lead to an increased rate of attacks, this would be detrimental to vendor profits and we recommend that vendors focus their attention on reducing patching costs to counter such perverse incentives.

One may desire to model a specific type of hacker motivation and endogenize that hacker's decisions, in hopes of yielding insights into how the vendor and consumers strategically interact with the hacker. There are two issues that arise. First, our model captures security externalities which cause a significant increase in complexity, e.g., the consumer market threshold characterization is governed by a nonlinear system and the thresholds themselves are the roots of higher order polynomials. Layering the vendor's pricing optimization problem on top of this foundation already requires asymptotic analysis. Adding more decision variables exacerbates this complexity issue and makes it impossible to characterize the equilibrium across a significant portion of the parameter space. Therefore, such an effort would heavily rely on numerical analysis. Second, one has to give up on other common hacker motivations in order to make that specification which results in a fundamental loss in model generality and applicability. That is, vendors and consumers are facing a diverse mix of hackers which makes it difficult to apply insights drawn from an analysis of a single hacker class. This becomes even more salient if one contrasts the incentives of a profit-motivated hacker to a state actor; their approach to residual losses would essentially be bipolar. The advantage of our model lies in its ability to examine any R and δ that could result from any practically-relevant model of hacker behavior - such a model would ultimately need to prescribe weights across hacker "types" (motivations) which would then result in a large range of expectations for ransoms and residual losses. This is exactly what our model can examine. Our intention in this research is to inform vendors and consumers on strategies and market outcomes in the presence of a generic ransomware threat, without zooming in on one particular type of hacker. An interesting and rich direction for future research would be a focused exploration of specific markets that are characterized primarily by a single hacker motivation (whether that is profit, disruption, etc.). For such markets, studying how a hacker specifies ransom amounts and residual damages may lead to further insights.

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Online Supplement for
"Economics of Ransomware: Risk Interdependence and Large-Scale Attacks"

Appendix A. Residual Losses and Robustness

A.1 Impact of Residual Losses

In the context of ransomware attacks on patchable vulnerabilities (RW), we explore how δ impacts the vendor's profit, expected aggregate losses incurred by the unpatched population, and aggregate consumer surplus. The latter two measures are defined by:

$$UL \triangleq \int_{\mathcal{V}} \mathbb{1}_{\{\sigma^*(v)=(B,NP,NR)\}} \pi_r u(\sigma^*) \alpha v dv + \int_{\mathcal{V}} \mathbb{1}_{\{\sigma^*(v)=(B,NP,R)\}} \pi_r u(\sigma^*)(R + \delta\alpha v) dv \,,$$

$$CS \triangleq \int_{\mathcal{V}} \mathbb{1}_{\{\sigma^*(v)\in\{(B,NP,NR),(B,NP,R),(B,P)\}\}} U_{RW}(v, \sigma^*) dv.$$

Proposition A.1. *There exists a bound $\tilde{\delta} > 0$ such that if $\delta < \tilde{\delta}$, $R \in (R_2, R_3)$ and $\pi_r > \bar{\pi}_r$*[31] *are satisfied, then:*

(a) the vendor's profit is increasing in δ;

(b) the market size, aggregate unpatched losses, and consumer surplus are all decreasing in δ.

Proposition A.1 characterizes a market scenario that falls within Region (C) of Figure 1, in which case $0 < v_{nr} < v_r < v_p < 1$ characterizes the equilibrium outcome. The results in Proposition A.1 can be observed in the range $0 < \delta < 0.301$ in Figure A.1. The vendor benefits from an increase in residual loss factor δ for the same reason for which it benefits when R increases (as discussed in part (c) of Proposition 1). An increase in δ only directly impacts ransom-paying unpatched consumers, providing a disincentive for them to adopt this strategy. As some consumers who were paying ransom switch to patching due to an increased risk of not receiving working decryption keys, the aggregate risk externality decreases and the vendor can increase its price while keeping the overall market relatively steady, extracting additional surplus.. When all market segments are present in equilibrium (depicted in the left-hand portion of the panels in Figure A.1), the vendor clearly prefers greater potential residual losses (whether stemming from failures with payment systems or decryption keys as well as mixed motivations of hackers) because it presents an unusual and counter-intuitive opportunity to charge a premium for higher potential residual losses in the market without losing too many consumers.

[31] The bounds $\bar{\pi}_r$, R_2, and R_3 are the same as in Proposition 1.

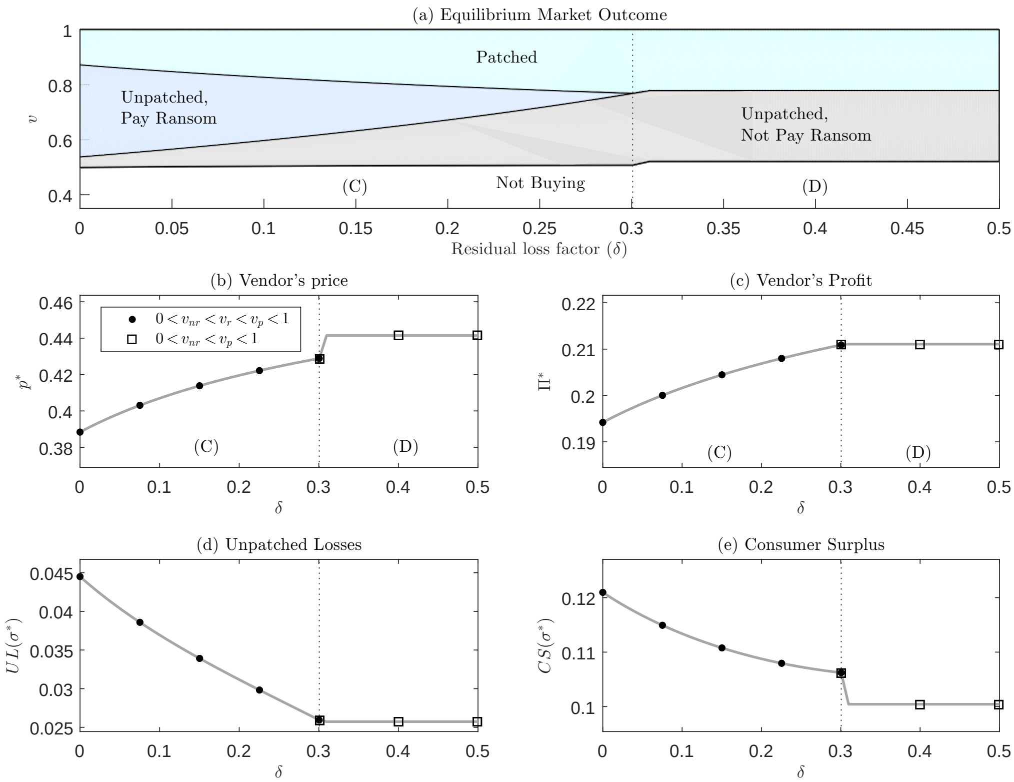

Figure A.1: Impact of the residual loss factor (δ) on the equilibrium market outcome, vendor's pricing, vendor's profit, aggregate unpatched consumer losses, and consumer surplus. The parameter values are $c_p = 0.12$, $\alpha = 0.8$, $R = 0.43$, and $\pi_r = 0.75$.

Furthermore, in this range of residual losses, the vendor would prefer that consumers have the worst possible perception regarding the trustworthiness of the attacker whereas an economically-driven hacker would prefer the opposite. Reports suggest that, compared to the early days of ransomware attacks, the market for such attacks has become efficient and the success rate in retrieving access to compromised assets following a ransom payment increased dramatically, highlighting prevalent economic motivations among attackers (Disparte 2018). But, in many cases, corporate victims that pay ransom do not publicize their actions (Cimpanu 2017), which makes it easier for the vendor to vilify attackers in an amplified way even when decryption keys are often returned. Even a small number of failed interactions can damage the hacker's reputation and effectively cut off its revenue stream.

Proposition A.1 further shows that the vendor's expected profit can be the lowest at the same δ that concomitantly gives the greatest losses to the unpatched population and the highest overall consumer surplus, as seen in panel (d) of Figure A.1. As δ increases, both the ransom-paying population shrinks as consumers at both ends of this segment choose different strategies (higher-valuation consumers choose to patch, while lower-valuation consumers choose not to pay ransom). Moreover, the overall unpatched population shrinks as well, thus lowering the security risk externality. The redistribution of consumers among segments and the reduced risk result in the expected losses to the unpatched population to be decreasing in δ. Even though these unpatched losses are decreasing, the vendor employs a higher price to further throttle the population size and help mitigate the increased magnitude of residual losses. Given the relatively stable (but slightly shrinking) size of the market when $\delta < 0.301$, the reduction in losses to the unpatched population is dominated by the larger premium, hence consumer surplus also decreases in δ.

We next turn our attention to the case where residual losses become high. An economically-motivated hacker may aptly be characterized as having a lower δ because revenue generation requires a mass of consumers to pay ransom in equilibrium. In particular, a lower δ helps to make this strategy incentive compatible for some. On the other hand, a politically-motivated hacker is less concerned with capping residual, as with attacks such as WannaCry or Not-Petya (Greenberg 2018). In that generating ransom payments is no of primary concern, the practical range of δ for hackers with such motivations is much broader.

Proposition A.2. *When the residual loss factor is high, satisfying $\delta > 1 - \frac{R}{\alpha}$, the vendor's equilibrium price, the size of the market, and equilibrium profit are constant in R and δ. The equilibrium outcome is given by $0 < v_{nr} < 1$ when patching costs are high ($\pi_r \alpha < c_p$) and $0 < v_{nr} < v_p < 1$ when patching costs are low ($\frac{c_p(2-3c_p)}{1-2c_p} < \pi_r \alpha$).*

Proposition A.2 demonstrates that as residual losses become high, consumers react in a predictable way in that paying ransom is no longer incentive compatible. Instead, consumers will either all remain unpatched and risk losses if the cost of patching is prohibitive or, otherwise, split between staying unpatched and patching. Note that in the absence of a ransom-paying population, these equilibrium structures are the only two that can arise, each being possible. As can be seen in the right-hand side of Figure A.1's panels, the relevant measures become constant in δ once the residual loss factor exceeds a threshold; for that particular parameter set, patching costs are low, leading to some consumers patching in equilibrium.

Because no one pays ransom, the ransomware and benchmark scenarios (as laid out in Section 5) essentially converge once δ exceeds a threshold. In other words, the remaining feasible strategies for consumers match in the scenarios, which gives rise to equivalent equilibrium measures in both cases. In particular, if $\delta > 1 - \frac{R}{\alpha}$, then $p^*_{RW} = p^*_{BM}$. And consequently, $\Pi^*_{RW} = \Pi^*_{BM}$ and $W_{RW} = W_{BM}$. Viewed differently, if politically-motivated hackers are utilizing ransomware attacks with high residual losses, then they need not employ ransomware at all; traditional attacks result in equivalent outcomes. However, given that high δ ransomware is actually deployed, this equivalence brings forth the question of whether such ransomware could be even more harmful depending on the motivation of the hacker. For example, if a politically-motivated hacker targeted total losses associated with the software, then lowering δ to induce some ransom payments would actually be more effective. Panel (a) of Figure A.2 illustrates a case where lowering δ to a medium range, i.e., $\delta \in (0.64, 0.76)$, increases expected losses relative to the higher range, i.e., $\delta > 0.76$.

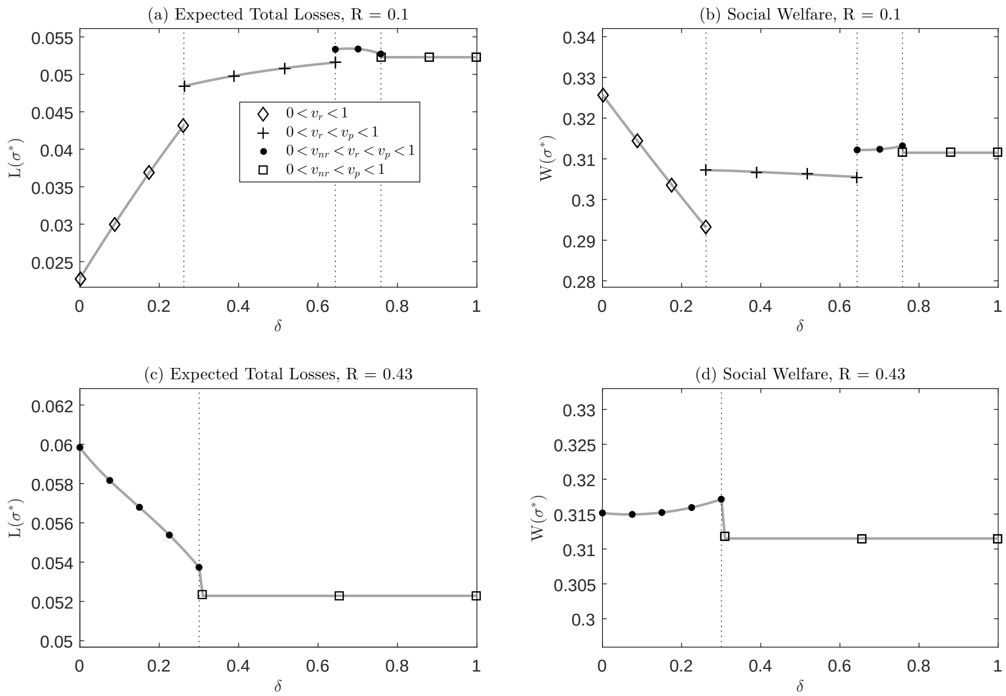

Figure A.2: Impact of residual loss factor (δ) on total expected losses and social welfare. The common parameter values are $c_p = 0.12$, $\alpha = 0.8$, and $\pi_r = 0.75$. Panels (a) and (b) illustrate comparative statics for $R = 0.1$, while panels (c) and (d) do the same for $R = 0.43$.

In contrast, at a higher level of ransom demand, a politically-motivated hacker focused on total expected losses might benefit from a minimal δ which greatly boosts total expected losses as is illustrated in panel (c) of Figure A.2. In this case, the behavior of politically-motivated and economically-motivated hackers actually coincides unlike in the case of a lower ransom demand shown in panel (a). On the other hand, as we discussed earlier, it is not easy to translate *politically-motivated* to an objective goal. For instance, perhaps politically-motivated could instead mean focused on reducing social welfare. In that case, a high level of residual losses where equivalence with the benchmark is achieved would be more effective at reducing welfare, which is illustrated in panel (d) of Figure A.2. This case highlights the inherent difficulty with hacker motivations: one gets polarizing predictions depending on how even a single motivation (such as being political) gets operationalized. Layering on that there are many diverse hacker motivations, this issue gets further compounded. In light of these issues, our analysis aims to provide insights across a broader set of these motivations by exploring varying levels of R and δ in different regimes. Panel (b) of Figure A.2 underscores the complexity that arises by depicting how a lower boundary value of δ is suddenly the most effective at reducing welfare once ransom demand falls to a lower level.

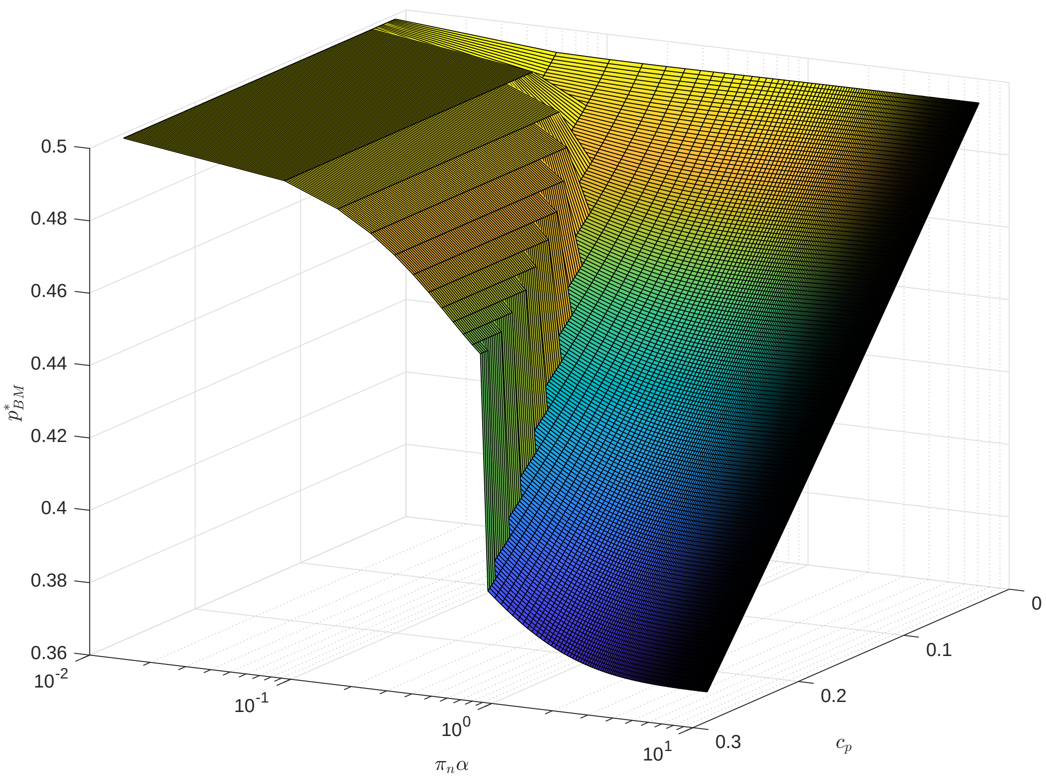

Figure A.3: Equilibrium price for the benchmark (BM) as influenced by the magnitude of the patching cost and effective loss factor.

A.2 Benchmark Case - Robustness of Pricing Strategy

In the main body of the paper, Proposition 7 formalizes a comparison of pricing behavior between ransomware and benchmark scenarios. In our discussion of these differences, depicted in Figure 7, an important contrast we highlight is that the vendor only drops price in response to increasing risk in the benchmark case whereas it may hike price in the ransomware case. In this section, we demonstrate that this contrasted pricing behavior observed in the benchmark case is robust over a wide range of risk and patching costs.

Figure A.3 captures a robustness analysis of the vendor's equilibrium price in the benchmark case over the entire space of patching costs (c_p) and the effective security loss factor ($\pi_n \alpha$). The figure illustrates that the only price jump that occurs is when the vendor significantly scales back price as the loss factor exceeds a threshold. At that point, the vendor's price provides the right incentives for a patching population to emerge in equilibrium.

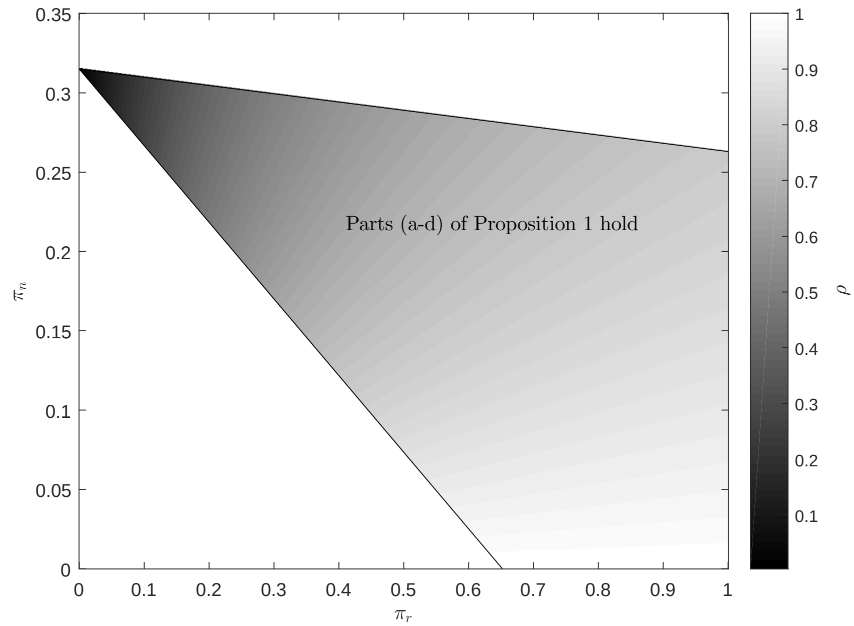

Figure A.4: Robustness of Proposition 1 to multiple classes of threats. The parameter values are $c_p = 0.12$, $\alpha = 0.8$, and $\delta = 0.05$.

A.3 Robustness of Results under Multiple Threats Scenarios

In this section, we investigate the robustness of several of our key findings on patch-mitigated ransomware threats with respect to the general scenario with multiple classes of threats presented in Section 5. Specifically, we examine to what extent the nature of our comparative statics results in R and π_r (Propositions 1, 2, 3, 7, and 8) continue to hold when we allow for the concomitant presence of more traditional threats in the security landscape (i.e., moving from RW to MT). Due to the additional complexity of MT, it is necessary to perform the comparisons using numerical analysis. To better illustrate our results, we define $\pi \triangleq \pi_r + \pi_n$ as the overall risk level and $\rho \triangleq \pi_r/\pi \in (0, 1]$ as the prevalence of ransomware in the threat landscape.

First, we explore the parameter region (π_r, π_n) under which parts (a)-(d) of Proposition 1 still hold in essence (i.e., we encounter precisely four regions with the same market structures, in the same sequence with respect to R, and the monotonicity of vendor's price, market size, and profits with respect to R is the same in each of these regions as in Proposition 1). As illustrated in Figure A.4, the results for Proposition 1 hold for a wide region of (π_r, π_n), with π_n spanning $[0, 0.31]$ and ρ spanning the entire interval $(0, 1]$ (i.e., all shades are observed). In particular, in the left portion of this region, risk is induced *predominantly* by non-ransomware threats, with a minority contribution from the ransomware threat ($\rho < 0.5$). This highlights that ransomware, while necessary in the threat landscape, need not be the most prevalent threat for our results to hold. Moreover, part (c) of Proposition 1 qualifies the existence of a region in R where all three consumer segments are present, which has significant practical relevance. In this region, the vendor's price, market size, and profits all increase in R, and notably the range of (π_r, π_n) where the equivalent region exists under MT (the generalized model) is considerably larger than the region shown in Figure A.4 (which is governed by a stricter requirement).

Similarly, we have numerically investigated and confirmed that the results in Propositions 2 and 3 are robust to the presence of traditional threats. For example, under the same parameters as in Figure 3, the essence of Propositions 2 and 3 is satisfied for $\pi_n \in [0, 0.30]$. For levels of π_n in this range, the behavior exhibited is similar to that in Figure 3 (hence, we omit a matching illustration for brevity).

Last, we explore the robustness of our results in Propositions 7 and 8. For the comparisons to be sensible, we compare and contrast the general model (MT) outcomes under parameters (π_r, π_n) to the outcomes under the benchmark model (BM) with a matching aggregate risk factor of π. Controlling for overall risk enables us to tease out how the presence of a ransom paying option alters market dynamics. We explore the differences between outcomes under the two scenarios for a continuum of ratios $\rho \in (0, 1]$ as well as a continuum of overall risk levels, $\pi \in (0, 1]$. The results of this exploration are depicted in panels (a) and (b) of Figure A.5, with the overall risk level, π, on the x-axis, and prevalence of ransomware threat relative to overall risk, ρ, on the y-axis.[32]

The essence of Propositions 7 and 8 can be visualized by the top, horizontal regions (when $\rho = 1$) of panels (a) and (b) of Figure A.5. As we move away from a ransomware only scenario toward a mixed-threat scenario MT, we incorporate a risk that is a weighted combination of the individual risks in the RW and BM models. As can be seen, the dynamics induced by the presence of a ransom option (at all levels of $\rho > 0$) lead to similar strategic pricing decisions and welfare outcomes as the ones described in Section 5. Panel (a) highlights the robustness of Proposition 7. When the overall risk in the market is low, then nobody patches and MT and BM scenarios induce the same pricing and market size. As the overall risk in the market increases, we notice at all levels of $\rho > 0$ the presence of a region where $p^*_{MT} > p^*_{BM}$ and this region shrinks in width as ρ decreases; this can be expected since, as ρ approaches 0, scenario MT converges to BM. The boundary between the leftmost region and middle region is a straight line, corresponding to the π value at which p^*_{BM} drops (which is independent of the change in ρ characterizing the mix of threats under MT). As the overall risk in the market grows large, we observe $p^*_{MT} < p^*_{BM}$ because the presence of a ransom option provides consumers with a means to mitigate risk. This remains true even when ransomware is not the only threat, provided that the security externality permits expansion at the lower end of the market. In panel (b), we observe the same sequence of welfare ordering as we formally

[32] In Figure A.5, we illustrate the analysis for the same parameter set that was used to generate Figures 7 and 8. We also conducted a separate sensitivity analysis on each parameter and found that the nature of the regions depicted remains consistent over a broad range of each parameter's values.

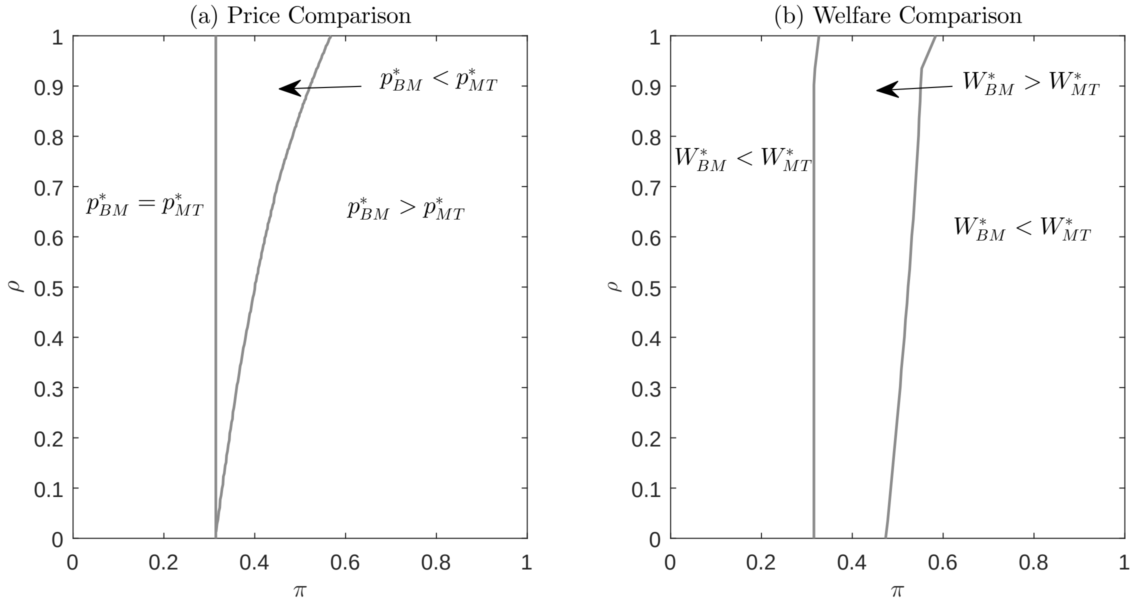

Figure A.5: Price and welfare comparison of ransomware and benchmark settings across risk scenarios. The parameter values are $c_p = 0.12$, $\alpha = 0.8$, $R = 0.43$, and $\delta = 0.05$.

established in Proposition 8 for all levels of ρ. For low and high overall risk in the market, scenario MT induces a higher social welfare compared to BM, whereas the opposite is true for intermediate levels of overall risk. We have extensively discussed the market forces that lead to these outcomes in our analysis of the RW and BM scenarios in Section 5. Similar arguments continue to motivate the outcomes in the generalized model.

Appendix B. Proofs of Lemmas and Propositions

B.1 Main model RW

We begin with our main model of ransomware. We solve the equilibrium via backward induction. In Section B.1.1, we present the consumption subgame (the last stage of the game) and characterize consumer choices for every possible price and set of parameters. Proposition B.1 sets up the groundwork for the vendor's pricing problem by establishing the price intervals (and the associated consumer segments they induce) that the vendor may select from for each set of the nine distinct, exogenous parameter regions organized by R and π_r. In Section B.1.2, for each region of the parameter space, we then solve the pricing subgame to characterize the unique equilibrium price and consumer market outcome.

B.1.1 Consumption Subgame

Proposition B.1. *There exist sets of mutually exclusive conditions on R, α, δ, π_r and c_p that cover the parameter space and organize the equilibrium outcome by price. Under each of these parameter sets, the feasible space for price p can be split into adjacent intervals, each of them with a single structure that characterizes the unique consumption subgame outcome.*

(i) $R \geq \alpha(1-\delta)$ *and* $\pi_r \leq \frac{c_p}{\alpha}$*:*

- $0 \leq p < 1 : (0 < v_{nr} < 1)$

(ii) $R \geq \alpha(1-\delta)$ *and* $\pi_r > \frac{c_p}{\alpha}$*:*

- $0 \leq p < \frac{(1-c_p)(-c_p+\pi_r\alpha)}{\pi_r\alpha} : (0 < v_{nr} < v_p < 1)$
- $\frac{(1-c_p)(-c_p+\pi_r\alpha)}{\pi_r\alpha} \leq p < 1 : (0 < v_{nr} < 1)$

(iii) $\left(R \leq \alpha(c_p-\delta) \text{ or } \pi_r \leq \frac{c_p}{R+\alpha\delta}\right)$ *and* $\pi_r < \frac{1-\delta}{-R+\alpha(1-\delta)}$ *and* $R < \alpha(1-\delta)$*:*

- $0 \leq p < \frac{R(1-\delta+(R-\alpha(1-\delta))\pi_r)}{\alpha(1-\delta)^2} : (0 < v_{nr} < v_r < 1)$
- $\frac{R(1-\delta+(R-\alpha(1-\delta))\pi_r)}{\alpha(1-\delta)^2} \leq p < 1 : (0 < v_r < 1)$

(iv) $\left(\pi_r \leq \frac{c_p\alpha(1-\delta)^2}{R^2} \text{ or } R \leq c_p\alpha(1-\delta)\right)$ *and* $R > \alpha(c_p-\delta)$ *and* $\frac{c_p}{R+\alpha\delta} < \pi_r \leq \frac{c_p\alpha(1-\delta)}{(-R+\alpha(1-\delta))(R+\alpha\delta)}$*:*

- $0 \leq p < \frac{(R+\alpha(-c_p+\delta))(-c_p+(R+\alpha\delta)\pi_r)}{(R+\alpha\delta)^2\pi_r} : (0 < v_{nr} < v_r < v_p < 1)$

- $\frac{(R+\alpha(-c_p+\delta))(-c_p+(R+\alpha\delta)\pi_r)}{(R+\alpha\delta)^2\pi_r} \leq p < \frac{R(1-\delta+(R-\alpha(1-\delta))\pi_r)}{\alpha(1-\delta)^2} : (0 < v_{nr} < v_r < 1)$
- $\frac{R(1-\delta+(R-\alpha(1-\delta))\pi_r)}{\alpha(1-\delta)^2} \leq p < 1 : (0 < v_r < 1)$

(v) $c_p\alpha(1-\delta) < R < \alpha(1-\delta)$ *and* $\frac{c_p\alpha(1-\delta)^2}{R^2} < \pi_r \leq \frac{c_p\alpha(1-\delta)}{(-R+\alpha(1-\delta))(R+\alpha\delta)}$*:*

- $0 \leq p \leq \frac{(-R+c_p\alpha(1-\delta))(c_p\alpha(1-\delta)^2-R^2\pi_r)}{R^2\pi_r\alpha(1-\delta)} : (0 < v_{nr} < v_p < 1)$
- $\frac{(-R+c_p\alpha(1-\delta))(c_p\alpha(1-\delta)^2-R^2\pi_r)}{R^2\pi_r\alpha(1-\delta)} < p < \frac{(R+\alpha(-c_p+\delta))(-c_p+(R+\alpha\delta)\pi_r)}{(R+\alpha\delta)^2\pi_r} : (0 < v_{nr} < v_r < v_p < 1)$
- $\frac{(R+\alpha(-c_p+\delta))(-c_p+(R+\alpha\delta)\pi_r)}{(R+\alpha\delta)^2\pi_r} \leq p < \frac{R(1-\delta+(R-\alpha(1-\delta))\pi_r)}{\alpha(1-\delta)^2} : (0 < v_{nr} < v_r < 1)$
- $\frac{R(1-\delta+(R-\alpha(1-\delta))\pi_r)}{\alpha(1-\delta)^2} \leq p < 1 : (0 < v_r < 1)$

(vi) $c_p\alpha(1-\delta) < R < \alpha(1-\delta)$ *and* $\pi_r > \max\left(\frac{c_p\alpha(1-\delta)}{(-R+\alpha(1-\delta))(R+\alpha\delta)}, \frac{c_p\alpha(1-\delta)^2}{R^2}\right)$*:*

- $0 \leq p \leq \frac{(-R+c_p\alpha(1-\delta))(c_p\alpha(1-\delta)^2-R^2\pi_r)}{R^2\pi_r\alpha(1-\delta)} : (0 < v_{nr} < v_p < 1)$
- $\frac{(-R+c_p\alpha(1-\delta))(c_p\alpha(1-\delta)^2-R^2\pi_r)}{R^2\pi_r\alpha(1-\delta)} < p < \frac{R\left(2(1-\delta)\delta+R\pi_r-\sqrt{\pi_r(4c_p\alpha\delta(1-\delta)^2+R^2\pi_r)}\right)}{2\alpha\delta(1-\delta)^2}$:
  $(0 < v_{nr} < v_r < v_p < 1)$
- $\frac{R\left(2(1-\delta)\delta+R\pi_r-\sqrt{\pi_r(4c_p\alpha\delta(1-\delta)^2+R^2\pi_r)}\right)}{2\alpha\delta(1-\delta)^2} \leq p < 1-c_p-\frac{c_p(R+\alpha\delta(1-c_p))}{(R+\alpha\delta)^2\pi_r} : (0 < v_r < v_p < 1)$
- $1-c_p-\frac{c_p(R+\alpha\delta(1-c_p))}{(R+\alpha\delta)^2\pi_r} \leq p < 1 : (0 < v_r < 1)$

(vii) $\left(\left(\pi_r < \frac{(1-\delta)\delta}{-R+c_p\alpha(1-\delta)} \text{ and } R \leq c_p\alpha(1-\delta)\right) \text{ or } \left(R > c_p\alpha(1-\delta)\right)\right)$ *and* $\pi_r > \frac{c_p\alpha(1-\delta)}{(-R+\alpha(1-\delta))(R+\alpha\delta)}$ *and* $\left(R \leq c_p\alpha(1-\delta) \text{ or } \pi_r \leq \frac{c_p\alpha(1-\delta)^2}{R^2}\right)$ *and* $R < \alpha(1-\delta)$*:*

- $0 \leq p < \frac{R\left(2(1-\delta)\delta+R\pi_r-\sqrt{\pi_r(4c_p\alpha\delta(1-\delta)^2+R^2\pi_r)}\right)}{2\alpha\delta(1-\delta)^2} : (0 < v_{nr} < v_r < v_p < 1)$
- $\frac{R\left(2(1-\delta)\delta+R\pi_r-\sqrt{\pi_r(4c_p\alpha\delta(1-\delta)^2+R^2\pi_r)}\right)}{2\alpha\delta(1-\delta)^2} \leq p < 1-c_p-\frac{c_p(R+\alpha\delta(1-c_p))}{(R+\alpha\delta)^2\pi_r} : (0 < v_r < v_p < 1)$
- $1-c_p-\frac{c_p(R+\alpha\delta(1-c_p))}{(R+\alpha\delta)^2\pi_r} \leq p < 1 : (0 < v_r < 1)$

(viii) $R \leq c_p\alpha(1-\delta)$ *and* $\pi_r \geq \frac{(1-\delta)\delta}{-R+c_p\alpha(1-\delta)}$ *and* $\pi_r > \frac{c_p(R+\alpha\delta(1-c_p))}{(1-c_p)(R+\alpha\delta)^2}$*:*

- $0 \leq p < 1-c_p-\frac{c_p(R+\alpha\delta(1-c_p))}{(R+\alpha\delta)^2\pi_r} : (0 < v_r < v_p < 1)$
- $1-c_p-\frac{c_p(R+\alpha\delta(1-c_p))}{(R+\alpha\delta)^2\pi_r} \leq p < 1 : (0 < v_r < 1)$

(ix) $R < \alpha(1-\delta)$ *and* $\frac{1-\delta}{-R+\alpha(1-\delta)} \leq \pi_r \leq \frac{c_p(R+(1-c_p)\alpha\delta)}{(1-c_p)(R+\alpha\delta)^2}$*:*

- $0 \leq p < 1 : (0 < v_r < 1)$

Under each of these parameter sets, setting $p = 1$ *leads to the trivial outcome of no one purchasing.*

Remarks
As can be seen above, in each of the nine sets of mutually exclusive conditions on parameters, there is anywhere from one to four contiguous intervals of price that give rise to a single type of equilibrium characterization. For example, in part (*ix*), only $0 < v_r < 1$ can arise which is to say that all consumers who are in the market remain unpatched and pay ransom. The parameter set that defines a region, together with a specific price interval, compose the validity conditions under which the corresponding structure (associated with that price interval) is in play. In determining the equilibrium price for a particular region, the vendor will compare profits across the price intervals (hence across the structures in play) to determine the price that maximizes profits. In proving our main propositions, we will establish the conditions under which the optimal price falls into any interval (hence giving rise to its associated structure in equilibrium) and refer to these as profit-maximizing conditions.

To prove Proposition B.1, we take the following strategy. First, in Lemma B.1, we establish the conditions under which each of the six consumer market structures (with positive consumption) can arise. Each set of conditions presented in this lemma come from the consumer's utility maximization problem. Specifically, we show that a threshold market structure arises in equilibrium and that only one of six market outcomes can emerge for any given price $p < 1$. Second, in Lemma B.2, we provide a simplification of these sets of conditions that makes it easier to see how we arrive at the nine parameter sub-regions listed in Proposition B.1. Finally, we derive those sub-regions and prove that they form a partition of the parameter space so that any set of conditions gives rise to one and only one region outcome in the equilibrium of the consumption subgame.

Lemma B.1. *The complete threshold characterization of the consumption subgame is as follows:*

(I) $(0 < v_{nr} < 1)$, *where* $v_{nr} = \frac{\pi_r\alpha - 1 + \sqrt{1+\pi_r\alpha(-2+4p+\pi_r\alpha)}}{2\pi_r\alpha}$ *if the following conditions hold:*

- *(a)* $p < 1$,
- *(b)* $R \geq \alpha(1-\delta)$,
- *(c)* $1 + \pi_r\alpha \leq 2c_p + \sqrt{1+\pi_r\alpha(-2+4p+\pi_r\alpha)}$;

(II) $(0 < v_{nr} < v_p < 1)$, *where* v_{nr} *is the largest positive root of the cubic* $f_1(x) \triangleq \pi_r\alpha x^3 + (1-(c_p+p)\pi_r\alpha)x^2 - 2px + p^2$ *and* $v_p = v_{nr} + \frac{v_{nr}-p}{\pi_r\alpha v_{nr}}$, *if the following conditions hold:*

- *(a)* $c_p\alpha(R - c_p\alpha(1-\delta))(1-\delta)^2 \leq R^2(R - \alpha(c_p+p)(1-\delta))\pi_r$,
- *(b)* $R - c_p\alpha(1-\delta) > 0$,
- *(c)* $(-1+c_p+p)\pi_r\alpha < -c_p + c_p^2$;

(III) $(0 < v_{nr} < v_r < 1)$, *where* $v_{nr} = \frac{\pi_r\alpha - 1 + \sqrt{1+\pi_r\alpha(-2+4p+\pi_r\alpha)}}{2\pi_r\alpha}$ *and* $v_r = \frac{R}{\alpha(1-\delta)}$, *if the following conditions hold:*

(a) $-2R\pi_r + (1-\delta)(-1 + \pi_r\alpha + \sqrt{1 + \pi_r\alpha(-2 + 4p + \pi_r\alpha)}) < 0$,

(b) $R < \alpha(1-\delta)$,

(c) $2c_p\alpha + (R + \alpha\delta)(\sqrt{1 + \pi_r\alpha(-2 + 4p + \pi_r\alpha)} - (1 + \pi_r\alpha)) \geq 0$;

(IV) $(0 < v_{nr} < v_r < v_p < 1)$, *where* v_{nr} *is the largest positive root of the cubic* $f_2(x) \triangleq \delta\pi_r\alpha x^3 + (\delta + R\pi_r - (c_p + p\delta)\pi_r\alpha)x^2 - (2\delta + R\pi_r)px + p^2\delta$ *and* $v_r = \frac{R}{\alpha(1-\delta)}$ *and* $v_p = v_{nr} + \frac{v_{nr} - p}{\pi_r\alpha v_{nr}}$, *if the following conditions hold:*

(a) $R > p\alpha(1-\delta)$,

(b) $R^2(R - (c_p + p)\alpha(1-\delta))\pi_r > -(R - p\alpha(1-\delta))^2\delta(1-\delta)$,

(c) $\alpha(c_p - \delta) + \alpha^2\pi_r(c_p(\pi_r\alpha + 2p - 2) + \delta(\alpha(p-1)\pi_r - 3p + 2)) + \sqrt{\pi_r\alpha(\pi_r\alpha + 4p - 2) + 1}(\alpha(\pi_r\alpha(c_p + \delta(p-1)) - c_p + \delta) + \alpha(p-1)\pi_r R + R) + R(\pi_r\alpha(\alpha(p-1)\pi_r - 3p + 2) - 1) < 0$,

(d) *Either*

- $\alpha c_p(\delta-1)^3(2\pi_r\alpha p + 1) + \sqrt{(\delta-1)^2 + 2(\delta-1)\pi_r(2\alpha(\delta-1)p + R) + \pi_r^2R^2} \times (\alpha c_p(\delta-1)^2 + (\delta-1)R(\pi_r\alpha(c_p + p) + 1) + \pi_r R^2) + (\delta-1)\pi_r R^2(\pi_r\alpha(c_p + p) + 2) + (\delta-1)^2R(2\pi_r\alpha c_p + 3\pi_r\alpha p + 1) + \pi_r^2R^3 < 0$, *and*
- $\delta + \sqrt{4\alpha(\delta-1)^2p\pi_r + (\delta + \pi_r R - 1)^2} < \pi_r R + 1$,

or

- $\alpha c_p(\delta-1)^3(2\pi_r\alpha p + 1) + \sqrt{(\delta-1)^2 + 2(\delta-1)\pi_r(2\alpha(\delta-1)p + R) + \pi_r^2R^2} \times (-\alpha c_p(\delta-1)^2 - (\delta-1)R(\pi_r\alpha(c_p + p) + 1) - \pi_r R^2) + (\delta-1)\pi_r R^2(\pi_r\alpha(c_p + p) + 2) + (\delta-1)^2R(2\pi_r\alpha c_p + 3\pi_r\alpha p + 1) + \pi_r^2R^3 > 0$, *and*
- $\pi_r(R - 2\alpha(1-\delta)p) > -\delta + \sqrt{4\alpha(\delta-1)^2p\pi_r + (\delta + \pi_r R - 1)^2} + 1$;

(V) $(0 < v_r < 1)$, *where* $v_r = \frac{-1 - R\pi_r + \delta\pi_r\alpha + \sqrt{4\delta\pi_r\alpha(p + R\pi_r) + (1 + R\pi_r - \delta\pi_r\alpha)^2}}{2\delta\pi_r\alpha}$, *if the following conditions hold:*

(a) $p < 1$,

(b) $c_p\left(\alpha\delta\pi_r + \sqrt{2\pi_r(\alpha\delta(2p-1) + R) + \pi_r^2(\alpha\delta + R)^2 + 1} + \pi_r R + 1\right) \geq 2(1-p)\pi_r(\alpha\delta + R)$,

(c) *Either*

- $2\delta + \sqrt{4\pi_r\alpha\delta(p + \pi_r R) + (-\pi_r\alpha\delta + \pi_r R + 1)^2} \leq \pi_r\alpha\delta + \pi_r R + 1$,

or

- $2\delta + \sqrt{4\pi_r\alpha\delta(p + \pi_r R) + (-\pi_r\alpha\delta + \pi_r R + 1)^2} > \pi_r\alpha\delta + \pi_r R + 1$, *and*
- $\frac{\pi_r\alpha\delta + \sqrt{4\pi_r\alpha\delta(p + \pi_r R) + (-\pi_r\alpha\delta + \pi_r R + 1)^2} - R\pi_r - 1}{2\pi_r\alpha\delta} \leq -\frac{2\delta p}{(\pi_r\alpha - 2)\delta - \sqrt{4\pi_r\alpha\delta(p + \pi_r R) + (-\pi_r\alpha\delta + \pi_r R + 1)^2} + \pi_r R + 1}$;

(VI) $(0 < v_r < v_p < 1)$, *where* v_r *is the largest positive root of the cubic* $f_3(x) = \alpha^2\delta^2\pi_r x^3 + (\alpha\delta(1+2R\pi_r-(c_p+p)\delta\pi_r\alpha))x^2+(R^2\pi_r-2\alpha\delta(p+(c_p+p)R\pi_r))x+p^2\alpha\delta-(c_p+p)R^2\pi_r$ *and* $v_p = v_r + \frac{v_r-p}{(R+v_r\alpha\delta)\pi_r}$, *if the following conditions hold:*

(a) $\left(\alpha\delta\pi_r + \sqrt{4\alpha\delta\pi_r(p+\pi_r R)+(-\alpha\delta\pi_r+\pi_r R+1)^2}+\pi_r R-1\right)^2 \times \left(\pi_r(-(\alpha\delta(2c_p+2p-1)+R))+\sqrt{4\alpha\delta\pi_r(p+\pi_r R)+(-\alpha\delta\pi_r+\pi_r R+1)^2}-1\right)+ 2\left(\alpha\delta\pi_r-2\alpha\delta p\pi_r+\sqrt{4\alpha\delta\pi_r(p+\pi_r R)+(-\alpha\delta\pi_r+\pi_r R+1)^2}-R\pi_r-1\right)^2>0,$

(b) $\frac{\alpha\delta\pi_r+\sqrt{4\alpha\delta\pi_r(p+\pi_r R)+(-\alpha\delta\pi_r+\pi_r R+1)^2}-R\pi_r-1}{2\alpha\delta\pi_r} > p,$

(c) Either

- $\pi_r\alpha c_p + \delta^2 \geq \alpha\delta\pi_r(c_p+p)+\delta+\pi_r R,$

or

- $\pi_r\alpha c_p + \delta^2 < \alpha\delta\pi_r(c_p+p)+\delta+\pi_r R$, *and*
- *either*
  - $\left(\frac{R}{\alpha(1-\delta)} \leq p\right),$

  or
  - $\frac{R}{\alpha(1-\delta)} > p$, *and*
  - $\pi_r R^2(\alpha(\delta-1)(c_p+p)+R) \leq (\delta-1)\delta(\alpha(\delta-1)p+R)^2$, *and*
  - $\frac{R}{\alpha-\alpha\delta} < c_p+p.$

(VII) $(0 < 1)$ *(in which no one purchases), if the following condition holds:*

- $p = 1$

Proof of Lemma B.1: First, we establish the general threshold-type equilibrium structure. Given the size of unpatched population u, the net payoff of the consumer with type v for strategy profile σ is written as

$$U_{RW}(v,\sigma) \triangleq \begin{cases} v-p-c_p & if \quad \sigma(v) = (B,P)\,; \\ v-p-\pi_r u(\sigma)(R+\delta\alpha v) & if \quad \sigma(v) = (B,NP,R)\,; \\ v-p-\pi_r\alpha u(\sigma)v & if \quad \sigma(v) = (B,NP,NR)\,; \\ 0 & if \quad \sigma(v) = (NB,NP)\,, \end{cases} \tag{B.1}$$

where

$$u_{RW}(\sigma) \triangleq \int_{\mathcal{V}} \mathbb{1}_{\{\sigma(v)\in\{(B,NP,R),(B,NP,NR)\}\}}\,dv\,. \tag{B.2}$$

Note $\sigma(v)=(B,P)$ if and only if

$$v-p-c_p \geq v-p-\pi_r u(\sigma)(R+\delta\alpha v) \Leftrightarrow v \geq \frac{c_p - R\pi_r u(\sigma)}{\delta\pi_r\alpha u(\sigma)}, \text{ and}$$

$$v-p-c_p \geq v-p-\pi_r\alpha u(\sigma)v \Leftrightarrow v \geq \frac{c_p}{\pi_r\alpha u(\sigma)}, \text{ and}$$

$$v-p-c_p \geq 0 \Leftrightarrow v \geq c_p + p,$$

which can be summarized as

$$v \geq \max\left(\frac{c_p - R\pi_r u(\sigma)}{\delta\pi_r\alpha u(\sigma)}, \frac{c_p}{\pi_r\alpha u(\sigma)}, c_p + p\right). \tag{B.3}$$

By (B.3), if a consumer with valuation v_0 buys and patches the software, then every consumer with valuation $v > v_0$ will also do so. Hence, there exists a threshold $v_p \in (0,1]$ such that for all $v \in \mathcal{V}$, $\sigma^*(v)=(B,P)$ if and only if $v \geq v_p$. Similarly, $\sigma(v) \in \{(B,P),(B,NR),(B,R)\}$, i.e., the consumer of valuation v purchases one of these alternatives, if and only if

$$v-p-c_p \geq 0 \Leftrightarrow v \geq c_p + p, \text{ or}$$

$$v-p-\pi_r\alpha u(\sigma)v \geq 0 \Leftrightarrow v \geq \frac{p}{1-\pi_r\alpha u(\sigma)}, \text{ or}$$

$$v-p-\pi_r u(\sigma)(R+\delta\alpha v) \geq 0 \Leftrightarrow v \geq \frac{p + R\pi_r u(\sigma)}{1-\delta\pi_r\alpha u(\sigma)},$$

which can be summarized as

$$v \geq \min\left(c_p + p, \frac{p}{1-\pi_r\alpha u(\sigma)}, \frac{p + R\pi_r u(\sigma)}{1-\delta\pi_r\alpha u(\sigma)}\right). \tag{B.4}$$

Let $0 < v_1 \leq 1$ and $\sigma^*(v_1) \in \{(B,P),(B,NR),(B,R)\}$, then by (B.4), for all $v > v_1$, $\sigma^*(v) \in \{(B,P),(B,NR),(B,R)\}$, and hence there exists a $\underline{v} \in (0,1]$ such that a consumer with valuation $v \in \mathcal{V}$ will purchase the software if and only if $v \geq \underline{v}$.

By (B.3) and (B.4), $\underline{v} \leq v_p$ holds. Moreover, if $\underline{v} < v_p$, consumers with types in $[\underline{v}, v_p]$ choose either (B,NR) or (B,R). A purchasing consumer with valuation v will prefer (B,R) over (B,NR) if and only if

$$v-p-\pi_r u(\sigma)(R+\delta\alpha v) \geq v-p-\pi_r\alpha u(\sigma)v \Leftrightarrow v \geq \frac{R}{\alpha(1-\delta)}. \tag{B.5}$$

Next, we characterize in more detail each outcome that can arise in equilibrium of the consumption subgame, as well as the corresponding parameter regions. For Case (I), in which all consumers who purchase choose to be unpatched and not pay ransom, i.e., $0 < v_{nr} < 1$, based on the threshold-type equilibrium structure, we have $u(\sigma) = 1 - v_{nr}$. We prove the following claim related to the corresponding parameter region in which Case (I) arises.

Claim 1. *The subgame outcome that corresponds to case (I) arises if and only if the following conditions are satisfied:*

$$p < 1 \text{ and } R \geq \alpha(1-\delta) \text{ and } 1+\pi_r\alpha \leq 2c_p + \sqrt{1+\pi_r\alpha(-2+4p+\pi_r\alpha)}. \quad \text{(B.6)}$$

The consumer indifferent between not purchasing at all and purchasing and remaining unpatched, v_{nr}, satisfies $v_{nr} - p - \pi_r\alpha u(\sigma)v_{nr} = 0$. To solve for the threshold v_{nr}, using $u(\sigma) = 1 - v_{nr}$, we solve

$$v_{nr} = \frac{p}{1-\pi_r\alpha u(\sigma)} = \frac{p}{1-\pi_r\alpha(1-v_{nr})}. \quad \text{(B.7)}$$

For this to be an equilibrium, we have that $v_{nr} \geq 0$. This rules out the smaller root of the quadratic as a solution. Given the underlying model assumptions, the other root is strictly positive, so the root characterizing v_{nr} is

$$v_{nr} = \frac{\pi_r\alpha - 1 + \sqrt{1+\pi_r\alpha(-2+4p+\pi_r\alpha)}}{2\pi_r\alpha} \quad \text{(B.8)}$$

For this to be an equilibrium, the necessary and sufficient conditions are that $0 < v_{nr} < 1$, type $v = 1$ weakly prefers (B, NR) to both (B, R) and (B, P).

For $v_{nr} < 1$, it is equivalent to have $p < 1$.

For $v = 1$ to prefer (B, NR) over (B, P), we need $1 \leq \frac{c_p}{\pi_r\alpha(1-v_{nr})}$. Simplifying, this becomes $1 + \pi_r\alpha \leq 2c_p + \sqrt{1+\pi_r\alpha(-2+4p\pi_r\alpha)}$.

For $v = 1$ to prefer (B, NR) over (B, R), we need $1 \leq \frac{R}{\alpha(1-\delta)}$. Simplifying, this becomes $R \geq \alpha(1-\delta)$. The conditions above are given in (B.6). □

Next, for case (II), in which the lower tier of purchasing consumers is unpatched and does not pay ransom while the upper tier patches, i.e., $0 < v_{nr} < v_p < 1$, we have $u = v_p - v_{nr}$. Following the same steps as before, we prove the following claim related to the corresponding parameter region in which case (II) arises.

Claim 2. *The subgame outcome that corresponds to case (II) arises if and only if the following conditions are satisfied:*

$$c_p\alpha(R - c_p\alpha(1-\delta))(1-\delta)^2 \leq R^2(R - \alpha(c_p+p)(1-\delta))\pi_r \text{ and}$$
$$R - c_p\alpha(1-\delta) > 0 \text{ and } (-1+c_p+p)\pi_r\alpha < -c_p + c_p^2. \quad \text{(B.9)}$$

To solve for the thresholds v_{nr} and v_p, using $u = v_p - v_{nr}$, note that they solve

$$v_{nr} = \frac{p}{1-\pi_r\alpha(v_p - v_{nr})}, \text{ and} \quad \text{(B.10)}$$

$$v_p = \frac{c_p}{\pi_r \alpha (v_p - v_{nr})}. \tag{B.11}$$

Solving for v_p in terms of v_{nr} in (B.10), we have

$$v_p = v_{nr} + \frac{v_{nr} - p}{\pi_r \alpha v_{nr}}. \tag{B.12}$$

Substituting this into (B.11), we have that v_{nr} must be a zero of the cubic equation:

$$f_1(x) \triangleq \pi_r \alpha x^3 + (1 - \pi_r \alpha (c_p + p))x^2 - 2px + p^2. \tag{B.13}$$

To find which root of the cubic v_{nr} must be, note that the cubic's highest order term is $\pi_r \alpha x^3$, so $\lim_{x\to-\infty} f_1(x) = -\infty$ and $\lim_{x\to\infty} f_1(x) = \infty$. We find $f_1(0) = p^2 > 0$, and $f_1(p) = -c_p \pi_r \alpha p^2 < 0$. Since $v_{nr} - p > 0$ in equilibrium, we have that v_{nr} is uniquely defined as the largest root of the cubic, lying past p. Then (B.12) characterizes v_p.

For this to be an equilibrium, the necessary and sufficient conditions are $0 < v_{nr} < v_p < 1$ and type $v = v_p$ weakly prefers (B, P) over (B, R). Type $v = v_p$ preferring (B, P) over (B, R) ensures $v > v_p$ also prefer (B, P) over (B, R), by (B.3). Moreover, type $v = v_p$ is indifferent between (B, P) and (B, NR), so this implies that $v = v_p$ weakly prefers (B, NR) over (B, R). This implies that all $v < v_{nr}$ strictly prefer (B, NR) over (B, R) by (B.5).

For $v_p < 1$, first note that from (B.10), we have $\pi_r \alpha (v_p - v_{nr}) = 1 - \frac{p}{v_{nr}}$ while from (B.11) we have $\pi_r \alpha (v_p - v_{nr}) = \frac{c_p}{v_p}$. So then solving for v_p, we have

$$v_p = \frac{c_p v_{nr}}{v_{nr} - p}. \tag{B.14}$$

Then using (B.14), a necessary and sufficient condition for $v_p < 1$ to hold is $v_{nr} > \frac{p}{1-c_p}$. This is equivalent to $f_3(\frac{p}{1-c_p}) < 0$, since $\frac{p}{1-c_p} > p$. This simplifies to $(-1 + c_p + p)\pi_r \alpha < -c_p + c_p^2$.

For $v_p > v_{nr}$, no conditions are necessary since v_p was defined in (B.12), and $v_{nr} > p$ by definition of v_{nr} as the largest root of (B.13).

Similarly, $v_{nr} > 0$, by definition of v_{nr}.

To ensure that no consumer has incentive to pay ransom, it suffices to make sure that $v = v_p$ prefers to not pay ransom over paying ransom. By (B.5), we will need $v_p \leq \frac{R}{\alpha(1-\delta)}$. Using (B.14), this is equivalent to $v_{nr}(R - c_p \alpha(1 - \delta)) \geq Rp$. If $R - c_p \alpha(1 - \delta) \leq 0$, then no v_{nr} would satisfy this condition in equilibrium. Hence, $R - c_p \alpha(1 - \delta) > 0$ is a necessary condition and we need $v_{nr} \geq \frac{Rp}{R - c_p \alpha(1-\delta)}$. This simplifies to $f[\frac{Rp}{R - c_p \alpha(1-\delta)}] \geq 0$, which is equivalent to $c_p \alpha(R - c_p \alpha(1 - \delta))(1 - \delta)^2 \leq R^2(R - \alpha(c_p + p)(1 - \delta))\pi_r$. The conditions above are summarized in (B.9). $\square$

Next, for case (III), in which there are no patched users while the lower tier chooses to not pay ransom and the upper tier pays ransom, i.e., $0 < v_{nr} < v_r < 1$, we have $u = 1 - v_{nr}$. Following the same steps as before, we prove the following claim related to the corresponding parameter region in which case (III) arises.

Claim 3. *The subgame outcome that corresponds to case (III) arises if and only if the following conditions are satisfied:*

$$\begin{aligned} &p > 0 \text{ and } \alpha\big(-2R\pi_r + (1-\delta)(-1+\pi_r\alpha+\sqrt{1+\pi_r\alpha(-2+4p+\pi_r\alpha)})\big) < 0 \text{ and} \\ &R < \alpha(1-\delta) \text{ and } 2c_p\alpha + (R+\alpha\delta)(\sqrt{1+\pi_r\alpha(-2+4p+\pi_r\alpha)} - (1+\pi_r\alpha)) \geq 0. \end{aligned} \tag{B.15}$$

To solve for the thresholds v_{nr} and v_r, using $u = 1 - v_{nr}$, note that they solve

$$v_{nr} = \frac{p}{1-\pi_r\alpha(1-v_{nr})}, \text{ and} \tag{B.16}$$

$$v_r = \frac{R}{\alpha(1-\delta)}, \tag{B.17}$$

where the expression in (B.17) comes from (B.5).

Solving for v_{nr} in (B.16), we have

$$v_{nr} = \frac{-1+\pi_r\alpha \pm \sqrt{1+\pi_r\alpha(-2+4p+\pi_r\alpha)}}{2\pi_r\alpha}. \tag{B.18}$$

Note that $\frac{-1+\pi_r\alpha-\sqrt{1+\pi_r\alpha(-2+4p+\pi_r\alpha)}}{2\pi_r\alpha} < p$ while $\frac{-1+\pi_r\alpha+\sqrt{1+\pi_r\alpha(-2+4p+\pi_r\alpha)}}{2\pi_r\alpha} > p$, and since $v_{nr} > p$ in equilibrium, it follows that

$$v_{nr} = \frac{-1+\pi_r\alpha + \sqrt{1+\pi_r\alpha(-2+4p+\pi_r\alpha)}}{2\pi_r\alpha}. \tag{B.19}$$

For this to be an equilibrium, the necessary and sufficient conditions are $0 < v_{nr} < v_r < 1$ and that type $v = 1$ weakly prefers (B,R) over (B,P). Type $v = 1$ preferring (B,R) over (B,P) ensures $v < 1$ also prefer (B,R) over (B,P), by (B.3). Moreover, type $v = v_r$ is indifferent between (B,R) and (B,NR), so this implies that $v = v_r$ strictly prefers (B,NR) over (B,P). This implies that all $v < v_{nr}$ strictly prefer (B,NR) over (B,P) as well, again by (B.3).

Note $v_{nr} > 0$ is satisfied if $p > 0$ since $v_{nr} > p$ under the preliminary model assumptions.

For $v_{nr} < v_r$, from (B.19) and (B.17), this simplifies to $\alpha\big(-2R\pi_r + (1-\delta)(-1+\pi_r\alpha + \sqrt{1+\pi_r\alpha(-2+4p+\pi_r\alpha)})\big) < 0$.

For $v_r < 1$, from (B.17), this simplifies to $R < \alpha(1-\delta)$.

For $v = 1$ to weakly prefer (B,R) over (B,P), we need $1 \leq \frac{c_p - R\pi_r(1-v_{nr})}{\pi_r\alpha\delta(1-v_{nr})}$. Substituting in (B.19) and simplifying, this becomes $2c_p\alpha+(R+\alpha\delta)(\sqrt{1+\pi_r\alpha(-2+4p+\pi_r\alpha)}-(1+\pi_r\alpha)) \geq 0$. The conditions above are summarized in (B.15). □

Next, for case (IV), in which the top tier of consumers patches, the middle tier pays ransom, and the bottom tier remains unpatched and does not pay ransom, i.e., $0 < v_{nr} <$

$v_r < v_p < 1$, we have $u = v_p - v_{nr}$. Following the same steps as before, we prove the following claim related to the corresponding parameter region in which case (IV) arises.

Claim 4. *The subgame outcome that corresponds to case (IV) arises if and only if the following conditions are satisfied:*

$$R > p\alpha(1-\delta) \text{ and } R^2(R-(c_p+p)\alpha(1-\delta))\pi_r > -(R-p\alpha(1-\delta))^2\delta(1-\delta) \text{ and}$$
$$\alpha(c_p-\delta)+\alpha^2\pi_r(c_p(\pi_r\alpha+2p-2)+\delta(\alpha(p-1)\pi_r-3p+2))+$$
$$\sqrt{\pi_r\alpha(\pi_r\alpha+4p-2)+1}(\alpha(\pi_r\alpha(c_p+\delta(p-1))-c_p+\delta)+\alpha(p-1)\pi_r R+R)+$$
$$R(\pi_r\alpha(\alpha(p-1)\pi_r-3p+2)-1)<0 \text{ and}$$
$$\text{either } \Bigg(\Big(\alpha c_p(\delta-1)^3(2\pi_r\alpha p+1)+\sqrt{(\delta-1)^2+2(\delta-1)\pi_r(2\alpha(\delta-1)p+R)+\pi_r^2R^2}\times$$
$$\big(\alpha c_p(\delta-1)^2+(\delta-1)R(\pi_r\alpha(c_p+p)+1)+\pi_r R^2\big)+(\delta-1)\pi_r R^2(\pi_r\alpha(c_p+p)+2)+$$
$$(\delta-1)^2R(2\pi_r\alpha c_p+3\pi_r\alpha p+1)+\pi_r^2R^3<0\Big) \text{ and}$$
$$\Big(\delta+\sqrt{4\alpha(\delta-1)^2p\pi_r+(\delta+\pi_r R-1)^2}<\pi_r R+1\Big)\Bigg), \text{ or}$$
$$\Bigg(\Big(\alpha c_p(\delta-1)^3(2\pi_r\alpha p+1)+\sqrt{(\delta-1)^2+2(\delta-1)\pi_r(2\alpha(\delta-1)p+R)+\pi_r^2R^2}\times$$
$$\big(-\alpha c_p(\delta-1)^2-(\delta-1)R(\pi_r\alpha(c_p+p)+1)-\pi_r R^2\big)+(\delta-1)\pi_r R^2(\pi_r\alpha(c_p+p)+2)+$$
$$(\delta-1)^2R(2\pi_r\alpha c_p+3\pi_r\alpha p+1)+\pi_r^2R^3>0\Big) \text{ and}$$
$$\Big(\pi_r(R-2\alpha(1-\delta)p)>-\delta+\sqrt{4\alpha(\delta-1)^2p\pi_r+(\delta+\pi_r R-1)^2}+1\Big)\Bigg). \quad \text{(B.20)}$$

To solve for the thresholds v_{nr} and v_p, using $u = v_p - v_{nr}$, note that they solve

$$v_{nr} = \frac{p}{1-\pi_r\alpha(v_p-v_{nr})}, \text{ and} \quad \text{(B.21)}$$

$$v_p = \frac{c_p - R\pi_r(v_p-v_{nr})}{\delta\pi_r\alpha(v_p-v_{nr})}, \quad \text{(B.22)}$$

Note that v_r solves

$$v_r = \frac{R}{\alpha(1-\delta)}, \quad \text{(B.23)}$$

where the expression in (B.23) comes from (B.5).

Solving for v_p in (B.21), we have

$$v_p = v_{nr} + \frac{v_{nr}-p}{v_{nr}\pi_r\alpha}. \quad \text{(B.24)}$$

Substituting this into (B.22), we have that v_{nr} must be a zero of the cubic equation:

$$f_2(x) \triangleq \delta\pi_r\alpha x^3 + (\delta + \pi_r(R - \alpha(c_p + \delta p)))x^2 - p(2\delta + R\pi_r)x + p^2\delta. \tag{B.25}$$

To find which root of the cubic v_{nr} must be, note that the cubic's highest order term is $\delta\pi_r\alpha x^3$, so $\lim_{x\to-\infty} f_2(x) = -\infty$ and $\lim_{x\to\infty} f_2(x) = \infty$. We find $f_2(0) = \delta p^2 > 0$, and $f_2(p) = -c_p\pi_r\alpha p^2 < 0$. Since $v_{nr} - p > 0$ in equilibrium, we have that v_{nr} is uniquely defined as the largest root of the cubic, lying past p. Then using (B.24), we solve for v_p.

For this to be an equilibrium, the necessary and sufficient conditions are $0 < v_{nr} < v_r < v_p < 1$.

The condition $v_{nr} > 0$ is satisfied without further conditions, since v_{nr} is the largest root of the cubic greater than p by definition.

For $v_{nr} < v_r$ to hold, we need $v_r > p$ and $f_2(v_r) > 0$. These conditions are equivalent to $R > p\alpha(1-\delta)$ and $R^2(R - (c_p + p)\alpha(1-\delta))\pi_r > -(R - p\alpha(1-\delta))^2\delta(1-\delta)$.

For $v_r < v_p$ to hold, we need $\frac{R}{\alpha(1-\delta)} < v_{nr} + \frac{v_{nr}-p}{\pi_r\alpha v_{nr}}$ by (B.23) and (B.24). Simplifying, this becomes $(1-\delta)\pi_r\alpha v_{nr}^2 + (1 - \delta - R\pi_r)v_{nr} - (1-\delta)p > 0$. Then v_{nr} needs to be larger than the larger root of this quadratic or smaller than the smaller root. The two roots of the quadratic are given by $\frac{-1+\delta+R\pi_r\pm\sqrt{4p\pi_r\alpha(1-\delta)^2+(1-\delta-R\pi_r)^2}}{2\pi_r\alpha(1-\delta)}$. If v_{nr} is larger than the larger root, then a necessary condition is that this larger root is smaller than v_r. On the other hand, if v_{nr} is smaller than the smaller root, then a necessary condition is that the smaller root is larger than p, since by definition $v_{nr} > p$.

Consider the first sub-case in which v_{nr} is larger than the larger root of the quadratic. The conditions are $v_{nr} > \frac{-1+\delta+R\pi_r+\sqrt{4p\pi_r\alpha(1-\delta)^2+(1-\delta-R\pi_r)^2}}{2\pi_r\alpha(1-\delta)}$ and $\frac{-1+\delta+R\pi_r+\sqrt{4p\pi_r\alpha(1-\delta)^2+(1-\delta-R\pi_r)^2}}{2\pi_r\alpha(1-\delta)} < \frac{R}{\alpha(1-\delta)}$. For $v_{nr} > \frac{-1+\delta+R\pi_r+\sqrt{4p\pi_r\alpha(1-\delta)^2+(1-\delta-R\pi_r)^2}}{2\pi_r\alpha(1-\delta)}$, either $\frac{-1+\delta+R\pi_r+\sqrt{4p\pi_r\alpha(1-\delta)^2+(1-\delta-R\pi_r)^2}}{2\pi_r\alpha(1-\delta)} \le p$, or $\frac{-1+\delta+R\pi_r+\sqrt{4p\pi_r\alpha(1-\delta)^2+(1-\delta-R\pi_r)^2}}{2\pi_r\alpha(1-\delta)} > p$ and $f_2(\frac{-1+\delta+R\pi_r+\sqrt{4p\pi_r\alpha(1-\delta)^2+(1-\delta-R\pi_r)^2}}{2\pi_r\alpha(1-\delta)}) < 0$. The condition $\frac{-1+\delta+R\pi_r+\sqrt{4p\pi_r\alpha(1-\delta)^2+(1-\delta-R\pi_r)^2}}{2\pi_r\alpha(1-\delta)} \le p$ simplifies to $R \le \alpha p(1-\delta)$. However, $R > p\alpha(1-\delta)$ from $v_r > p$. Since $R > \alpha p(1-\delta)$, a necessary condition is $f_2(\frac{-1+\delta+R\pi_r+\sqrt{4p\pi_r\alpha(1-\delta)^2+(1-\delta-R\pi_r)^2}}{2\pi_r\alpha(1-\delta)}) < 0$, which simplifies to $\alpha c_p(\delta-1)^3(2\pi_r\alpha p + 1) + \sqrt{(\delta-1)^2 + 2(\delta-1)\pi_r(2\alpha(\delta-1)p + R) + \pi_r^2R^2} \times (\alpha c_p(\delta-1)^2 + (\delta-1)R(\pi_r\alpha(c_p+p)+1) + \pi_rR^2) + (\delta-1)\pi_rR^2(\pi_r\alpha(c_p+p)+2) + (\delta-1)^2R(2\pi_r\alpha c_p + 3\pi_r\alpha p + 1) + \pi_r^2R^3 < 0$. Lastly for this sub-case, we need that the quadratic root $\frac{-1+\delta+R\pi_r+\sqrt{4p\pi_r\alpha(1-\delta)^2+(1-\delta-R\pi_r)^2}}{2\pi_r\alpha(1-\delta)} < v_r = \frac{R}{\alpha(1-\delta)}$. This condition simplifies to $\delta + \sqrt{4\alpha(\delta-1)^2p\pi_r + (\delta + \pi_rR - 1)^2} < \pi_rR + 1$. Altogether, these form the first set of conditions in (C) of case (IV).

In the second sub-case in which v_{nr} is smaller than the smaller root of the quadratic, the necessary and sufficient conditions are that $\frac{-1+\delta+R\pi_r+\sqrt{4p\pi_r\alpha(1-\delta)^2+(1-\delta-R\pi_r)^2}}{2\pi_r\alpha(1-\delta)} > p$ and $v_{nr} < \frac{-1+\delta+R\pi_r+\sqrt{4p\pi_r\alpha(1-\delta)^2+(1-\delta-R\pi_r)^2}}{2\pi_r\alpha(1-\delta)}$. Note that the second condition is equivalent to

$f_2(\frac{-1+\delta+R\pi_r+\sqrt{4p\pi_r\alpha(1-\delta)^2+(1-\delta-R\pi_r)^2}}{2\pi_r\alpha(1-\delta)}) > 0$ since $f_2(x) > 0$ for any $x > v_{nr}$. The condition that $\frac{-1+\delta+R\pi_r+\sqrt{4p\pi_r\alpha(1-\delta)^2+(1-\delta-R\pi_r)^2}}{2\pi_r\alpha(1-\delta)} > p$ simplifies to $\pi_r(R-2\alpha(1-\delta)p) > -\delta+\sqrt{4\alpha(\delta-1)^2p\pi_r+(\delta+\pi_rR-1)^2}+1$.

The condition that $f_2(\frac{-1+\delta+R\pi_r+\sqrt{4p\pi_r\alpha(1-\delta)^2+(1-\delta-R\pi_r)^2}}{2\pi_r\alpha(1-\delta)}) > 0$ simplifies to $\alpha c_p(\delta-1)^3(2\pi_r\alpha p+1)+\sqrt{(\delta-1)^2+2(\delta-1)\pi_r(2\alpha(\delta-1)p+R)+\pi_r^2R^2}\times(-\alpha c_p(\delta-1)^2-(\delta-1)R(\pi_r\alpha(c_p+p)+1)-\pi_rR^2)+(\delta-1)\pi_rR^2(\pi_r\alpha(c_p+p)+2)+(\delta-1)^2R(2\pi_r\alpha c_p+3\pi_r\alpha p+1)+\pi_r^2R^3 > 0$. Altogether, these form the second set of conditions in (C) of case (IV).

Lastly, we need $v_p < 1$. Using (B.24), this simplifies to $\pi_r\alpha v_{nr}^2+(1-\pi_r\alpha)v_{nr} < p$. Then v_{nr} needs to be between the two roots of that quadratic, $\frac{-1+\pi_r\alpha\pm\sqrt{1-2\pi_r\alpha+4p\pi_r\alpha+(\pi_r\alpha)^2}}{2\pi_r\alpha}$. But note that the smaller of the roots not positive, from $0 \le p \le 1$ and $\pi_r\alpha > 0$. Therefore, $v_{nr} > \frac{-1+\pi_r\alpha-\sqrt{1-2\pi_r\alpha+4p\pi_r\alpha+(\pi_r\alpha)^2}}{2\pi_r\alpha}$ is satisfied without further conditions. For $v_{nr} < \frac{-1+\pi_r\alpha+\sqrt{1-2\pi_r\alpha+4p\pi_r\alpha+(\pi_r\alpha)^2}}{2\pi_r\alpha}$, $f_2(\frac{-1+\pi_r\alpha+\sqrt{1-2\pi_r\alpha+4p\pi_r\alpha+(\pi_r\alpha)^2}}{2\pi_r\alpha}) > 0$ is a necessary and sufficient condition since $\frac{-1+\pi_r\alpha+\sqrt{1-2\pi_r\alpha+4p\pi_r\alpha+(\pi_r\alpha)^2}}{2\pi_r\alpha} > p$. This simplifies to $\alpha(c_p-\delta)+\alpha^2\pi_r(c_p(\pi_r\alpha+2p-2)+\delta(\alpha(p-1)\pi_r-3p+2))+\sqrt{\pi_r\alpha(\pi_r\alpha+4p-2)+1}(\alpha(\pi_r\alpha(c_p+\delta(p-1))-c_p+\delta)+\alpha(p-1)\pi_rR+R)+R(\pi_r\alpha(\alpha(p-1)\pi_r-3p+2)-1) < 0$, which is condition (B) of case (IV). Altogether, the conditions above are given in (B.20). $\square$

Next, for case (V), in which there are no patched users while all consumers who purchase are unpatched and pay ransom, i.e., $0 < v_r < 1$, we have $u = 1-v_r$. Following the same steps as before, we prove the following claim related to the corresponding parameter region in which case (V) arises.

Claim 5. *The subgame outcome that corresponds to case (V) arises if and only if the following conditions are satisfied:*

$$
\begin{aligned}
&p < 1 \text{ and}\\
&c_p\left(\alpha\delta\pi_r+\sqrt{2\pi_r(\alpha\delta(2p-1)+R)+\pi_r^2(\alpha\delta+R)^2+1}+\pi_rR+1\right) \ge 2(1-p)\pi_r(\alpha\delta+R),\\
&\quad\text{and either } 2\delta+\sqrt{4\pi_r\alpha\delta(p+\pi_rR)+(-\pi_r\alpha\delta+\pi_rR+1)^2} \le \pi_r\alpha\delta+\pi_rR+1, \text{ or}\\
&\qquad 2\delta+\sqrt{4\pi_r\alpha\delta(p+\pi_rR)+(-\pi_r\alpha\delta+\pi_rR+1)^2} > \pi_r\alpha\delta+\pi_rR+1 \text{ and}\\
&\qquad \frac{\pi_r\alpha\delta+\sqrt{4\pi_r\alpha\delta(p+\pi_rR)+(-\pi_r\alpha\delta+\pi_rR+1)^2}-R\pi_r-1}{2\pi_r\alpha\delta} \le\\
&\qquad -\frac{2\delta p}{\pi_r\alpha\delta-2\delta-\sqrt{4\pi_r\alpha\delta(p+\pi_rR)+(-\pi_r\alpha\delta+\pi_rR+1)^2}+\pi_rR+1}.
\end{aligned}
\tag{B.26}
$$

To solve for the thresholds v_r, using $u = 1-v_r$, note it solves

$$
v_r = \frac{p+R\pi_r(1-v_r)}{1-\delta\pi_r\alpha(1-v_r)} \tag{B.27}
$$

Then v_r is one of the two roots of the equation above, $\frac{-1-R\pi_r+\delta\pi_r\alpha\pm\sqrt{4\delta\pi_r\alpha(p+R\pi_r)+(1+R\pi_r-\delta\pi_r\alpha)^2}}{2\delta\pi_r\alpha}$. However, the smaller of the two roots is negative, so v_r must be the larger of the two roots in equilibrium. Hence, we have

$$v_r = \frac{-1 - R\pi_r + \delta\pi_r\alpha + \sqrt{4\delta\pi_r\alpha(p + R\pi_r) + (1 + R\pi_r - \delta\pi_r\alpha)^2}}{2\delta\pi_r\alpha}. \tag{B.28}$$

For this to be an equilibrium, the necessary and sufficient conditions are $p < v_r < 1$, and no consumer prefers to patch or not pay ransom over paying ransom.

For $v_r > p$, using (B.28), this simplifies to $p < 1$. For $v_r > p$, using (B.28), this also simplifies to $p < 1$. Similarly, $v_r < 1$ also simplifies to $p < 1$.

For no consumer to strictly prefer patching over paying ransom, it suffices to have type $v = 1$ weakly prefer paying ransom to patching. This is given as $1 \le \frac{c_p - R\pi_r(1-v_r)}{\delta\pi_r\alpha(1-v_r)}$. Using (B.28), this simplifies to $c_p\left(\alpha\delta\pi_r + \sqrt{2\pi_r(\alpha\delta(2p-1)+R) + \pi_r^2(\alpha\delta+R)^2 + 1} + \pi_r R + 1\right) \ge 2(1-p)\pi_r(\alpha\delta + R)$.

For no consumer to strictly prefer not paying ransom over paying ransom, it suffices to have $v = v_r$ weakly prefer not to buy over buying and not paying ransom (since type $v = v_r$ is indifferent between the option of not purchasing and the option of purchasing, remaining unpatched, and paying ransom). Now if $1 - \pi_r\alpha u[\sigma] \le 0$, then $v(1 - \pi_r\alpha u[\sigma]) - p < 0$, so that everyone would prefer (NB, NP) over (B, NP, NR). In this case, no further conditions are needed. On the other hand, if $1 - \pi_r\alpha u[\sigma] > 0$, then we will need the condition $v_r \le \frac{p}{1-\pi_r\alpha(1-v_r)}$ for $v = v_r$ to weakly prefer not buying over buying but not paying ransom.

In the first sub-case, the condition $v(1 - \pi_r\alpha u[\sigma]) - p < 0$ simplifies to $2\delta + \sqrt{4\pi_r\alpha\delta(p + \pi_r R) + (-\pi_r\alpha\delta + \pi_r R + 1)^2} \le \pi_r\alpha\delta + \pi_r R + 1$, using (B.28).

In the second sub-case, the conditions $1 - \pi_r\alpha u[\sigma] > 0$ and $v_r \le \frac{p}{1-\pi_r\alpha(1-v_r)}$ simplify to $2\delta + \sqrt{4\pi_r\alpha\delta(p + \pi_r R) + (-\pi_r\alpha\delta + \pi_r R + 1)^2} > \pi_r\alpha\delta + \pi_r R + 1$ and $\frac{\pi_r\alpha\delta+\sqrt{4\pi_r\alpha\delta(p+\pi_r R)+(-\pi_r\alpha\delta+\pi_r R+1)^2}-R\pi_r-1}{2\pi_r\alpha\delta} \le -\frac{2\delta p}{\pi_r\alpha\delta-2\delta-\sqrt{4\pi_r\alpha\delta(p+\pi_r R)+(-\pi_r\alpha\delta+\pi_r R+1)^2}+\pi_r R+1}$. The conditions above are summarized in (B.26). $\square$

Lastly, for case (VI), in which the top tier patches while lower tier of the market remains unpatched but pays the ransom, i.e., $0 < v_r < v_p < 1$, we have $u = v_p - v_r$. Following the same steps as before, we prove the following claim related to the corresponding parameter region in which case (VI) arises.

Claim 6. *The subgame outcome that corresponds to case (VI) arises if and only if the* *following conditions are satisfied:*

$$\left(\alpha\delta\pi_r + \sqrt{4\alpha\delta\pi_r(p+\pi_r R) + (-\alpha\delta\pi_r + \pi_r R + 1)^2} + \pi_r R - 1\right)^2 \times$$
$$\left(\pi_r(-(\alpha\delta(2c_p + 2p - 1) + R)) + \sqrt{4\alpha\delta\pi_r(p+\pi_r R) + (-\alpha\delta\pi_r + \pi_r R + 1)^2} - 1\right) +$$
$$2\left(\alpha\delta\pi_r - 2\alpha\delta p\pi_r + \sqrt{4\alpha\delta\pi_r(p+\pi_r R) + (-\alpha\delta\pi_r + \pi_r R + 1)^2} - R\pi_r - 1\right)^2 > 0 \textit{ and}$$
$$\frac{\alpha\delta\pi_r + \sqrt{4\alpha\delta\pi_r(p+\pi_r R) + (-\alpha\delta\pi_r + \pi_r R + 1)^2} - R\pi_r - 1}{2\alpha\delta\pi_r} > p \textit{ and either}$$
$$\left(\pi_r\alpha c_p + \delta^2 \geq \alpha\delta\pi_r(c_p+p) + \delta + \pi_r R\right), \textit{ or}$$
$$\left(\left(\pi_r\alpha c_p + \delta^2 < \alpha\delta\pi_r(c_p+p) + \delta + \pi_r R\right) \textit{ and } \left(\left(\frac{R}{\alpha(1-\delta)} \leq p\right) \textit{ or}\right.\right.$$
$$\left.\left.\left(\frac{R}{\alpha(1-\delta)} > p \textit{ and } \pi_r R^2(\alpha(\delta-1)(c_p+p)+R) \leq (\delta-1)\delta(\alpha(\delta-1)p+R)^2 \textit{ and } \frac{R}{\alpha-\alpha\delta} < c_p+p\right)\right)\right). \tag{B.29}$$

To solve for the thresholds v_r and v_p, using $u(\sigma) = v_p - v_{nr}$, we solve

$$v_r = \frac{p + R\pi_r u(\sigma)}{1 - \delta\pi_r\alpha u(\sigma)} = \frac{p + R\pi_r(v_p - v_r)}{1 - \delta\pi_r\alpha(v_p - v_r)}, \text{ and} \tag{B.30}$$

$$v_p = \frac{c_p - R\pi_r u(\sigma)}{\delta\pi_r\alpha u(\sigma)} = \frac{c_p - R\pi_r(v_p - v_r)}{\delta\pi_r\alpha(v_p - v_r)}. \tag{B.31}$$

Solving for v_p in terms of v_r in (B.30), we have

$$v_p = v_r + \frac{v_r - p}{(R + v_r\alpha\delta)\pi_r}. \tag{B.32}$$

Substituting this into (B.31), we have that v_r must be a zero of the cubic equation:

$$f_3(x) \triangleq \delta^2\alpha^2\pi_r x^3 - \alpha\delta(-1 - 2R\pi_r + (c_p+p)\delta\pi_r\alpha)x^2 + (R^2\pi_r - 2\alpha\delta(p + (c_p+p)R\pi_r))x + p^2\alpha\delta - (c_p+p)R^2\pi_r. \tag{B.33}$$

To find which root of the cubic v_r must be, note that the cubic's highest order term is $\delta^2\alpha^2\pi_r x^3 > 0$, so $\lim_{x\to-\infty} f_3(x) = -\infty$ and $\lim_{x\to\infty} f_3(x) = \infty$. Note $f_3(-\frac{R}{\alpha\delta}) = \alpha\delta\left(p + \frac{R}{\alpha\delta}\right)^2 > 0$, $f_3(p) = -c_p(R + p\alpha\delta)^2\pi_r < 0$, and $f_3(c_p + p) = c_p^2\alpha\delta > 0$. Then the root between p and $c_p + p$ is the largest positive root of the cubic. Since $v_r - p > 0$ in equilibrium, we have that v_r is uniquely defined as the largest root of the cubic, lying past p. Then using (B.32) to define v_p, we have v_p.

For this to be an equilibrium, the necessary and sufficient conditions are $0 < v_r < v_p < 1$ and no consumer strictly prefers to not pay the ransom over either (B, P) or (B, R).

First, note that $v_r > p$ implies both $v_r > 0$ and $v_p > v_r$, from (B.32).

For $v_p < 1$, using (B.32), this is equivalent to $\delta\pi_r\alpha v_r^2 + (1 + R\pi_r - \delta\pi_r\alpha)v_r - R\pi_r < p$. For this quadratic in v_r to be less than a constant, v_r needs to be between the two roots of the quadratic, $\frac{-1-R\pi_r+\delta\pi_r\alpha\pm\sqrt{4\delta\pi_r\alpha(p+R\pi_r)+(1+R\pi_r-\delta\pi_r\alpha)^2}}{2\delta\pi_r\alpha}$. Both roots exist since the radicand is strictly positive.

Note that since $p \leq 1$, then $\frac{-1-R\pi_r+\delta\pi_r\alpha-\sqrt{4\delta\pi_r\alpha(p+R\pi_r)+(1+R\pi_r-\delta\pi_r\alpha)^2}}{2\delta\pi_r\alpha} \leq p$. Since we already have conditions for $v_r > p$, this implies that v_r is larger than the smaller root of the quadratic above.

Then the conditions we need for $v_p < 1$ are $v_r < \frac{-1-R\pi_r+\delta\pi_r\alpha+\sqrt{4\delta\pi_r\alpha(p+R\pi_r)+(1+R\pi_r-\delta\pi_r\alpha)^2}}{2\delta\pi_r\alpha}$ and $\frac{-1-R\pi_r+\delta\pi_r\alpha+\sqrt{4\delta\pi_r\alpha(p+R\pi_r)+(1+R\pi_r-\delta\pi_r\alpha)^2}}{2\delta\pi_r\alpha} > p$. The latter condition is given in (B) of case (VI). With $\frac{-1-R\pi_r+\delta\pi_r\alpha+\sqrt{4\delta\pi_r\alpha(p+R\pi_r)+(1+R\pi_r-\delta\pi_r\alpha)^2}}{2\delta\pi_r\alpha} > p$, it follows that a necessary and sufficient for $v_r < \frac{-1-R\pi_r+\delta\pi_r\alpha+\sqrt{4\delta\pi_r\alpha(p+R\pi_r)+(1+R\pi_r-\delta\pi_r\alpha)^2}}{2\delta\pi_r\alpha}$ is $f_3(\frac{-1-R\pi_r+\delta\pi_r\alpha+\sqrt{4\delta\pi_r\alpha(p+R\pi_r)+(1+R\pi_r-\delta\pi_r\alpha)^2}}{2\delta\pi_r\alpha}) > 0$. This is given in (A) of case (VI).

Lastly, we need to ensure that no consumer has an incentive to choose to not patch and not pay ransom. If $1 - \pi_r\alpha u(\sigma) \leq 0$, then $v(1 - \pi_r\alpha u(\sigma)) \leq 0$ for all v so that everyone would weakly prefer (NB, NP) over (B, NR). In this case, we do not need further conditions. Specifically, using (B.32), we have that $u(\sigma) = v_p - v_r = \frac{v_r-p}{(R+v_r\delta\alpha)\pi_r}$. So the condition that $1 - \pi_r\alpha u(\sigma) \leq 0$ is equivalent to $v_r \geq \frac{R+p\alpha}{\alpha(1-\delta)}$. Since $\frac{R+p\alpha}{\alpha(1-\delta)} > p$, this is equivalent to $f_3(\frac{R+p\alpha}{\alpha(1-\delta)}) \leq 0$, which boils down to $\pi_r\alpha c_p + \delta^2 \geq \alpha\delta\pi_r(c_p + p) + \delta + \pi_r R$.

On the other hand, if $\pi_r\alpha c_p + \delta^2 < \alpha\delta\pi_r(c_p + p) + \delta + \pi_r R$ so that $1 - \pi_r\alpha u(\sigma) > 0$, then a necessary and sufficient condition for no one to strictly prefer (B, NR) over the other options is for type $v = v_r$ to weakly prefer (NB, NP) over (B, NR). This would imply that all $v < v_r$ also have the same preference, from (B.4). Also, since $v = v_r$ is indifferent between (NB, NP) and (B, R), it follows that $v = v_r$ weakly prefers (B, R) over (B, NR). Then since only higher-valuation consumers would prefer paying ransom from (B.5), it follows that all $v > v_r$ would also have the same preference. The condition that $v = v_r$ weakly prefers (NB, NP) over (B, NR) is $v_r \leq \frac{p}{1-\pi_r\alpha(\frac{v_r-p}{(R+v_r\alpha\delta)\pi_r})}$. This simplifies to $v_r \geq \frac{R}{\alpha(1-\delta)}$.

Now if $\frac{R}{\alpha(1-\delta)} \leq p$, then no further conditions are needed since $v_r > p$ by definition of v_r. On the other hand, if $\frac{R}{\alpha(1-\delta)} > p$, then a necessary and sufficient condition for $v_r \geq \frac{R}{\alpha(1-\delta)}$ is for $f_3(\frac{R}{\alpha(1-\delta)}) \leq 0$ and $\frac{R}{\alpha(1-\delta)} < c_p$ (since $v_r < c_p$ by construction). This simplifies to $\pi_r R^2(\alpha(\delta - 1)(c_p + p) + R) \leq (\delta - 1)\delta(\alpha(\delta - 1)p + R)^2$ and $\frac{R}{\alpha-\alpha\delta} < c_p + p$.

Claim 7. *The subgame outcome that corresponds to case (VII) arises if and only if the following conditions are satisfied:*

$$p = 1. \tag{B.34}$$

When $p = 1$, then the choice NB dominates all the other choices for every consumer in $[0, 1]$. Consequently, the outcome when $p = 1$ is that no one purchases.

This concludes the proof of the equilibrium characterization of the consumption subgame. ■

Proof of Lemma 1: This follows directly from Lemma B.1. ■

Next, we provide a simpler characterization of the conditions provided in Lemma B.1. This will be helpful for proving Proposition B.1.

Lemma B.2. *The complete threshold characterization of the consumption subgame is as follows:*

(I) $(0 < v_{nr} < 1)$, *where* $v_{nr} = \frac{\pi_r\alpha - 1 + \sqrt{1+\pi_r\alpha(-2+4p+\pi_r\alpha)}}{2\pi_r\alpha}$, *iff the following conditions hold:*

(a) $\frac{(1-c_p)(-c_p+\pi_r\alpha)}{\pi_r\alpha} \leq p < 1$, *and*

(b) $R \geq \alpha(1-\delta)$.

(II) $(0 < v_{nr} < v_p < 1)$, *where* v_{nr} *is the largest positive root of the cubic* $f_1(x) \triangleq \pi_r\alpha x^3 + (1-(c_p+p)\pi_r\alpha)x^2 - 2px + p^2$ *and* $v_p = v_{nr} + \frac{v_{nr}-p}{\pi_r\alpha v_{nr}}$, *iff the following conditions hold:*

(a) Either $\left(R \geq \alpha(1-\delta) \text{ and } 0 \leq p < \frac{(1-c_p)(-c_p+\pi_r\alpha)}{\pi_r\alpha}\right)$, *or*

(b) $\left(c_p\alpha(1-\delta) < R < \alpha(1-\delta) \text{ and } 0 \leq p \leq \frac{(-R+c_p\alpha(1-\delta))(c_p\alpha(1-\delta)^2-R^2\pi_r)}{R^2\pi_r\alpha(1-\delta)}\right)$

(III) $(0 < v_{nr} < v_r < 1)$, *where* $v_{nr} = \frac{\pi_r\alpha - 1 + \sqrt{1+\pi_r\alpha(-2+4p+\pi_r\alpha)}}{2\pi_r\alpha}$ *and* $v_r = \frac{R}{\alpha(1-\delta)}$, *iff the following conditions hold:*

(a) Either $\left(\pi_r \leq \frac{2c_p-1}{\alpha} \text{ and } R < \alpha(1-\delta) \text{ and } 0 \leq p < \frac{R(1-\delta+\pi_r(R-\alpha(1-\delta)))}{\alpha(1-\delta)^2}\right)$, *or*

(b) $\left(\pi_r > \frac{2c_p-1}{\alpha} \text{ and } R \leq \alpha\left(\frac{2c_p}{\alpha\pi_r+1} - \delta\right) \text{ and } 0 \leq p < \frac{R(1-\delta+\pi_r(R-\alpha(1-\delta)))}{\alpha(1-\delta)^2}\right)$, *or*

(c) $\left(\pi_r > \frac{2c_p-1}{\alpha} \text{ and } \alpha\left(\frac{2c_p}{\alpha\pi_r+1} - \delta\right) < R < \alpha(1-\delta) \text{ and } \frac{(\alpha(\delta-c_p)+R)(\pi_r(\alpha\delta+R)-c_p)}{\pi_r(\alpha\delta+R)^2} \leq p < \frac{R(1-\delta+\pi_r(R-\alpha(1-\delta)))}{\alpha(1-\delta)^2}\right)$

(IV) $(0 < v_{nr} < v_r < v_p < 1)$, *where* v_{nr} *is the largest positive root of the cubic* $f_2(x) \triangleq \delta\pi_r\alpha x^3 + (\delta + R\pi_r - (c_p+p\delta)\pi_r\alpha)x^2 - (2\delta + R\pi_r)px + p^2\delta$, $v_r = \frac{R}{\alpha(1-\delta)}$, *and* $v_p = v_{nr} + \frac{v_{nr}-p}{\pi_r\alpha v_{nr}}$, *iff the following conditions hold:*

(a) *Either* $\left(\alpha\left(\frac{2c_p}{\alpha\pi_r+1}-\delta\right)<R\le\frac{(1-\delta)\left(\sqrt{8\alpha c_p\pi_r+1}-1\right)}{2\pi_r}\right.$ *and*

$0\le p<\min\left(\frac{(\alpha(\delta-c_p)+R)(\pi_r(\alpha\delta+R)-c_p)}{\pi_r(\alpha\delta+R)^2},\frac{R\left(-\sqrt{\pi_r(4\alpha c_p(1-\delta)^2\delta+\pi_rR^2)}+2(1-\delta)\delta+\pi_rR\right)}{2\alpha(1-\delta)^2\delta}\right)\Bigg)$, *or*

(b) $\left(R>\max\left(\alpha\left(\frac{2c_p}{\alpha\pi_r+1}-\delta\right),\frac{(1-\delta)\left(\sqrt{8\alpha c_p\pi_r+1}-1\right)}{2\pi_r}\right)\right.$ *and* $\frac{(-R+\alpha c_p(1-\delta))\left(\alpha c_p(1-\delta)^2-\pi_rR^2\right)}{\alpha(1-\delta)\pi_rR^2}<$

$p<\min\left(\frac{(\alpha(\delta-c_p)+R)(\pi_r(\alpha\delta+R)-c_p)}{\pi_r(\alpha\delta+R)^2},\frac{R\left(-\sqrt{\pi_r(4\alpha c_p(1-\delta)^2\delta+\pi_rR^2)}+2(1-\delta)\delta+\pi_rR\right)}{2\alpha(1-\delta)^2\delta}\right)\Bigg)$

(V) $(0<v_r<1)$, *where* $v_r=\frac{-1-R\pi_r+\delta\pi_r\alpha+\sqrt{4\delta\pi_r\alpha(p+R\pi_r)+(1+R\pi_r-\delta\pi_r\alpha)^2}}{2\delta\pi_r\alpha}$, *iff the following conditions hold:*

(a) $R<\alpha(1-\delta)$ *and* $\max\left(\frac{R(1-\delta+\pi_r(R-\alpha(1-\delta)))}{\alpha(1-\delta)^2},1-c_p-\frac{c_p(\alpha(1-c_p)\delta+R)}{\pi_r(\alpha\delta+R)^2}\right)\le p<1$

(VI) $(0<v_r<v_p<1)$, *where* v_r *is the largest positive root of the cubic* $f_3(x)=\alpha^2\delta^2\pi_rx^3+(\alpha\delta(1+2R\pi_r-(c_p+p)\delta\pi_r\alpha))x^2+(R^2\pi_r-2\alpha\delta(p+(c_p+p)R\pi_r))x+p^2\alpha\delta-(c_p+p)R^2\pi_r$ *and* $v_p=v_r+\frac{v_r-p}{(R+v_r\alpha\delta)\pi_r}$, *iff the following conditions hold:*

(a) $\frac{R\left(-\sqrt{\pi_r(4\alpha c_p(1-\delta)^2\delta+\pi_rR^2)}+2(1-\delta)\delta+\pi_rR\right)}{2\alpha(1-\delta)^2\delta}\le p<1-c_p-\frac{c_p(\alpha(1-c_p)\delta+R)}{\pi_r(\alpha\delta+R)^2}$

(VII) $(0<1)$ *(in which no one purchases), if the following condition holds:*

- $p=1$

Proof of Lemma B.2: First, we simplify the conditions from condition set (I) of Lemma B.1. Consider the condition $1+\pi_r\alpha\le 2c_p+\sqrt{1+\pi_r\alpha(-2+4p+\pi_r\alpha)}$. For the radicand to be non-negative, we just need $p\ge-\frac{(1-\pi_r\alpha)^2}{4\pi_r\alpha}$, which is always true since $\alpha>0$ and $\pi_r>0$. If $1+\pi_r\alpha-2c_p\le 0$, then no further conditions are needed. Otherwise, we can square both sides and rewrite the condition as $p\ge\frac{(1-c_p)(-c_p+\pi_r\alpha)}{\pi_r\alpha}$. Note that this lower bound on p is no longer positive when $\pi_r\ge\frac{c_p}{\alpha}$. Moreover, $\frac{c_p}{\alpha}<\frac{-1+2c_p}{\alpha}$ from $0<c_p<1$, so then this condition altogether just becomes $p\ge\frac{(1-c_p)(-c_p+\pi_r\alpha)}{\pi_r\alpha}$. Combining this with the conditions $R\ge\alpha(1-\delta)$ and $p<1$ from (I) of Lemma B.1 yields the conditions for condition set (I) of Lemma B.2.

Next, we simplify the conditions from condition set (II) of Lemma B.1. In this case, there are two upper bounds on p, which can be written as $p<\frac{(1-c_p)(-c_p+\pi_r\alpha)}{\pi_r\alpha}$ (which requires $\pi_r>\frac{c_p}{\alpha}$ for the upper bound to be positive) and $p\le\frac{(-R+c_p\alpha(1-\delta))(c_p\alpha(1-\delta)^2-R^2\pi_r)}{R^2\pi_r\alpha(1-\delta)}$ (which requires $R>(1-\delta)\sqrt{\frac{c_p\alpha}{\pi_r}}$ and $R>c_p\alpha(1-\delta)$ for the upper bound to be positive). Then we compare these two bounds on p. The inequality $\frac{(-R+c_p\alpha(1-\delta))(c_p\alpha(1-\delta)^2-R^2\pi_r)}{R^2\pi_r\alpha(1-\delta)}\ge\frac{(1-c_p)(-c_p+\pi_r\alpha)}{\pi_r\alpha}$ only holds if $R\ge\alpha(1-\delta)$. Note that $\alpha(1-\delta)>c_p\alpha(1-\delta)$, so the conditions then simplify into two cases, depending on the range of R. If $R\ge\alpha(1-\delta)$, then $\frac{(1-c_p)(-c_p+\pi_r\alpha)}{\pi_r\alpha}$ is the smaller upper bound on p. On the other hand, if $c_p\alpha(1-\delta)<R<\alpha(1-\delta)$, then

$\frac{(-R+c_p\alpha(1-\delta))(c_p\alpha(1-\delta)^2-R^2\pi_r)}{R^2\pi_r\alpha(1-\delta)}$ is the smaller upper bound on p. This comprises the two cases for condition set (II) of Lemma B.2.

Next, we simplify the conditions from condition set (III) of Lemma B.1. The condition $-2R\pi_r + (1-\delta)(-1+\pi_r\alpha+\sqrt{1+\pi_r\alpha(-2+4p+\pi_r\alpha)}) < 0$ simplifies to the conditions $p < \frac{R(1-\delta+\pi_r(R-\alpha(1-\delta)))}{\alpha(1-\delta)^2}$. The other condition on p of condition set (III) of Lemma B.1 is given as $2c_p\alpha+(R+\alpha\delta)(\sqrt{1+\pi_r\alpha(-2+4p+\pi_r\alpha)}-(1+\pi_r\alpha)) \geq 0$. This simplifies to either $R \leq \alpha\left(-\delta+\frac{2c_p}{1+\pi_r\alpha}\right)$ or $p \geq \frac{(\alpha(\delta-c_p)+R)(\pi_r(\alpha\delta+R)-c_p)}{\pi_r(\alpha\delta+R)^2}$. This then gives a lower bound on p that we have to compare to the upper bound we found earlier. The inequality $\alpha(1-\delta) \leq \alpha(-\delta+\frac{2c_p}{1+\pi_r\alpha})$ holds if and only if $\pi_r \leq \frac{-1+2c_p}{\alpha}$. Then if $\pi_r \leq \frac{-1+2c_p}{\alpha}$, the bound $R \leq \alpha\left(-\delta+\frac{2c_p}{1+\pi_r\alpha}\right)$ is implied by the condition $R < \alpha(1-\delta)$. Otherwise, if $\pi_r > \frac{-1+2c_p}{\alpha}$, then whether the lower bound on p is less than 0 depends on whether $R > \alpha\left(-\delta+\frac{2c_p}{1+\pi_r\alpha}\right)$. Then altogether, these give the three mutually exclusive conditions provided in condition set (III) of Lemma B.2 into the three cases defined in the lemma.

Next, we simplify the conditions from condition set (IV) of Lemma B.1. Rewriting the condition in terms of p, condition (a) of condition set (IV) of Lemma B.1 can be rewritten $p < \frac{R}{\alpha(1-\delta)}$. Rewriting condition (b) gives $p < \frac{R\left(-\sqrt{\pi_r(4\alpha c_p(\delta-1)^2\delta+\pi_r R^2)}-2(\delta-1)\delta+\pi_r R\right)}{2\alpha(\delta-1)^2\delta}$. Note that $\frac{R\left(-\sqrt{\pi_r(4\alpha c_p(\delta-1)^2\delta+\pi_r R^2)}-2(\delta-1)\delta+\pi_r R\right)}{2\alpha(\delta-1)^2\delta} < \frac{R}{\alpha(1-\delta)}$ follows from $0 < c_p < 1$, $\alpha > 0$, $\pi_r > 0$, $R > 0$, and $0<\delta<1$, so condition (b) gives a tighter upper bound on p. For condition (c), we can use the substitution $u = -1+\pi_r\alpha+\sqrt{1+\pi_r\alpha(-2+4p+\pi_r\alpha)}$ to isolate p in terms of u, resulting in $p = \frac{u(2+u-2\pi_r\alpha)}{4\pi_r\alpha}$. Substituting this expression for p back into the original inequality yields $u^2(2c_p\alpha+(R+\alpha\delta)(u-2\pi_r\alpha)) < 0$. Simplifying and re-expressing this in terms of p again, this becomes $\sqrt{1+\pi_r\alpha(-2+4p+\pi_r\alpha)} < 1+\pi_r\alpha-\frac{2c_p\alpha}{R+\alpha\delta}$. This requires the right-hand side to be positive, which gives the condition $R > \alpha(-\delta+\frac{2c_p}{1+\pi_r\alpha})$. Then simplifying in terms of p, we have $p < \frac{(R-\alpha(c_p-\delta))(-c_p+(R+\alpha\delta)\pi_r)}{(R+\alpha\delta)^2\pi_r}$. So then altogether, condition (c) becomes $R > \alpha(-\delta+\frac{2c_p}{1+\pi_r\alpha})$ and $p < \frac{(R-\alpha(c_p-\delta))(-c_p+(R+\alpha\delta)\pi_r)}{(R+\alpha\delta)^2\pi_r}$. Similarly, solving for p in condition (d) from condition set (IV) of Lemma B.1 yields: $R \leq \frac{(1-\delta)\left(\sqrt{8\alpha c_p\pi_r+1}-1\right)}{2\pi_r}$ or $\left(R > \frac{(1-\delta)\left(\sqrt{8\alpha c_p\pi_r+1}-1\right)}{2\pi_r} \text{ and } p > \frac{(\alpha c_p(\delta-1)+R)\left(\alpha c_p(\delta-1)^2-\pi_r R^2\right)}{\alpha(\delta-1)\pi_r R^2}\right)$. Then altogether, this gives the two cases for condition set (IV) of Lemma B.2.

Next, we simplify the conditions from condition set (V) of Lemma B.1. Rewriting the conditions in terms of p, condition (b) of condition set (V) of Lemma B.1 can be rewritten as: $p \geq 1-c_p+\frac{c_p(\alpha(c_p-1)\delta-R)}{\pi_r(\alpha\delta+R)^2}$ or $p \geq 1+\frac{1}{2}c_p\left(-\frac{1}{\alpha\delta\pi_r+\pi_r R}-1\right)$. This can be further simplified by noting that $1-c_p+\frac{c_p(\alpha(c_p-1)\delta-R)}{\pi_r(\alpha\delta+R)^2} = 0$ for $\pi_r = \frac{c_p(R+\alpha\delta(1-c_p))}{(1-c_p)(\alpha\delta+R)^2}$ while $1+\frac{1}{2}c_p\left(-\frac{1}{\alpha\delta\pi_r+\pi_r R}-1\right) = 0$ for $\pi_r = \frac{c_p}{(2-c_p)(\alpha\delta+R)}$ and that $\frac{c_p(R+\alpha\delta(1-c_p))}{(1-c_p)(\alpha\delta+R)^2} > \frac{c_p}{(2-c_p)(\alpha\delta+R)}$ follows from $0 < c_p < 1$, $\alpha > 0$, $\delta > 0$, $\pi_r > 0$, and $R > 0$. Consequently, $1-c_p+\frac{c_p(\alpha(c_p-1)\delta-R)}{\pi_r(\alpha\delta+R)^2} \leq 1+\frac{1}{2}c_p\left(-\frac{1}{\alpha\delta\pi_r+\pi_r R}-1\right)$, and the smaller lower bound on p is $1-c_p+\frac{c_p(\alpha(c_p-1)\delta-R)}{\pi_r(\alpha\delta+R)^2}$. Then condition (b) of the original condition set can be written as $p \geq 1-c_p+\frac{c_p(\alpha(c_p-1)\delta-R)}{\pi_r(\alpha\delta+R)^2}$. Similarly, condition (c) of the original condition set can be written in terms of p as $\frac{R(-\delta+\pi_r(\alpha(\delta-1)+R)+1)}{\alpha(\delta-1)^2} \leq p < 1$ and $R < \alpha(1-\delta)$ (which guarantees $\frac{R(-\delta+\pi_r(\alpha(\delta-1)+R)+1)}{\alpha(\delta-1)^2} < 1$). Combining this with the previous condition, this becomes: $R < \alpha(1-\delta)$ and $\max\left(1 - c_p - \frac{c_p(\alpha(1-c_p)\delta+R)}{\pi_r(\alpha\delta+R)^2}, \frac{R(1-\delta+\pi_r(-\alpha(1-\delta)+R))}{\alpha(1-\delta)^2}\right) \leq p < 1$. Altogether, this gives the condition set (V) of Lemma B.2.

Lastly, we simplify the conditions from (VI) of Lemma B.1. From the condition (a) of that condition set is equivalent to $\max(-\frac{R}{\alpha\delta}, \frac{\pi_r\left(2\alpha\delta-\pi_r(\alpha\delta+R)^2-2R\right)-1}{4\alpha\delta\pi_r}) \leq p < \frac{c_p(\alpha(c_p-1)\delta-R)}{\pi_r(\alpha\delta+R)^2} - c_p + 1$. Since $0 > \max(-\frac{R}{\alpha\delta}, \frac{\pi_r\left(2\alpha\delta-\pi_r(\alpha\delta+R)^2-2R\right)-1}{4\alpha\delta\pi_r})$ follows from $R > 0$, $\alpha > 0$, $\delta > 0$, and $\pi_r > 0$, this condition simplifies to $p < \frac{c_p(\alpha(c_p-1)\delta-R)}{\pi_r(\alpha\delta+R)^2} - c_p + 1$. Next, condition (b) from condition set (VI) of Lemma B.1 simplifies to $p < 1$. This bound is implied by the previous one ($1 > \frac{c_p(\alpha(c_p-1)\delta-R)}{\pi_r(\alpha\delta+R)^2} - c_p + 1$), from $0 < c_p < 1$ and the other parameters being positive. Lastly, condition (c) from condition set (VI) of Lemma B.1 simplifies to either $p \geq \frac{R\left(-\sqrt{\pi_r(4\alpha c_p(\delta-1)^2\delta+\pi_r R^2)}-2(\delta-1)\delta+\pi_r R\right)}{2\alpha(\delta-1)^2\delta}$ or $R \leq \frac{(\delta-1)(\delta-\alpha c_p\pi_r)}{\pi_r}$. Note that $\frac{R\left(-\sqrt{\pi_r(4\alpha c_p(\delta-1)^2\delta+\pi_r R^2)}-2(\delta-1)\delta+\pi_r R\right)}{2\alpha(\delta-1)^2\delta} \leq 0$ follows from $R \leq \frac{(\delta-1)(\delta-\alpha c_p\pi_r)}{\pi_r}$, so we can simplify this logical 'or' condition to just the first condition $p \geq \frac{R\left(-\sqrt{\pi_r(4\alpha c_p(\delta-1)^2\delta+\pi_r R^2)}+2(1-\delta)\delta+\pi_r R\right)}{2\alpha(1-\delta)^2\delta}$. Altogether, this is $\frac{R\left(-\sqrt{\pi_r(4\alpha c_p(1-\delta)^2\delta+\pi_r R^2)}+2(1-\delta)\delta+\pi_r R\right)}{2\alpha(1-\delta)^2\delta} \leq p < 1 - c_p - \frac{c_p(\alpha(1-c_p)\delta-R)}{\pi_r(\alpha\delta+R)^2}$, the condition set (VI) of Lemma B.2. ■

Proof of Proposition B.1: The conditions of Proposition B.1 take the conditions of Lemma B.2 and group them to clearly see how the conditions span the parameter space. Each grouping comes from examining Lemma B.2 and observing which cases share a boundary in p. Tracing out the different ways in which the market outcomes can change gives the nine condition sets of Proposition B.1 for which $p < 1$. When $p = 1$ is a trivial case when no one purchases. The R and π_r conditions of each region comes from Lemma B.2 as well as conditions comparing the p bounds in each region to each other so that those ranges of p are non-empty.

To show that the ordering presented in Proposition B.1 forms a complete partition of the parameter space, first note that within each of the nine condition sets for which $p < 1$, the ranges of p form a partition of $[0, 1)$. Since this is true for each of the nine parameter regions in Proposition B.1 for which $p < 1$, it suffices to show that for any δ, c_p, and α, the R and π_r conditions form a partition of the $\{(R, \pi_r) : R \in (0, \infty), \pi_r \in (0, 1)\}$.

To show each region is mutually exclusive, we will consider three subregions separately: $R \geq \alpha(1-\delta)$, $\left(\alpha c_p(1-\delta) < R < \alpha - \alpha\delta \text{ and } \pi_r > \frac{\alpha c_p(1-\delta)^2}{R^2}\right)$, and $\left(R < \alpha - \alpha\delta \text{ and } \left(R \leq \alpha c_p(1-\delta) \text{ or } \pi_r \leq \frac{\alpha c_p(\delta-1)^2}{R^2}\right)\right)$ (which is the complement of the union of the other two subregions).

First, consider the region $R \geq \alpha(1-\delta)$. We will show that only Regions (i) and (ii) will appear in this area. With the exception of Region (iv), all of the other regions require either a condition of $R < \alpha(1-\delta)$ or $R \leq \alpha c_p(1-\delta)$, so none of those regions would overlap with any region with $R \geq \alpha(1-\delta)$.

For Region (iv), two conditions are $\left(\left(\pi_r \leq \frac{\alpha c_p(1-\delta)^2}{R^2} \text{ or } R \leq \alpha c_p(1-\delta)\right) \text{ and } \pi_r > \frac{c_p}{\alpha\delta+R}\right)$. If $R \leq \alpha c_p(1-\delta)$, then $R < \alpha(1-\delta)$, so this does not overlap with any region with $R \geq \alpha(1-\delta)$. On the other hand, suppose $\pi_r \leq \frac{\alpha c_p(1-\delta)^2}{R^2}$ held. Then for $\pi_r > \frac{c_p}{\alpha\delta+R}$ to hold as well, we need $\frac{c_p}{\alpha\delta+R} < \frac{\alpha c_p(1-\delta)^2}{R^2}$, which would require $R < \alpha - \alpha\delta$. Hence, other then Regions (i) and (ii), no other regions overlap with $R \geq \alpha(1-\delta)$.

Next, consider when $\alpha c_p(1-\delta) < R < \alpha - \alpha\delta$ and $\pi_r > \frac{\alpha c_p(1-\delta)^2}{R^2}$. We will show that only Regions (v) and (vi) are in this subregion. Regions (i) and (ii) are out due to $R \geq \alpha - \alpha\delta$ being necessary for those cases. Regions (iv) and (vii) are out due to $\pi_r \leq \frac{\alpha c_p(1-\delta)^2}{R^2}$ or $R \leq \alpha c_p(1-\delta)$ being necessary for those cases. Region (viii) is out due to $R \leq \alpha c_p(1-\delta)$ being necessary for Region (viii). Region (ix) is out due to $\pi_r \geq \frac{1-\delta}{\alpha(1-\delta)-R}$ and $\pi_r \leq \frac{c_p(\alpha(c_p-1)\delta-R)}{(c_p-1)(\alpha\delta+R)^2}$ being conditions, which implies $R \leq \alpha(c_p-\delta)$, which implies $R < \alpha c_p(1-\delta)$.

Comparing Regions (i) and (ii), they are disjoint because Region (i) requries $\pi_r \leq \frac{c_p}{\alpha}$ while Region (ii) requires $\pi_r > \frac{c_p}{\alpha}$. Comparing Regions (v) and (vi), they are disjoint because Region (v) requires $\pi_r \leq \frac{\alpha c_p(\delta-1)}{(\alpha(\delta-1)+R)(\alpha\delta+R)}$ while Region (vi) requires $\pi_r > \frac{\alpha c_p(\delta-1)}{(\alpha(\delta-1)+R)(\alpha\delta+R)}$.

To finish proving that the regions are mutually exclusive, we just need to show that Regions (iii), (iv), (vii), (viii), and (ix) are disjoint. The nine comparisons below show that no two regions of the five listed in the previous sentence have an overlap.

Region (iii) and Region (iv): Region (iii) requires $\left(R \leq \alpha(c_p-\delta) \text{ or } \pi_r \leq \frac{c_p}{\alpha\delta+R}\right)$ while Region (iv) requires $\left(R > \alpha(c_p-\delta) \text{ and } \pi_r > \frac{c_p}{\alpha\delta+R}\right)$.

Region (iii) and Region (vii): Region (vii) requires $\pi_r \geq \frac{\alpha c_p(\delta-1)}{(\alpha(\delta-1)+R)(\alpha\delta+R)}$ while Region (iii) requires $\pi_r \leq \frac{c_p}{\alpha\delta+R}$ or $R \leq \alpha(c_p-\delta)$. There is no overlap when $\pi_r \leq \frac{c_p}{\alpha\delta+R}$ since $\frac{c_p}{\alpha\delta+R} < \frac{\alpha c_p(\delta-1)}{(\alpha(\delta-1)+R)(\alpha\delta+R)}$ holds from $R < \alpha(1-\delta)$. If there is an overlap, then it must be when $R \leq \alpha(c_p-\delta)$. However, if $R \leq \alpha(c_p-\delta)$, then $R \leq \alpha c_p(1-\delta)$. Then in Region (vii), the following conditions are in play: $\pi_r < \frac{(1-\delta)\delta}{\alpha c_p(1-\delta)-R}$ and $R \leq \alpha c_p(1-\delta)$ and $\pi_r > \frac{\alpha c_p(\delta-1)}{(\alpha(\delta-1)+R)(\alpha\delta+R)}$. For $\frac{(1-\delta)\delta}{\alpha c_p(1-\delta)-R} > \frac{\alpha c_p(\delta-1)}{(\alpha(\delta-1)+R)(\alpha\delta+R)}$, $R < \alpha(c_p-\delta)$ has to hold. Then Region (vii) would be: $R < \alpha(c_p-\delta)$ and $\frac{\alpha c_p(\delta-1)}{(\alpha(\delta-1)+R)(\alpha\delta+R)} < \pi_r < \frac{(1-\delta)\delta}{\alpha c_p(1-\delta)-R}$. Region (iii) would be: $R \leq \alpha(c_p-\delta)$ and $\pi_r < \frac{1-\delta}{\alpha(1-\delta)-R}$ and $R < \alpha - \alpha\delta$. In this case, $\frac{1-\delta}{\alpha(1-\delta)-R} < \frac{\alpha c_p(\delta-1)}{(\alpha(\delta-1)+R)(\alpha\delta+R)}$, so then there is no overlap in the regions, since $\pi_r < \frac{1-\delta}{\alpha(1-\delta)-R}$ is needed for Region (iii) while $\pi_r > \frac{\alpha c_p(\delta-1)}{(\alpha(\delta-1)+R)(\alpha\delta+R)}$ is needed for Region (vii).

Region (iii) and Region (viii): Region (iii) requires $\left(R \leq \alpha(c_p-\delta) \text{ or } \pi_r \leq \frac{c_p}{\alpha\delta+R}\right)$ and$\pi_r < \frac{1-\delta}{\alpha(1-\delta)-R}$ while Region (viii) requires $\pi_r > \frac{c_p(\alpha(c_p-1)\delta-R)}{(c_p-1)(\alpha\delta+R)^2}$. If $R \leq \alpha(c_p-\delta)$, then $\frac{1-\delta}{\alpha(1-\delta)-R} \leq \frac{c_p(\alpha(c_p-1)\delta-R)}{(c_p-1)(\alpha\delta+R)^2}$, and so there is no overlap between Region (iii) and Region (viii) in this case, since Region (iii) requires $\pi_r < \frac{1-\delta}{\alpha(1-\delta)-R}$ while Region (viii) requires $\pi_r > \frac{c_p(\alpha(c_p-1)\delta-R)}{(c_p-1)(\alpha\delta+R)^2}$. On the other hand, suppose $\pi_r \leq \frac{c_p}{\alpha\delta+R}$. Then we also do not have an overlap in this case since $\pi_r \leq \frac{c_p}{\alpha\delta+R}$ and $\pi_r > \frac{c_p(\alpha(c_p-1)\delta-R)}{(c_p-1)(\alpha\delta+R)^2}$ cannot simultaneously hold, given $\frac{c_p}{\alpha\delta+R} < \frac{c_p(\alpha(c_p-1)\delta-R)}{(c_p-1)(\alpha\delta+R)^2}$.

Region (iii) and Region (ix): Region (iii) requires $\pi_r < \frac{1-\delta}{\alpha(1-\delta)-R}$ while Region (ix) requires $\pi_r \geq \frac{1-\delta}{\alpha(1-\delta)-R}$.

Region (iv) and Region (vii): Region (iv) requires $\pi_r \leq \frac{\alpha c_p(\delta-1)}{(\alpha(\delta-1)+R)(\alpha\delta+R)}$ while Region (vii) requires $\pi_r > \frac{\alpha c_p(\delta-1)}{(\alpha(\delta-1)+R)(\alpha\delta+R)}$.

Region (iv) and Region (viii): Region (iv) requires $\pi_r \leq \frac{\alpha c_p(\delta-1)}{(\alpha(\delta-1)+R)(\alpha\delta+R)}$ and $R > \alpha(c_p - \delta)$. Region (viii) requires $\pi_r \geq \frac{(1-\delta)\delta}{\alpha c_p(1-\delta)-R}$. $R > \alpha(c_p - \delta)$ is a condition of Region (iv), so then $\frac{\alpha c_p(\delta-1)}{(\alpha(\delta-1)+R)(\alpha\delta+R)} < \frac{(1-\delta)\delta}{\alpha c_p(1-\delta)-R}$ holds when the conditions of Case (iv) are met. Since, $\pi_r \geq \frac{(1-\delta)\delta}{\alpha c_p(1-\delta)-R}$ do not hold when the conditions for Region (iv) are met, there is no overlap between Region (iv) and Region (viii).

Region (iv) and Region (ix): Region (iv) requires $R > \alpha(c_p-\delta)$ while $\pi_r \geq \frac{1-\delta}{\alpha(1-\delta)-R}$ and $\pi_r \leq \frac{c_p(\alpha(c_p-1)\delta-R)}{(c_p-1)(\alpha\delta+R)^2}$ being conditions for Region (ix) implies $R \leq \alpha(c_p - \delta)$.

Region (vii) and Region (viii): Region (viii) has $R \leq \alpha c_p(1-\delta)$ and $\pi_r \geq \frac{(1-\delta)\delta}{\alpha c_p(1-\delta)-R}$ while Region (vii) requires $\left((\pi_r < \frac{(1-\delta)\delta}{\alpha c_p(1-\delta)-R} \text{ and } R \leq \alpha c_p(1-\delta)) \text{ or } (R > \alpha c_p(1-\delta))\right)$.

Region (vii) and Region (ix): $\pi_r \geq \frac{1-\delta}{\alpha(1-\delta)-R}$ and $\pi_r \leq \frac{c_p(\alpha(c_p-1)\delta-R)}{(c_p-1)(\alpha\delta+R)^2}$ being conditions for Region (ix) implies $R \leq \alpha(c_p-\delta)$ and $c_p > \delta$. We will show that $R > \alpha(c_p-\delta)$ is an implied condition for Region (vii). One condition for Region (vii) is $\left((\pi_r < \frac{(1-\delta)\delta}{\alpha c_p(1-\delta)-R} \text{ and } R \leq \alpha c_p(1-\delta)) \text{ or } (R > \alpha c_p(1-\delta))\right)$. Note that $\alpha c_p(1-\delta) > \alpha(c_p-\delta)$, so if $R > \alpha c_p(1-\delta)$, then that would imply $R > \alpha(c_p-\delta)$. Suppose that $R \leq \alpha c_p(1-\delta)$ in Region (vii). Then this would require $\pi_r < \frac{(1-\delta)\delta}{\alpha c_p(1-\delta)-R}$ to also hold. On the other hand, suppose that $\pi_r < \frac{(1-\delta)\delta}{\alpha c_p(1-\delta)-R}$ and $R \leq \alpha c_p(1-\delta)$ held. Another condition of Region (vii) is $\pi_r > \frac{\alpha c_p(\delta-1)}{(\alpha(\delta-1)+R)(\alpha\delta+R)}$. Comparing the π_r bounds: $\frac{(1-\delta)\delta}{\alpha c_p(1-\delta)-R} > \frac{\alpha c_p(\delta-1)}{(\alpha(\delta-1)+R)(\alpha\delta+R)}$ needs to hold. Then this implies $R > \alpha(c_p-\delta)$. In either case, this implies that $R > \alpha(c_p-\delta)$ for Region (vii) to hold, which means there is no overlap with Region (ix).

Region (viii) and Region (ix): Region (viii) has $\pi_r > \frac{c_p(\alpha(c_p-1)\delta-R)}{(c_p-1)(\alpha\delta+R)^2}$ while Region (ix) requires $\pi_r \leq \frac{c_p(\alpha(c_p-1)\delta-R)}{(c_p-1)(\alpha\delta+R)^2}$.

Altogether, the nine comparisons above along with the earlier comparisons for Regions (i), (ii), (v), and (vi) show that the nine regions are mutually exclusive.

Next, we will show that these nine parameter regions cover the entire parameter space. First off, note that the p conditions are clearly ordered within each region, so it suffices to show that given any δ, α, and c_p, the nine parameter regions sets cover (R, π_r) space.

Any point (R, π_r) such that $R \geq \alpha(1-\delta)$ is fully covered by Regions (i) and (ii), depending on whether $\pi_r \leq \frac{c_p}{\alpha}$. Any point (R, π_r) such that $\alpha c_p(1-\delta) < R < \alpha - \alpha\delta$ and $\pi_r > \frac{\alpha c_p(1-\delta)^2}{R^2}$ is fully covered by Regions (v) and (vi), depending on whether $\pi_r \leq \frac{\alpha c_p(\delta-1)}{(\alpha(\delta-1)+R)(\alpha\delta+R)}$.

It remains to show that any point with $R < \alpha-\alpha\delta$ and $\left(R \leq \alpha c_p(1-\delta) \text{ or } \pi_r \leq \frac{\alpha c_p(\delta-1)^2}{R^2}\right)$ is covered by one of the remaining regions. We will π_r bounds are ordered between the cases in such a way that any point is covered in (R, π_r) space. We will break the regions down into different regions of R: $\alpha c_p(1-\delta) < R < \alpha(1-\delta)$, $\max(0, \alpha(c_p - \delta)) < R \leq \alpha c_p(1-\delta)$, $R \leq \max(0, \alpha(c_p - \delta))$. We will show that for each subregion of R, each point in that subregion is in one of the parameter regions in the proposition (specifically, in one of the five remaining regions: Regions (iii), (iv), (vii), (viii), or (ix)).

First, consider when $R > \alpha c_p(1-\delta)$. We will first show that the region $R < \alpha(1-\delta)$ and $\pi_r \leq \frac{\alpha c_p(\delta-1)^2}{R^2}$ and $R > \alpha c_p(1-\delta)$ is fully covered by one of Regions (iii), (iv) or (vii). In Region (iii) (under the conditions of $R > \alpha c_p(1-\delta)$ and $R < \alpha(1-\delta)$), we have that $\pi_r \leq \frac{c_p}{\alpha\delta+R}$ is the tighter upper bound on π_r. Then for any point in this subregion of the parameter space, if $\pi_r \leq \frac{c_p}{\alpha\delta+R}$, then that point is in Region (iii). Region (iv)'s conditions simplified under the assumptions of this subregion becomes: $\frac{c_p}{\alpha\delta+R} < \pi_r \leq \frac{\alpha c_p(\delta-1)}{(\alpha(\delta-1)+R)(\alpha\delta+R)}$. Region (vii)'s conditions simplified under the assumptions of this subregion becomes: $\pi_r > \frac{\alpha c_p(\delta-1)}{(\alpha(\delta-1)+R)(\alpha\delta+R)}$. Hence, this subregion is fully covered by one of these three regions when $\alpha c_p(1-\delta) < R < \alpha(1-\delta)$ and $\pi_r \leq \frac{\alpha c_p(\delta-1)^2}{R^2}$.

Now consider $\max(0, \alpha(c_p - \delta)) < R \leq \alpha c_p(1-\delta)$ and $\pi_r \leq \frac{\alpha c_p(\delta-1)^2}{R^2}$. In this subregion of the parameter space, $\pi_r \leq \frac{c_p}{\alpha\delta+R}$ is still the tighter lower bound for Region (iii), so then Region (iii) still becomes: $\pi_r \leq \frac{c_p}{\alpha\delta+R}$. Region (iv) becomes: $\frac{\alpha c_p(\delta-1)}{(\alpha(\delta-1)+R)(\alpha\delta+R)} \geq \pi_r > \frac{c_p}{\alpha\delta+R}$. Region (vii) becomes: $\frac{\alpha c_p(\delta-1)}{(\alpha(\delta-1)+R)(\alpha\delta+R)} < \pi_r < \frac{(\delta-1)\delta}{\alpha c_p(\delta-1)+R}$. Region (viii) simplifies to: $\pi_r \geq \frac{(\delta-1)\delta}{\alpha c_p(\delta-1)+R}$ and $\pi_r > \frac{c_p(\alpha(c_p-1)\delta-R)}{(c_p-1)(\alpha\delta+R)^2}$. Note that $\frac{(\delta-1)\delta}{\alpha c_p(\delta-1)+R} > \frac{c_p(\alpha(c_p-1)\delta-R)}{(c_p-1)(\alpha\delta+R)^2}$ under the assumptions of this subregion, so Region (viii) simplifies to $\pi_r \geq \frac{(\delta-1)\delta}{\alpha c_p(\delta-1)+R}$. Then any (R, π_r) satisfying $R < \alpha(1-\delta)$ and $\pi_r \leq \frac{\alpha c_p(\delta-1)^2}{R^2}$ and $R < \alpha c_p(1-\delta)$ will fall into one of these four regions.

Lastly, consider the scenario with $R \leq \alpha(c_p - \delta)$, which is only under consideration when $c_p > \delta$. Region (iii) simplifies to: $\pi_r < \frac{1-\delta}{\alpha(1-\delta)-R}$. Region (ix) simplifies to: $\frac{1-\delta}{\alpha(1-\delta)-R} \leq \pi_r \leq \frac{c_p(\alpha(c_p-1)\delta-R)}{(c_p-1)(\alpha\delta+R)^2}$. Region (viii) simplifies to: $\pi_r > \frac{c_p(\alpha(c_p-1)\delta-R)}{(c_p-1)(\alpha\delta+R)^2}$ and $\pi_r \geq \frac{(\delta-1)\delta}{\alpha c_p(\delta-1)+R}$. Under the assumptions of this subregion, $\frac{c_p(\alpha(c_p-1)\delta-R)}{(c_p-1)(\alpha\delta+R)^2} \geq \frac{(\delta-1)\delta}{\alpha c_p(\delta-1)+R}$, so then Region (viii) simplifies to: $\pi_r > \frac{c_p(\alpha(c_p-1)\delta-R)}{(c_p-1)(\alpha\delta+R)^2}$. Altogether, this shows that when $R \leq \alpha(c_p - \delta)$, there is full coverage across all π_r. Altogether, for any δ, c_p, and α, for any subregion of R and π_r, for any range of p within a subregion, every point is covered by one of the parameter regions. This proves that the parameter regions in the proposition cover the entire parameter space and that there are no missing cases. ■

B.1.2 Pricing Subgame: Equilibrium Outcomes

In this section, we complete the characterization of the equilibrium by solving the first stage, i.e., the pricing subgame. More precisely, we characterize the optimal price and consumer market equilibrium outcome and how they vary across regions in R and π_r. In the proof, we leverage the validity conditions of Proposition B.1 to determine which consumer market outcomes are feasible as the vendor changes price. Further, we use the consumer threshold characterizations provided in Lemma B.2 (as well as in Lemma B.1) to determine the highest profits obtainable within each feasible outcome. We then compare these profits across each feasible outcome to determine the profit-maximizing equilibrium price.

Lemma B.3. *There exists a bound $\tilde{\delta} > 0$ such that if $\delta < \tilde{\delta}$ and $\pi_r > \bar{\pi}_r$ (where $\bar{\pi}_r$ is defined in the proof of the lemma) then:*

(a) if $0 < R < R_1$, then the equilibrium price is p_V^* and the equilibrium consumer market structure is $0 < v_r < 1$;

(b) if $R_1 \leq R < R_2$, then the equilibrium price is p_{VI}^* and the equilibrium consumer market structure is $0 < v_r < v_p < 1$;

(c) if $R_2 \leq R < R_3$, then the equilibrium price is p_{IV}^* and the equilibrium consumer market structure is $0 < v_{nr} < v_r < v_p < 1$;

(d) if $R_3 \leq R < \omega$, then the equilibrium price is $p_{boundary}$ and the equilibrium consumer market structure is $0 < v_{nr} < v_p < 1$;

(e) if $R \geq \omega$, then the equilibrium price is p_{II}^* and the equilibrium consumer market structure is $0 < v_{nr} < v_p < 1$,

where $R_1 = \frac{(2-c_p)c_p}{(1-c_p)^2\pi_r} + \kappa_1(\delta)$, $R_2 = \frac{\alpha}{2-c_p} + \kappa_2(\delta)$, and $R_3 = \frac{\alpha\sqrt{\pi_r}+\sqrt{\alpha(16c_p+\alpha\pi_r)}}{4\sqrt{\pi_r}} + \kappa_3(\delta)$. The prices p_V^*, p_{VI}^*, p_{IV}^*, and $p_{boundary}$ are characterized in the proof below, and the characterization of p_{II}^* is in Lemma B.5.

Proof of Lemma B.3: Given any set of parameters and a price, Lemma B.1 and the validity conditions of Proposition B.1 ensure that a unique equilibrium in the consumption subgame arises. Within each region of the parameter space defined by Proposition B.1, the thresholds $v_{nr}, v_r,$ and v_p are smooth functions of the parameters, as well as the vendor's price p. In the cases where the thresholds are given in closed-form, this is evident. In the cases where these thresholds are implicitly defined as the root of a polynomial, then the smoothness of the thresholds in the parameters follows from the Implicit Function Theorem. Specifically, for each of those cases, the threshold defined was the largest positive root v_{nr}^* (or v_r^*) of a cubic function of v_{nr} (or v_r), $f(v_{nr}, p) = 0$. Moreover, the cubic $f(v_{nr}, p)$ has two local extrema in v_{nr} and is negative to the left of v_{nr}^* and positive to the right of it ($f(v_{nr}^* - \epsilon, p) < 0$ and $f(v_{nr}^* + \epsilon, p) > 0$ for arbitrarily small $\epsilon > 0$). Therefore, $\frac{\partial f}{\partial v_{nr}}(v_{nr}, p) \neq 0$ so that the Implicit Function Theorem applies. The thresholds being smooth in p implies that the profit function for each case of the parameter space defined by Proposition B.1 is smooth in p. In our proofs, we use asymptotic analysis to characterize the equilibrium prices and profits when needed, using Taylor series representations in δ of the thresholds, price, and profit expressions. When writing the Taylor series, we will abuse notation by re-using the same notation for the remainder terms throughout the paper. This is just to simplify the notation. The remainder terms for Taylor series of different expressions are not the same.

In the proofs of the lemmas characterizing equilibrium outcomes, we find the profit-maximizing interior solution within the compact closure of each subcase to characterize the conditions under which each of the consumer market outcomes of Lemma B.2 could arise in under optimal pricing and verify that the second-order condition holds for each of these prices under the conditions of their respective regions. We then find regions of the parameter space where there is overlap between the different cases under optimal pricing to find the boundaries between different regions defining equilibrium outcomes. We compare the profits obtained under optimal pricing within each feasible market outcome: these are the profit-maximizing conditions. In the cases where a regime switch happens not at a discontinuous price change but from the vendor choosing a boundary price, the boundary price can be found from finding common region boundaries in Proposition B.1.

First, we specify the interior optimal price and vendor's profit at that interior optimal price for all market outcomes except for $0 < v_{nr} < v_p < 1$ (which is handled separately in Section B.1.3 of the Appendix).

Given a price p, the region of the parameter space defining $0 < v_{nr} < 1$ is given in part (I) of Lemma B.2. For this case, we have

$$v_{nr} = \frac{-1 + \pi_r\alpha + \sqrt{1 + \pi_r\alpha(-2 + 4p + \pi_r\alpha)}}{2\pi_r\alpha}. \tag{B.35}$$

The profit function in this case is $\Pi_I(p) = p(1 - v_{nr}(p))$. Let C_I be the compact closure of the region of the parameter space defining $0 < v_{nr} < 1$, given in part (I) of Lemma B.2. By the Weierstrass extreme value theorem, there exists p in C_I that maximizes $\Pi_I(p)$. If this p is interior to C_I, the unconstrained maximizer satisfies the first-order condition. The Weierstrass extreme value applies for all regions, and we will not state this for other regions.

Differentiating the profit function with respect to p, the first-order condition is given as:

$$\frac{1}{2} - \frac{p}{\sqrt{1 + \pi_r\alpha(-2 + 4p + \pi_r\alpha)}} - \frac{-1 + \sqrt{1 + \pi_r\alpha(-2 + 4p + \pi_r\alpha)}}{2\pi_r\alpha} = 0. \tag{B.36}$$

Solving for p in this equation and ruling out the negative root as a solution for p^*, we have that

$$p_I^* = \frac{1}{9}\left(4 - \frac{1}{\pi_r\alpha} - \pi_r\alpha + \frac{\sqrt{1 + \pi_r\alpha + (\pi_r\alpha)^3 + (\pi_r\alpha)^4}}{\pi_r\alpha}\right). \tag{B.37}$$

The second-derivative of the profit function with respect to p is given by:

$$\frac{d^2}{dp^2}\left[\Pi_I(p)\right] = -\frac{2(1 + \pi_r\alpha(-2 + 3p + \pi_r\alpha))}{(1 + \pi_r\alpha(-2 + 4p + \pi_r\alpha))^{\frac{3}{2}}}. \tag{B.38}$$

Substituting (B.37) in for p in the above expression, the second-derivative evaluates to:

$$\left(\frac{d^2}{dp^2}\left[\Pi_I(p)\right]\right)\Bigg|_{p=p_I^*} = -\frac{18(2+2\pi_r\alpha(-1+\pi_r\alpha)+\sqrt{1+\pi_r\alpha+(\pi_r\alpha)^3+(\pi_r\alpha)^4})}{(5+\pi_r\alpha(-2+5\pi_r\alpha)+4\sqrt{1+\pi_r\alpha+(\pi_r\alpha)^3+(\pi_r\alpha)^4})^{\frac{3}{2}}}. \quad \text{(B.39)}$$

This is negative using $\pi_r > 0$ and $\alpha > 0$, so the second-order condition is satisfied.

Given a price p, the region of the parameter space defining $0 < v_{nr} < v_r < 1$ is given in part (III) of Lemma B.2. For this case, we have

$$v_{nr} = \frac{-1+\pi_r\alpha+\sqrt{1+\pi_r\alpha(-2+4p+\pi_r\alpha)}}{2\pi_r\alpha}. \quad \text{(B.40)}$$

The profit function in this case is $\Pi_{III}(p) = p(1 - v_{nr}(p))$. Let C_{III} be the compact closure of the region of the parameter space defining $0 < v_{nr} < v_r < 1$, given in part (III) of Lemma B.2. As in the previous case, there exists p in C_{III} that maximizes $\Pi_{III}(p)$. If this p is interior to C_{III}, the unconstrained maximizer satisfies the first-order condition.

Differentiating the profit function with respect to p, the first-order condition is the same as in (B.36), and solving for the positive root of the quadratic, we have that (same as in (B.37)):

$$p_{III}^* = \frac{1}{9}\left(4 - \frac{1}{\pi_r\alpha} - \pi_r\alpha + \frac{\sqrt{1+\pi_r\alpha+(\pi_r\alpha)^3+(\pi_r\alpha)^4}}{\pi_r\alpha}\right). \quad \text{(B.41)}$$

This is the same as for Case I, so the second-derivative of the profit function is the same expression and the second-order condition is similarly satisfied at this optimal price given $\pi_r > 0$ and $\alpha > 0$.

The profit corresponding to this price for this case is given by:

$$\Pi_{III}^* = \frac{\left(3+3\alpha\pi_r-\sqrt{5+\pi_r\alpha(-2+5\pi_r\alpha)+4\sqrt{1+\pi_r\alpha+(\pi_r\alpha)^3+(\pi_r\alpha)^4}}\right)}{54(\pi_r\alpha)^2}\times \left(-1+\pi_r\alpha(4-\pi_r\alpha)+\sqrt{1+\pi_r\alpha+(\pi_r\alpha)^3+(\pi_r\alpha)^4}\right). \quad \text{(B.42)}$$

Given a price p, the region of the parameter space defining $0 < v_{nr} < v_r < v_p < 1$ is given in part (IV) of Lemma B.2.

From (B.25), we have that v_{nr} is the largest root of

$$f_2(x) \triangleq \delta\pi_r\alpha x^3 + (\delta + \pi_r(R - \alpha(c_p + \delta p)))x^2 - p(2\delta + R\pi_r)x + p^2\delta = 0. \quad \text{(B.43)}$$

A generalization of the Implicit Function Theorem gives that v_{nr} is not only a smooth function of the parameters, but it is also an analytic function of the parameters so that it can be represented locally as a Taylor series of its parameters. More specifically, since $f_2'(x) \neq 0$

at the root for which v_{nr} is defined, there exists a $\delta_1 > 0$ such that for $\delta < \delta_1$, $v_{nr} = \sum_{k=0}^{\infty} a_k \delta^k$ for some sequence of coefficients α_k. Substituting $v_{nr} = \sum_{k=0}^{\infty} a_k \delta^k$ into (B.43), we have that $-a_0(a_0\pi_r\alpha c_p - a_0 R\pi_r + pR\pi_r) + \sum_{k=1}^{\infty} a_k\delta^k = 0.$

Then $a_0 = 0$ or $a_0 = \frac{pR}{R-\alpha c_p}$ are the only solutions for a_0 that make the first term zero. Now, $a_0 \neq 0$, since otherwise $v_{nr} < p$ for sufficiently low δ, which cannot happen. So $a_0 = \frac{pR}{R-\alpha c_p}$. Then substituting $v_{nr} = \frac{pR}{R-\alpha c_p} + \sum_{k=1}^{\infty} a_k\delta^k$ into (B.43) and similarly solving for a_1, we have $a_1 = \frac{c_p p\alpha^2(c_p^2\pi_r\alpha - R\pi_r(c_p+pR\pi_r))}{R(R-c_p\alpha)^3\pi_r^2}$. Continuing on this way, we have that

$$\begin{aligned} v_{nr} = \frac{pR}{R-c_p\alpha} + \frac{c_p p\alpha^2(-c_pR + c_p^2\alpha - pR^2\pi_r)\delta}{R(R-c_p\alpha)^3\pi_r} - \\ \frac{c_p p\alpha^3(-c_pR + c_p^2\alpha - pR^2\pi_r)(c_p(-2R+c_p\alpha)(-R+c_p\alpha) + pR^2(R+c_p\alpha)\pi_r)\delta^2}{R^3(R-c_p\alpha)^5\pi_r^2} + \sum_{k=3}^{\infty} a_k\delta^k. \end{aligned} \tag{B.44}$$

The profit function in this case is $\Pi_{IV}(p) = p(1-v_{nr}(p))$. Let C_{IV} be the compact closure of the region of the parameter space defining $0 < v_{nr} < v_r < v_p < 1$, given in part (IV) of Lemma B.2. There exists p in C_{IV} that maximizes $\Pi_{IV}(p)$. If this p is interior to C_{IV}, the unconstrained maximizer satisfies the first-order condition. Viewing (B.44) as a function of p, substituting this into $\Pi_{IV}(p) = p(1 - v_{nr}(p))$, and differentiating with respect to p, the first-order condition can be written as:

$$\begin{aligned} -R^3(R-c_p\alpha)^4((-1+2p)R + c_p\alpha)\pi_r^2 + c_p pR^2\alpha^2(R-c_p\alpha)^2\pi_r(2c_p(R-c_p\alpha) + 3pR^2\pi_r)\delta + \\ c_p p\alpha^3(2c_p^2(R-c_p\alpha)^2(-2R+c_p\alpha) + 9c_p pR^3(-R+c_p\alpha)\pi_r - 4p^2R^4(R+c_p\alpha)\pi_r^2)\delta^2 + \sum_{k=3}^{\infty} a_k\delta^k = 0. \end{aligned} \tag{B.45}$$

Expanding the price as a Taylor series in δ, we can then characterize the asymptotic expansion of the optimal price in the same way we had done above with v_{nr}. Omitting the algebra, we have that the interior solution satisfying the FOC above is given by

$$\begin{aligned} p^*_{IV} = \frac{R-c_p\alpha}{2R} + \frac{c_p\alpha^2(4c_p + 3R\pi_r)\delta}{8R^3\pi_r} + \\ \frac{c_p\alpha^3(16c_p^3\alpha - 4R^3\pi_r^2 + c_pR^2\pi_r(-18+5\pi_r\alpha) + 2c_p^2R(-8+9\pi_r\alpha))\delta^2}{16R^5(R-c_p\alpha)\pi_r^2} + \sum_{k=3}^{\infty} a_k\delta^k. \end{aligned} \tag{B.46}$$

The asymptotic expression for the second-derivative of the profit function with respect to p is given by:

$$\frac{d^2}{dp^2}\left[\Pi_{IV}(p)\right] = -\frac{2R}{R - c_p\alpha} + \frac{2c_p\alpha^2(c_pR - c_p^2\alpha + 3pR^2\pi_r)}{R(R - c_p\alpha)^3\pi_r}\delta + \sum_{k=2}^{\infty} a_k\delta^k. \tag{B.47}$$

This is negative for any p for sufficiently small δ (including (B.46)) using $R > c_p\alpha$ (which is a condition for the price in (B.46)) to be positive for sufficiently small δ), so the second-order condition is satisfied for sufficient small δ when (B.46)) is positive.

The profit associated with this price is given by

$$\Pi^*_{IV} = \frac{R - c_p\alpha}{4R} + \frac{c_p\alpha^2(2c_p + R\pi_r)\delta}{8R^3\pi_r} + \frac{c_p\alpha^3(32c_p^3\alpha - 4R^3\pi_r^2 + 8c_p^2R(-4 + 3\pi_r\alpha) + c_pR^2\pi_r(-24 + 5\pi_r\alpha))\delta^2}{64R^5(R - c_p\alpha)\pi_r^2} + \sum_{k=3}^{\infty} a_k\delta^k. \tag{B.48}$$

Given a price p, the region of the parameter space defining $0 < v_r < 1$ is given in part (V) of Lemma B.2.

From (B.28), we have that

$$v_r = \frac{-1 - R\pi_r + \delta\pi_r\alpha + \sqrt{4\delta\pi_r\alpha(p + R\pi_r) + (1 + R\pi_r - \delta\pi_r\alpha)^2}}{2\delta\pi_r\alpha}. \tag{B.49}$$

The vendor's profit function is given as $\Pi_V(p) = p(1 - v_r(p))$. Differentiating the profit function with respect to p, the first-order condition is given as:

$$1 - \frac{p}{\sqrt{4\alpha\delta\pi_r(p + R\pi_r) + (1 + R\pi_r - \alpha\delta\pi_r)^2}} - \frac{-1 - R\pi_r + \alpha\delta\pi_r + \sqrt{4\alpha\delta\pi_r(p + R\pi_r) + (1 + R\pi_r - \alpha\delta\pi_r)^2}}{2\alpha\delta\pi_r} = 0. \tag{B.50}$$

Solving for p in the first-order condition and looking at only the positive root gives

$$p^*_V = \Bigg(-1 - 2R\pi_r + 4\delta\pi_r\alpha - R^2\pi_r^2 - 2R\delta\alpha\pi_r^2 - (\delta\pi_r\alpha)^2 + (1 + R\pi_r + \delta\pi_r\alpha)\sqrt{1 + \pi_r(2R - \delta\alpha + (R + \delta\alpha)^2\pi_r)}\Bigg)\Bigg(9\delta\pi_r\alpha\Bigg)^{-1}. \tag{B.51}$$

The asymptotic expression for the second-derivative of the profit function with respect to p is given by:

$$\frac{d^2}{dp^2}\left[\Pi_V(p)\right] = -\frac{2}{1 + R\pi_r} + \frac{2\pi_r\alpha(-1 + 3p + R\pi_r)}{(1 + R\pi_r)^3}\delta + \sum_{k=2}^{\infty} a_k\delta^k. \tag{B.52}$$

This is negative for any p (including (B.51)) for sufficiently small δ, so the second-order condition is satisfied.

Substituting (B.51) into the profit function of this case yields the associated maximal profit of this case. Characterizing this profit expression in terms of a Taylor Series expansion,

$$\Pi_V^* = \frac{1}{4(1+R\pi_r)} + \sum_{k=1}^{\infty} a_k \delta^k. \tag{B.53}$$

Given a price p, the region of the parameter space defining $0 < v_r < v_p < 1$ is given in part (VI) of Lemma B.2. From (B.33), we have that v_r is the largest root of

$$\delta^2\alpha^2\pi_r x^3 - \alpha\delta(-1 - 2R\pi_r + (c_p+p)\delta\pi_r\alpha)x^2 + (R^2\pi_r - 2\alpha\delta(p + (c_p+p)R\pi_r))x + p^2\alpha\delta - (c_p+p)R^2\pi_r = 0. \tag{B.54}$$

Characterizing the asymptotic expansion of v_r as we had done for earlier cases, we have that

$$v_r = c_p + p - \frac{c_p^2\alpha\delta}{R^2\pi_r} + \frac{2(c_p^3\alpha^2 + c_p^3R\alpha^2\pi_r + c_p^2pR\alpha^2\pi_r)\delta}{R^4\pi_r^2} + \sum_{k=3}^{\infty} a_k \delta^k. \tag{B.55}$$

The profit function for this case is given by $\Pi_{VI} = p(1 - v_r(p))$. Viewing (B.55) as a function of p, substituting this into $\Pi_{VI}(p) = p(1 - v_r(p))$, and differentiating with respect to p, the first-order condition can be written as:

$$(1 - c_p - 2p) + \frac{c_p^2\alpha}{R^2\pi_r}\delta - \frac{2(c_p^2\alpha^2(c_p + (c_p+2p)R\pi_r))}{R^4\pi_r^2}\delta^2 + \sum_{k=3}^{\infty} a_k \delta^k = 0. \tag{B.56}$$

Expanding the price as a Taylor series in δ, we can then characterize the asymptotic expansion of the optimal price in the same way we had done above with v_r. The root of (B.56) is characterized by:

$$p_{VI}^* = \frac{1-c_p}{2} + \frac{c_p^2\alpha}{2R^2\pi_r}\delta + \sum_{k=2}^{\infty} a_k \delta^k. \tag{B.57}$$

The asymptotic expression for the second-derivative of the profit function with respect to p is given by:

$$\frac{d^2}{dp^2}\left[\Pi_{VI}(p)\right] = -2 - \frac{4(c_p\alpha)^2}{R^3\pi_r}\delta + \sum_{k=2}^{\infty} a_k \delta^k. \tag{B.58}$$

This is negative for any p (including (B.57)) for sufficiently small δ, so the second-order condition is satisfied.

The corresponding profit is given by

$$\Pi_{VI}^{*}=\frac{1}{4}\left(1-c_p\right)^2+\frac{(1-c_p)c_p^2\alpha}{2R^2\pi_r}\delta+\sum_{k=2}^{\infty}a_k\delta^k. \tag{B.59}$$

Now that we have found the interior optimal prices for these regions, we use Proposition B.1 to find conditions under which the interior optimal price for a case lies within the set of conditions defining that case. When more than one interior optimal price is within the price range defining that case, then we have to compare the profits of those cases. In what follows below, we will go through each relevant region in Proposition B.1.

First, we examine Region (i) of Proposition B.1. Below are the conditions from Region (i) of Proposition B.1 to help the reader.

(i) $R\geq\alpha(1-\delta)$ and $\pi_r\leq\frac{c_p}{\alpha}$:

- $0\leq p<1:(0<v_{nr}<1)$

This region cannot arise under the assumptions of Proposition 1 (namely the π_r condition). Specifically, when $\pi_r>\bar{\pi}_r$ (defined in later (B.64)) for sufficiently small δ, then $\pi_r\leq\frac{c_p}{\alpha}$ cannot hold since $\bar{\pi}_r>\frac{c_p}{\alpha}$ for sufficiently small δ. This rules out this region from arising under the conditions of this lemma (which are the same as the conditions of Proposition 1).

Next, we examine Region (ii) of Proposition B.1. Below are the conditions from Region (ii) of Proposition B.1 to help the reader.

(ii) $R\geq\alpha(1-\delta)$ and $\pi_r>\frac{c_p}{\alpha}$:

- $0\leq p<\frac{(1-c_p)(-c_p+\pi_r\alpha)}{\pi_r\alpha}:(0<v_{nr}<v_p<1)$
- $\frac{(1-c_p)(-c_p+\pi_r\alpha)}{\pi_r\alpha}\leq p<1:(0<v_{nr}<1)$

We will show that under the conditions of the lemma, the interior optimal price of $0<v_{nr}<1$ would lie below the range of p that defines the case in this region. Specifically, we will show that for sufficiently low δ under the conditions of this region, p_I^* (from B.37) is below $\frac{(1-c_p)(-c_p+\pi_r\alpha)}{\pi_r\alpha}$. Then we show that the vendor's price in the case of $0<v_{nr}<v_p<1$ is be interior to the range of p that defines it in this region to show that this is the equilibrium outcome in this region.

To have $p_I^*\geq\frac{(1-c_p)(-c_p+\pi_r\alpha)}{\pi_r\alpha}$ would be equivalent to the condition $\pi_r\leq\frac{(2-3c_p)c_p}{\alpha(1-2c_p)}$. However, $\frac{(2-3c_p)c_p}{\alpha(1-2c_p)}<\frac{(2-c_p)^2c_p}{(1-c_p)^2\alpha}$ using the focal region assumptions on c_p (namely, $0<c_p<2-\sqrt{3}$), so then $\frac{(2-3c_p)c_p}{\alpha(1-2c_p)}<\bar{\pi}_r$ for sufficiently low δ.

Next, we will show that $\frac{(1-c_p)(-c_p+\pi_r\alpha)}{\pi_r\alpha}$ is above an upper bound for the equilibrium price when $0<v_{nr}<v_p<1$ arises. By Lemma B.6, $v_{nr}\leq\frac{1+c_p}{2}$ when $0<v_{nr}<v_p<1$ arises in equilibrium, and by (B.97) the vendor's optimal price can be expressed as in the following way: $p^*_{II} = \frac{1}{2}v_{nr}\left(2+\pi_r\alpha v_{nr} - \sqrt{\pi_r\alpha(4c_p+\pi_r\alpha v_{nr}^2)}\right)$. In Lemma B.5, we show that this expression as a function of v_{nr} is increasing in v_{nr}. Consequently, an upper bound on the vendor's price in this case can be found by substituting $v_{nr} = \frac{1+c_p}{2}$ into the price expression. This is given below:

$$\bar{p}_{II} = \frac{1}{8}(1+c_p)\left(4+(1+c_p)\pi_r\alpha - \sqrt{\pi_r\alpha(16c_p+(1+c_p)^2\pi_r\alpha)}\right). \tag{B.60}$$

For sufficiently small δ, when $\pi_r > \bar{\pi}_r$, then $\bar{p}_{II} < \frac{(1-c_p)(-c_p+\pi_r\alpha)}{\pi_r\alpha}$. Altogether, this shows that under the assumptions of Proposition 1, when we are in Region (ii), then $0<v_{nr}<v_p<1$ is the equilibrium outcome.

Next, we examine Region (iii) of Proposition B.1. Below are the conditions from Region (iii) of Proposition B.1 to help the reader.

(iii) $\left(R \le \alpha(c_p-\delta) \text{ or } \pi_r \le \frac{c_p}{R+\alpha\delta}\right)$ and $\pi_r < \frac{1-\delta}{-R+\alpha(1-\delta)}$ and $R < \alpha(1-\delta)$:

- $0 \le p < \frac{R(1-\delta+(R-\alpha(1-\delta))\pi_r)}{\alpha(1-\delta)^2} : (0 < v_{nr} < v_r < 1)$
- $\frac{R(1-\delta+(R-\alpha(1-\delta))\pi_r)}{\alpha(1-\delta)^2} \le p < 1 : (0 < v_r < 1)$

First, we will show that for sufficiently small δ, under the conditions of this region (as well as the focal region assumptions and the assumptions of this lemma), the interior optimal price of $0<v_{nr}<v_r<1$ is above the range of p that defines it above. Then we will show that, under the same assumptions, the interior optimal price of $0<v_r<1$ is inside of the range of p that defines it in this region to show that this is the equilibrium outcome in this region.

The interior optimal price of $0<v_{nr}<v_r<1$ was given in (B.41). We will show that this is larger than $\frac{R(1-\delta+(R-\alpha(1-\delta))\pi_r)}{\alpha(1-\delta)^2}$ for sufficiently small δ. For sufficiently small δ, this bound would become arbitrarily close to $\frac{R(1+(R-\alpha)\pi_r)}{\alpha}$. The condition $p^*_{III} < \frac{R(1+(R-\alpha)\pi_r)}{\alpha}$ is equivalent to $R > \frac{-3+3\pi_r\alpha+\sqrt{5+\pi_r\alpha(-2+5\pi_r\alpha)+4\sqrt{1+\pi_r\alpha+(\pi_r\alpha)^3+(\pi_r\alpha)^4}}}{6\pi_r}$. Under the focal region assumptions on c_p and α, this bound as a function of π_r is strictly increasing. The this implies that if $p^*_{III} < \frac{R(1+(R-\alpha)\pi_r)}{\alpha}$, then a lower bound on R is just the bound above evaluated at $\pi_r = \bar{\pi}_r$. Specifically, this implies that for $p^*_{III} < \frac{R(1+(R-\alpha)\pi_r)}{\alpha}$ to hold for $\pi_r > \bar{\pi}_r$ for sufficiently small δ, we have $R > \left.\left(\frac{-3+3\pi_r\alpha+\sqrt{5+\pi_r\alpha(-2+5\pi_r\alpha)+4\sqrt{1+\pi_r\alpha+(\pi_r\alpha)^3+(\pi_r\alpha)^4}}}{6\pi_r}\right)\right|_{\pi_r=\bar{\pi}_r}$. Define this bound as $\bar{R}_{III}$.

At the same time, a condition of Region (iii) is $(R \le \alpha(c_p-\delta)$ or $\pi_r \le \frac{c_p}{R+\alpha\delta})$. First, note that $c_p\alpha < \bar{R}_{III}$ under the focal region assumptions of c_p and α, so if $R < c_p\alpha$ holds, then p^*_{III} cannot be interior to the range of p defining this case for sufficiently small δ. On the other hand, suppose $\pi_r \le \frac{c_p}{R+\alpha\delta})$ holds. Along with $\pi_r > \bar{\pi}_r$, this implies that a condition of this region for sufficiently small δ is $R < \frac{(1-c_p)\alpha}{(2-c_p)^2}$. However, $\frac{(1-c_p)\alpha}{(2-c_p)^2} < \bar{R}_{III}$ under the focal region assumptions. Therefore, when Region (iii) occurs for sufficiently low δ under $\pi_r > \bar{\pi}_r$, then p^*_{III} cannot be interior to the range of p defining that case.

Next, we will show that p^*_V is interior to the range of p that defines it in Region (iii). The p^*_V was given in (B.51), and an asymptotic expression of this for sufficiently small δ can be given as

$$p^*_V = \frac{1}{2} - \frac{\alpha\pi_r}{8(1+R\pi_r)^2}\delta + \sum_{k=2}^{\infty} a_k\delta^k. \tag{B.61}$$

We want to show that for sufficiently small δ, this falls in $\frac{R(1-\delta+(R-\alpha(1-\delta))\pi_r)}{\alpha(1-\delta)^2} \le p < 1$. That $p^*_V < 1$ is satisfied for sufficiently small δ follows from $\frac{1}{2} < 1$. To have $p^*_V \ge \frac{R(1-\delta+(R-\alpha(1-\delta))\pi_r)}{\alpha(1-\delta)^2}$ for sufficiently small δ, this is equivalent to $\pi_r > \frac{-2R+\alpha}{2R(R-\alpha)}$. That $\frac{-2R+\alpha}{2R(R-\alpha)} < \frac{(2-c_p)^2c_p}{(1-c_p)^2\alpha}$ follows from the focal region assumptions, $\pi_r > \bar{\pi}_r$, and $\left(R \le \alpha c_p \text{ or } \pi_r \le \frac{c_p}{R}\right)$ (which comes from the conditions of Region (iii) for sufficiently small δ). Therefore, for sufficiently small δ, $\pi_r > \bar{\pi}_r$ implies that $p^*_V \ge \frac{R(1-\delta+(R-\alpha(1-\delta))\pi_r)}{\alpha(1-\delta)^2}$. Consequently, when $\pi_r > \bar{\pi}_r$ and the Region (iii) conditions hold, then $0 < v_r < 1$ is the equilibrium outcome with p^*_V as the vendor's price.

Next, we examine Region (iv) of Proposition B.1. Below are the conditions from Region (iv) of Proposition B.1 to help the reader.

(iv) $\left(\pi_r \le \frac{c_p\alpha(1-\delta)^2}{R^2} \text{ or } R \le c_p\alpha(1-\delta)\right)$ and $R > \alpha(c_p-\delta)$ and $\frac{c_p}{R+\alpha\delta} < \pi_r \le \frac{c_p\alpha(1-\delta)}{(-R+\alpha(1-\delta))(R+\alpha\delta)}$:

- $0 \le p < \frac{(R+\alpha(-c_p+\delta))(-c_p+(R+\alpha\delta)\pi_r)}{(R+\alpha\delta)^2\pi_r} : (0 < v_{nr} < v_r < v_p < 1)$
- $\frac{(R+\alpha(-c_p+\delta))(-c_p+(R+\alpha\delta)\pi_r)}{(R+\alpha\delta)^2\pi_r} \le p < \frac{R(1-\delta+(R-\alpha(1-\delta))\pi_r)}{\alpha(1-\delta)^2} : (0 < v_{nr} < v_r < 1)$
- $\frac{R(1-\delta+(R-\alpha(1-\delta))\pi_r)}{\alpha(1-\delta)^2} \le p < 1 : (0 < v_r < 1)$

Under the conditions of this region, we will show that interior optimal prices for $0 < v_{nr} < v_r < 1$ and $0 < v_{nr} < v_r < v_p < 1$ and are both outside of the ranges of p that define those cases. Then we show that the interior optimal price for $0 < v_r < 1$ is interior to the range of p that defines that case to show that this is the equilibrium outcome in this region.

First, we will show that $p^*_{IV} > \frac{(R+\alpha(-c_p+\delta))(-c_p+(R+\alpha\delta)\pi_r)}{(R+\alpha\delta)^2\pi_r}$ for sufficiently small δ. The asymptotic expression for p^*_{IV} in δ is given in (B.46). Taking the limit of this as $\delta \to 0$, the limit of p^*_{IV} is $\frac{R-c_p\alpha}{2R}$. The limit of the upper bound of p for this case is $\frac{(R-c_p\alpha)(-c_p+R\pi_r)}{R^2\pi_r}$. The inequality $\frac{R-c_p\alpha}{2R} > \frac{(R-c_p\alpha)(-c_p+R\pi_r)}{R^2\pi_r}$ is equivalent to $(R-c_p\alpha)(-2c_p+R\pi_r) < 0$. Taking the limit as $\delta \to 0$ of the region conditions gives that we need $R > c_p\alpha$ and $\pi_r \le \min\left(\frac{c_p\alpha}{\pi_r^2}, \frac{c_p\alpha}{R(\alpha-R)}\right)$ to hold for sufficiently small δ to be in this region. Under these conditions, we have that $(R-c_p\alpha)(-2c_p+R\pi_r) < 0$. Therefore, for sufficiently small δ, $p^*_{IV} > \frac{(R+\alpha(-c_p+\delta))(-c_p+(R+\alpha\delta)\pi_r)}{(R+\alpha\delta)^2\pi_r}$ holds, and so the vendor's price in this region is not p^*_{IV}.

To show that p^*_{III} is larger than the upper bound of the range of p that defines that case is identical to the analysis in the previous region and will be omitted for brevity. Combined with the previous analysis, this means that the vendor's price in this region is at least $\frac{R(1-\delta+(R-\alpha(1-\delta))\pi_r)}{\alpha(1-\delta)^2}$ (the price boundary between $0<v_{nr}<v_r<1$ and $0<v_r<1$).

To show that the vendor's price is not at the boundary, we show that $p^*_V \in (\frac{R(1-\delta+(R-\alpha(1-\delta))\pi_r)}{\alpha(1-\delta)^2}, 1)$ (the range of p that defines this case). For sufficiently small δ, for p^*_V to be interior to the range of p defining it requires the same π_r condition as in the previous region since the p boundary for this case in this region is identical to that of the previous region. Specifically, we need $\pi_r > \frac{-2R+\alpha}{2R(R-\alpha)}$ to hold. This condition follows from the focal region assumptions, $\pi_r > \bar{\pi}_r$, and $\left(R \leq \alpha c_p \text{ or } \pi_r \leq \frac{c_p}{R}\right)$ (which comes from the conditions of Region (iv) for sufficiently small δ). Therefore, for sufficiently small δ, $\pi_r > \bar{\pi}_r$ implies that $p^*_V \geq \frac{R(1-\delta+(R-\alpha(1-\delta))\pi_r)}{\alpha(1-\delta)^2}$.

Altogether, when $\pi_r > \bar{\pi}_r$ and the Region (iv) conditions hold, then $0<v_r<1$ is the equilibrium outcome with p^*_V as the vendor's price.

Next, we examine Region (v) of Proposition B.1. Below are the conditions from Region (v) of Proposition B.1 to help the reader.

(v) $c_p\alpha(1-\delta) < R < \alpha(1-\delta)$ and $\frac{c_p\alpha(1-\delta)^2}{R^2} < \pi_r \leq \frac{c_p\alpha(1-\delta)}{(-R+\alpha(1-\delta))(R+\alpha\delta)}$:

- $0 \leq p \leq \frac{(-R+c_p\alpha(1-\delta))(c_p\alpha(1-\delta)^2-R^2\pi_r)}{R^2\pi_r\alpha(1-\delta)} : (0 < v_{nr} < v_p < 1)$
- $\frac{(-R+c_p\alpha(1-\delta))(c_p\alpha(1-\delta)^2-R^2\pi_r)}{R^2\pi_r\alpha(1-\delta)} < p < \frac{(R+\alpha(-c_p+\delta))(-c_p+(R+\alpha\delta)\pi_r)}{(R+\alpha\delta)^2\pi_r} : (0 < v_{nr} < v_r < v_p < 1)$
- $\frac{(R+\alpha(-c_p+\delta))(-c_p+(R+\alpha\delta)\pi_r)}{(R+\alpha\delta)^2\pi_r} \leq p < \frac{R(1-\delta+(R-\alpha(1-\delta))\pi_r)}{\alpha(1-\delta)^2} : (0 < v_{nr} < v_r < 1)$
- $\frac{R(1-\delta+(R-\alpha(1-\delta))\pi_r)}{\alpha(1-\delta)^2} \leq p < 1 : (0 < v_r < 1)$

First, we will show that $0<v_{nr}<v_r<1$ has its price outside of the range defining this case (specifically, that it is lower than the lower bound on p defining this case in this region). Then we characterize the conditions under which p^*_{IV} is interior to the range of p defining this case. Specifically, we show that p^*_{IV} is always below its upper bound but may not be above its lower bound if R is high enough. We show that when p^*_{IV} is interior, then the equilibrium price of $0<v_{nr}<v_p<1$ is above the upper bound of the range of p defining that case. Lastly, we compare Π^*_V to Π^*_{IV} as well as to the profit at the price boundary between $0<v_{nr}<v_p<1$ and $0<v_{nr}<v_r<v_p<1$ to show that Π^*_V is dominated by Π^*_{IV} when p^*_{IV} is interior and dominated by a boundary profit when p^*_{IV} is below its lower bound. Altogether in this region, if p^*_{IV} is interior to the range of p defining it in this region (namely, when R is smaller than an R bound R_3 we will characterize), then $0<v_{nr}<v_r<v_p<1$ is the outcome. Then slightly above R_3, we will be at a boundary price (since neither $0<v_{nr}<v_p<1$ nor $0<v_{nr}<v_r<v_p<1$ have their interior optimal prices within the ranges of p defining these cases in some range of R above R_3). For some sufficiently large R (ω in Proposition 1), then the price will be interior to the range of p defining $0<v_{nr}<v_p<1$ since the upper bound of the range of p for $0<v_{nr}<v_p<1$ goes to ∞ as $R \to \infty$.

First, we show that $0<v_{nr}<v_r<1$ is lower than the lower bound of the range of p defining this case. For p^*_{III} (in (B.41)) to be larger than the lower bound of p for this case for sufficiently small δ, we need $p^*_{III} > \frac{(R-c_p\alpha)(-c_p+R\pi_r)}{R^2\pi_r}$. This is equivalent to the R condition $\frac{6c_p\alpha}{3+3\pi_r\alpha+\sqrt{5+\pi_r\alpha(-2+5\pi_r\alpha)+4\sqrt{1+\pi_r\alpha+(\pi_r\alpha)^3+(\pi_r\alpha)^4}}} < R < \frac{6c_p\alpha}{3+3\pi_r\alpha-\sqrt{5+\pi_r\alpha(-2+5\pi_r\alpha)+4\sqrt{1+\pi_r\alpha+(\pi_r\alpha)^3+(\pi_r\alpha)^4}}}$. Under the conditions of the focal region, the upper bound on this R range is decreasing in π_r. This means that an upper bound on R for this to hold for any $\pi_r > \bar{\pi}_r$ is that upper bound evaluated at $\pi_r = \bar{\pi}_r$. Define $\bar{R}_{VI} = \left(\frac{6c_p\alpha}{3+3\pi_r\alpha-\sqrt{5+\pi_r\alpha(-2+5\pi_r\alpha)+4\sqrt{1+\pi_r\alpha+(\pi_r\alpha)^3+(\pi_r\alpha)^4}}}\right)\Bigg|_{\pi_r=\bar{\pi}_r}$, so that we need $R < \bar{R}_{VI}$ for p^*_{III} to be interior for $\pi_r > \bar{\pi}_r$.

At the same time, the conditions for this region as $\delta \to 0$ are $c_p\alpha < R < \alpha$ and $\frac{c_p\alpha}{R^2} < \pi_r \leq \frac{c_p\alpha}{R(\alpha-R)}$. We also need $\pi_r > \bar{\pi}_r$ to hold (since that is the π_r condition for this lemma and for Proposition 1). Together with this region's conditions, this implies that $R > \max\left(\frac{(1-c_p)\alpha}{2-c_p}, \frac{\alpha(2-c_p+\sqrt{(4-3c_p)c_p})}{2(2-c_p)}\right)$. Call this lower bound $\underline{R}_{VI}$, so that we need $R > \underline{R}_{VI}$ for the conditions of this region to hold for $\pi_r > \bar{\pi}_r$. Then note that $\bar{R}_{VI} < \underline{R}_{VI}$ under the focal region assumptions on c_p and α. Therefore, $R < \bar{R}_{VI}$ and $R > \underline{R}_{VI}$ cannot simultaneously hold for $\pi_r > \bar{\pi}_r$, and so p^*_{III} will be less than the lower bound of the range of p defining this case in this region.

Next, we find the conditions under which p^*_{IV} is interior for sufficiently low δ. For sufficiently low δ, $\frac{(-R+c_p\alpha(1-\delta))(c_p\alpha(1-\delta)^2-R^2\pi_r)}{R^2\pi_r\alpha(1-\delta)} < p < \frac{(R+\alpha(-c_p+\delta))(-c_p+(R+\alpha\delta)\pi_r)}{(R+\alpha\delta)^2\pi_r}$ becomes the condition $\frac{(-R+c_p\alpha)(c_p\alpha-R^2\pi_r)}{R^2\pi_r\alpha} < p < \frac{(R-c_p\alpha)(-c_p+R\pi_r)}{R^2\pi_r}$. For this to hold with $p = p^*_{IV}$ for sufficiently low δ, with $p^*_{IV} \to \frac{R-c_p\alpha}{2R}$ as $\delta \to 0$, we need $R\pi_r > 2c_p$ for $\frac{R-c_p\alpha}{2R} < \frac{(R-c_p\alpha)(-c_p+R\pi_r)}{R^2\pi_r}$ and $R < \frac{\alpha}{4}+\frac{\sqrt{\alpha(16c_p+\pi_r\alpha)}}{4\sqrt{\pi_r}}$ for $\frac{R-c_p\alpha}{2R} > \frac{(-R+c_p\alpha)(c_p\alpha-R^2\pi_r)}{R^2\pi_r\alpha}$. The condition of this region $\pi_r \leq \frac{c_p\alpha}{R(\alpha-R)}$ is equivalent to $R \geq \frac{\pi_r\alpha+\sqrt{\pi_r\alpha(-4c_p+\pi_r\alpha)}}{2\pi_r}$. Since $\frac{\pi_r\alpha+\sqrt{\pi_r\alpha(-4c_p+\pi_r\alpha)}}{2\pi_r} > \frac{2c_p}{\pi_r}$ under this region's conditions along with $\pi_r > \bar{\pi}_r$, then $R > \frac{2c_p}{\pi_r}$ is implied by $\pi_r \leq \frac{c_p\alpha}{R(\alpha-R)}$. Therefore, p^*_{IV} is always below its upper bound. However, in this region, it may be the case that p^*_{IV} is below its lower bound, depending on R. Specifically, in this region, there is a bound R_3 where $R_3 \to \frac{\alpha}{4} + \frac{\sqrt{\alpha(16c_p+\pi_r\alpha)}}{4\sqrt{\pi_r}}$ as $\delta \to 0$ such that p^*_{IV} is interior for $R < R_3$.

Next, we show that under the conditions of this region, if the price of $0<v_{nr}<v_r<v_p<1$ is interior, then the vendor's interior optimal price in the case of $0<v_{nr}<v_p<1$ is outside of the range that defines it (in particular, it is bigger than the upper bound of the range of p defining $0<v_{nr}<v_p<1$). This shows that if p^*_{IV} is interior to the range of p defining it in this region (namely, when $R < R_3$), then Π^*_{IV} is greater than the profit at any price in $0<v_{nr}<v_p<1$. For $0<v_{nr}<v_p<1$, the range of p is given as $0 \leq p \leq \frac{(-R+c_p\alpha(1-\delta))(c_p\alpha(1-\delta)^2-R^2\pi_r)}{R^2\pi_r\alpha(1-\delta)}$. Taking the limit as $\delta \to 0$ of the upper bound, this becomes $\frac{(-R+c_p\alpha)(c_p\alpha-R^2\pi_r)}{R^2\pi_r\alpha}$. Note that this is the lower bound of $0<v_{nr}<v_r<v_p<1$ since they are adjacent market outcomes in p. Call this price $p_{boundary}$, the boundary price between $0<v_{nr}<v_p<1$ and $0<v_{nr}<v_r<v_p<1$ (written below for sufficiently small δ outside of the limit):

$$p_{boundary} = \frac{(R - c_p\alpha)(-c_p\alpha + R^2\pi_r)}{R^2\alpha\pi_r} + \sum_{k=1}^{\infty} a_k\delta^k. \tag{B.62}$$

The proof of Lemma B.5 shows that the price $p^*_{II}(v_{nr})$ is increasing in v_{nr}, and so a lower bound on the equilibrium price in the case of $0 < v_{nr} < v_p < 1$ is $p^*(v_{nr})$ evaluated at a lower bound on v_{nr}. Lemma B.5 shows that a lower bound on v_{nr} is $\frac{1}{2}$. Using (B.97) evaluated at $v_{nr} = \frac{1}{2}$, we have that a lower bound on p^*_{II} is $\frac{1}{8}\left(4 + \pi_r\alpha - \sqrt{\pi_r\alpha(16c_p + \pi_r\alpha)}\right)$. The condition $\frac{1}{8}\left(4 + \pi_r\alpha - \sqrt{\pi_r\alpha(16c_p + \pi_r\alpha)}\right) \geq \frac{(-R+c_p\alpha)(c_p\alpha - R^2\pi_r)}{R^2\pi_r\alpha}$ is equivalent to $R > \frac{\alpha}{2}$ and $\pi_r \leq \frac{2c_p\alpha}{2R^2 - R\alpha}$ under the conditions of this region. But the condition $\pi_r \leq \frac{2c_p\alpha}{2R^2 - R\alpha}$ with $R > \frac{\alpha}{2}$ is equivalent to the condition $R \leq \frac{\alpha}{4} + \frac{\sqrt{\alpha(16c_p + \pi_r\alpha)}}{4\sqrt{\pi_r}}$. Since $R < \frac{\alpha}{4} + \frac{\sqrt{\alpha(16c_p + \pi_r\alpha)}}{4\sqrt{\pi_r}}$ for sufficiently small δ implies that p^*_{IV} is interior to the range of p that defines $0 < v_{nr} < v_r < v_p < 1$, this means that whenever p^*_{IV} is interior, then p^*_{II} is above the range that defines $0 < v_{nr} < v_p < 1$.

Next, we compare the profit at the interior optimal solution p^*_V with the profit at the interior optimal solution p^*_{IV} as well as with the profit at the boundary of $0 < v_{nr} < v_p < 1$ and $0 < v_{nr} < v_r < v_p < 1$ to show that Π^*_V is dominated in this region. First, suppose that $R < R_3$ so that p^*_{IV} is interior. Then comparing Π^*_V (shown in (B.53)) with Π^*_{IV} (shown in (B.48)) as $\delta \to 0$, we want to show that $\frac{R - c_p\alpha}{4R} > \frac{1}{4(1 + R\pi_r)}$ holds. This is equivalent to $R > \frac{1}{2}\left(c_p\alpha + \sqrt{\frac{c_p\alpha(4 + c_p\pi_r\alpha)}{\pi_r}}\right)$. Recall again that the region condition (in the limit as $\delta \to 0$) $\pi_r \leq \frac{c_p\alpha}{R(\alpha - R)}$ is equivalent to $R \geq \frac{\pi_r\alpha + \sqrt{\pi_r\alpha(-4c_p + \pi_r\alpha)}}{2\pi_r}$. That $\frac{\pi_r\alpha + \sqrt{\pi_r\alpha(-4c_p + \pi_r\alpha)}}{2\pi_r} > \frac{1}{2}\left(c_p\alpha + \sqrt{\frac{c_p\alpha(4 + c_p\pi_r\alpha)}{\pi_r}}\right)$ follows from the focal region assumptions, $\pi_r > \bar{\pi}_r$, and this region's conditions. Therefore, when p^*_{IV} is interior, then $\Pi^*_{IV} > \Pi^*_V$.

On the other hand, suppose that $R \geq R_3$. First, assume that $R = R_3$. We will show that at this point, the boundary profit between $0 < v_{nr} < v_p < 1$ and $0 < v_{nr} < v_r < v_p < 1$ is higher than Π^*_V. Then we will show that this boundary profit strictly increases in R up to the point that it reaches p^*_{II} (the interior optimal price of $0 < v_{nr} < v_p < 1$) while Π^*_V strictly decreases in R. This will show that the boundary profit dominates Π^*_V even for $R > R_3$. At p^*_{IV}, the equilibrium v_{nr} in $0 < v_{nr} < v_r < v_p < 1$ approaches $\frac{1}{2}$ as $\delta \to 0$ (from substituting (B.46) into (B.44) and taking the limit). Then evaluating the profit function (as a function of v_{nr}, written in (B.99)) of $0 < v_{nr} < v_p < 1$ at $v_{nr} = \frac{1}{2}$, comparing that to Π^*_V, and expressing that in terms of R, we want to show that $R > \frac{-\alpha(1-4c_p) + \sqrt{\alpha\left(\alpha + \frac{16c_p}{\pi_r}\right)}}{4 + 2(1 - 2c_p)\pi_r\alpha}$. That $\frac{\alpha}{4} + \frac{\sqrt{\alpha(16c_p + \pi_r\alpha)}}{4\sqrt{\pi_r}} > \frac{-\alpha(1-4c_p) + \sqrt{\alpha\left(\alpha + \frac{16c_p}{\pi_r}\right)}}{4 + 2(1 - 2c_p)\pi_r\alpha}$ for all $\pi_r > \bar{\pi}_r$ follows from the focal region assumptions on c_p and α. Therefore, at $R = R_3$, then $R > \frac{-\alpha(1-4c_p) + \sqrt{\alpha\left(\alpha + \frac{16c_p}{\pi_r}\right)}}{4 + 2(1 - 2c_p)\pi_r\alpha}$ holds.

Then for $R > R_3$, note that profit at the boundary price between the $0 < v_{nr} < v_p < 1$ and $0 < v_{nr} < v_r < v_p < 1$ weakly increases as the upper bound on the range of p for $0 < v_{nr} < v_p < 1$ increases in R (the feasible ramge of p expands in R). For some sufficiently large R (denoted

ω in Proposition 1), then the interior optimal price of $0 < v_{nr} < v_p < 1$ will indeed be interior to the range of p defining this case since the upper bound of the range of p for $0 < v_{nr} < v_p < 1$ goes to ∞ as $R \to \infty$. We can write ω as:

$$\omega = \{R : p_{boundary}(R) = p^*_{II}\}. \tag{B.63}$$

There is a unique R in that set defining ω since $p_{boundary}(R)$ is an increasing function of R under this region's conditions and the focal region assumptions while p^*_{II} does not depend in R. At the same time Π^*_V strictly decreases in R for sufficiently small δ. Hence, even for $R > R_3$, Π^*_V is dominated by either the boundary profit or the profit at the interior optimal p^*_{II}.

Altogether, if the conditions of this region are met, then if $R < R_3$ where $R_3 \to \frac{\alpha}{4} + \frac{\sqrt{\alpha(16c_p+\pi_r\alpha)}}{4\sqrt{\pi_r}}$ as $\delta \to 0$, then $0 < v_{nr} < v_r < v_p < 1$ is the equilibrium outcome. If $R_3 \leq R < \omega$, then the outcome is $0 < v_{nr} < v_p < 1$ (with the price being at $p_{boundary}$). If $R \geq \omega$, then the outcome is $0 < v_{nr} < v_p < 1$ with the price being the interior optimal price of this case.

Next, we examine Region (vi) of Proposition B.1. Below are the conditions from Region (vi) of Proposition B.1 to help the reader.

(vi) $c_p\alpha(1-\delta) < R < \alpha(1-\delta)$ and $\pi_r > \max\left(\frac{c_p\alpha(1-\delta)}{(-R+\alpha(1-\delta))(R+\alpha\delta)}, \frac{c_p\alpha(1-\delta)^2}{R^2}\right)$:

- $0 \leq p \leq \frac{(-R+c_p\alpha(1-\delta))(c_p\alpha(1-\delta)^2-R^2\pi_r)}{R^2\pi_r\alpha(1-\delta)} : (0 < v_{nr} < v_p < 1)$
- $\frac{(-R+c_p\alpha(1-\delta))(c_p\alpha(1-\delta)^2-R^2\pi_r)}{R^2\pi_r\alpha(1-\delta)} < p < \frac{R\left(2(1-\delta)\delta+R\pi_r-\sqrt{\pi_r(4c_p\alpha\delta(1-\delta)^2+R^2\pi_r)}\right)}{2\alpha\delta(1-\delta)^2}$: $(0 < v_{nr} < v_r < v_p < 1)$
- $\frac{R\left(2(1-\delta)\delta+R\pi_r-\sqrt{\pi_r(4c_p\alpha\delta(1-\delta)^2+R^2\pi_r)}\right)}{2\alpha\delta(1-\delta)^2} \leq p < 1-c_p-\frac{c_p(R+\alpha\delta(1-c_p))}{(R+\alpha\delta)^2\pi_r} : (0 < v_r < v_p < 1)$
- $1 - c_p - \frac{c_p(R+\alpha\delta(1-c_p))}{(R+\alpha\delta)^2\pi_r} \leq p < 1 : (0 < v_r < 1)$

We will rely on the analysis of the previous region to show that $0 < v_{nr} < v_p < 1$ will be the outcome for $R \geq R_3$. Then for $R < R_3$ for each of the remaining possible outcomes, there are overlaps in the region of the parameter space when the price of case is interior to the range of p defining it. When there is an overlap in which more than one interior solution is indeed interior to the range of p defining that case, we compare the profits at those interior solutions to find the conditions under which one profit dominates.

The price boundary between $0 < v_{nr} < v_p < 1$ and $0 < v_{nr} < v_r < v_p < 1$ is the same as in the previous case. Relying on the analysis of the previous region, we have that when $R > R_3$, then p^*_{IV} will not be interior to the range of p defining $0 < v_{nr} < v_r < v_p < 1$. For $R > R_3$, the analysis is the same as in Region (v), with $R = \omega$ defined to be the point at which p^*_{II} is interior to the range of p defining $0 < v_{nr} < v_p < 1$. When $R \leq R_3$, then p^*_{II} of $0 < v_{nr} < v_p < 1$ will not be interior to the range of p defining the case.

Focusing on $R < R_3$, then one of the other cases will arise. For $0 < v_{nr} < v_r < v_p < 1$, the interior optimal price p^*_{IV} needs to satisfy the condition $\frac{(-R+c_p\alpha(1-\delta))(c_p\alpha(1-\delta)^2-R^2\pi_r)}{R^2\pi_r\alpha(1-\delta)} < p < \frac{R\left(2(1-\delta)\delta+R\pi_r-\sqrt{\pi_r(4c_p\alpha\delta(1-\delta)^2+R^2\pi_r)}\right)}{2\alpha\delta(1-\delta)^2}$ to indeed be interior to the range of p defining this case. Taking the limit as $\delta \to 0$, this condition in the limit becomes $\frac{(R-c_p\alpha)(-c_p\alpha+R^2\pi_r)}{R^2\pi_r\alpha} < p^*_{IV} < -c_p + \frac{R}{\alpha}$. The condition $\frac{(R-c_p\alpha)(-c_p\alpha+R^2\pi_r)}{R^2\pi_r\alpha} < p^*_{IV}$ as $\delta \to 0$ leads to the condition $R < R_3$. The condition $p^*_{IV} < -c_p + \frac{R}{\alpha}$ as $\delta \to 0$ leads to $R > \tilde{R}_2$, where $\tilde{R}_2 \to \frac{\alpha}{2}$ as $\delta \to 0$. That $\frac{\alpha}{2} < R_3$ follows from the focal region conditions along with $\pi_r > \bar{\pi}_r$.

For $0 < v_r < v_p < 1$, the interior optimal price p^*_{VI} (given in (B.57)) needs to fall in the price range given by $\frac{R\left(2(1-\delta)\delta+R\pi_r-\sqrt{\pi_r(4c_p\alpha\delta(1-\delta)^2+R^2\pi_r)}\right)}{2\alpha\delta(1-\delta)^2} \leq p < 1-c_p-\frac{c_p(R+\alpha\delta(1-c_p))}{(R+\alpha\delta)^2\pi_r}$ to indeed be interior to the range of p defining this case. Taking the limit as $\delta \to 0$, this condition in the limit becomes $-c_p + \frac{R}{\alpha} < p^*_{VI} < 1 - c_p\left(1 + \frac{1}{R\pi_r}\right)$. The condition $-c_p + \frac{R}{\alpha} < p^*_{VI}$ as $\delta \to 0$ leads to the condition $R < \tilde{R}_3$ where $\tilde{R}_3 \to \frac{\alpha}{2}(1 + c_p)$ as $\delta \to 0$. The condition $p^*_{VI} < 1 - c_p\left(1 + \frac{1}{R\pi_r}\right)$ as $\delta \to 0$ leads to $R > \tilde{R}_4$, where $\tilde{R}_4 \to \frac{2c_p}{\pi_r(1-c_p)}$ as $\delta \to 0$. That $\tilde{R}_4 < \tilde{R}_3$ follows from the focal region conditions along with $\pi_r > \bar{\pi}_r$.

For $0 < v_r < 1$, the interior optimal price p^*_V (given in (B.51)) needs to fall in the price range given by $1 - c_p - \frac{c_p(R+\alpha\delta(1-c_p))}{(R+\alpha\delta)^2\pi_r} \leq p < 1$ to indeed be interior to the range of p defining this case. Taking the limit as $\delta \to 0$, this condition in the limit becomes $1 - c_p\left(1 + \frac{1}{R\pi_r}\right) < p^*_V < 1$. The condition $1 - c_p\left(1 + \frac{1}{R\pi_r}\right) < p^*_V$ as $\delta \to 0$ leads to the condition $R < \tilde{R}_5$ where $\tilde{R}_5 \to \frac{2c_p}{\pi_r(1-2c_p)}$ as $\delta \to 0$. The condition $p^*_V < 1$ as $\delta \to 0$ leads to $\frac{1}{2} < 1$, which is satisfied without further conditions.

From the above, we see that there will be an overlap in the parameter space for when p^*_V is interior and p^*_{VI} is interior, since $\tilde{R}_5 > \tilde{R}_4$ under the focal region assumptions on c_p and α. There will also be an overlap when p^*_{VI} is interior and p^*_{IV} is interior since $\tilde{R}_2 < \tilde{R}_3$ under the focal region assumptions on c_p and α. There may also be an overlap between when p^*_V and p^*_{IV} are interior, but as was the case in Region (v), $\Pi^*_{IV} > \Pi^*_V$, so when p^*_{IV} is interior, then Π^*_{IV} dominates the profit of $0 < v_r < 1$.

Consequently, we need to compare Π^*_V with Π^*_{VI} and Π^*_{VI} with Π^*_{IV}. These expressions are given in (B.53), (B.59), and (B.48). Comparing (B.59) and (B.48) in the limit as $\delta \to 0$, we have that $\frac{R-c_p\alpha}{4R} > \frac{1}{4}(1-c_p)^2$ if $R > \frac{\alpha}{2-c_p}$, so for sufficiently small δ, $\Pi^*_{IV} > \Pi^*_{VI}$ if $R > R_2$ where $R_2 \to \frac{\alpha}{2-c_p}$ as $\delta \to 0$. That $\tilde{R}_2 < R_2 < \tilde{R}_3$ for sufficiently small δ follows from the focal region assumptions on c_p and α.

Comparing (B.59) and (B.53) in the limit as $\delta \to 0$, we have that $\frac{1}{4(1+R\pi_r)} > \frac{1}{4}(1 - c_p)^2$ if $R < \frac{(2-c_p)c_p}{(1-c_p)^2\pi_r}$, so for sufficiently small δ, $\Pi^*_V > \Pi^*_{VI}$ if $R < R_1$ where $R_1 \to \frac{(2-c_p)c_p}{(1-c_p)^2\pi_r}$ as $\delta \to 0$. That $\tilde{R}_4 < R_1 < \tilde{R}_5$ for sufficiently small δ follows from the focal region assumptions on c_p and α. That $R_2 < R_3$ for sufficiently small δ follows from the focal region assumptions on c_p and α. That $R_1 < R_2$ follows from $\pi_r > \bar{\pi}_r$ where $\bar{\pi}_r \to \frac{(2-c_p)^2c_p}{(1-c_p)^2\alpha}$ as $\delta \to 0$. We can define the π_r bound as:

$$\bar{\pi}_r = \frac{(2-c_p)^2 c_p}{(1-c_p)^2\alpha} + \sum_{k=1}^{\infty} a_k \delta^k, \tag{B.64}$$

Altogether for this region, if $R < R_1$, then $0<v_r<1$ is the equilibrium outcome. If $R_1 \leq R < R_2$, then $0<v_r<v_p<1$ is the equilibrium outcome. If $R_2 \leq R < R_3$, then $0<v_{nr}<v_r<v_p<1$ is the equilibrium outcome. If $R_3 \leq R < \omega$, then $0<v_{nr}<v_p<1$ is the equilibrium outcome, with the price at $p_{boundary}$ (the boundary price between $0<v_{nr}<v_p<1$ and $0<v_{nr}<v_r<v_p<1$). If $R \geq \omega$, then $0<v_{nr}<v_p<1$ is the equilibrium outcome, with the price at the interior optimal solution of $0<v_{nr}<v_p<1$.

Next, we examine Region (vii) of Proposition B.1. Below are the conditions from Region (vii) of Proposition B.1 to help the reader.

(vii) $\left(\left(\pi_r < \frac{(1-\delta)\delta}{-R+c_p\alpha(1-\delta)} \text{ and } R \leq c_p\alpha(1-\delta)\right) \text{ or } \left(R > c_p\alpha(1-\delta)\right)\right)$ and $\pi_r > \frac{c_p\alpha(1-\delta)}{(-R+\alpha(1-\delta))(R+\alpha\delta)}$ and $\left(R \leq c_p\alpha(1-\delta) \text{ or } \pi_r \leq \frac{c_p\alpha(1-\delta)^2}{R^2}\right)$ and $R < \alpha(1-\delta)$:

- $0 \leq p < \frac{R\left(2(1-\delta)\delta+R\pi_r-\sqrt{\pi_r(4c_p\alpha\delta(1-\delta)^2+R^2\pi_r)}\right)}{2\alpha\delta(1-\delta)^2} : (0 < v_{nr} < v_r < v_p < 1)$
- $\frac{R\left(2(1-\delta)\delta+R\pi_r-\sqrt{\pi_r(4c_p\alpha\delta(1-\delta)^2+R^2\pi_r)}\right)}{2\alpha\delta(1-\delta)^2} \leq p < 1-c_p-\frac{c_p(R+\alpha\delta(1-c_p))}{(R+\alpha\delta)^2\pi_r} : (0 < v_r < v_p < 1)$
- $1-c_p-\frac{c_p(R+\alpha\delta(1-c_p))}{(R+\alpha\delta)^2\pi_r} \leq p < 1 : (0 < v_r < 1)$

We will show that the interior optimal price of $0<v_{nr}<v_r<v_p<1$ is outside of the range of p that defines it in this region. Then we will find the R boundary between when $0<v_r<1$ and $0<v_r<v_p<1$ arise, since both outcomes can have their respective prices interior to the range of p defining each of them under same parameter conditions.

First, note that for sufficiently small δ, the upper bound of the range of p for $0<v_{nr}<v_r<v_p<1$ is $\frac{R-c_p\alpha}{\alpha}$. Also note that for sufficiently small δ, the conditions of this region becomes $c_p\alpha < R < \alpha$ and $\frac{c_p\alpha}{R(-R+\alpha)} < \pi_r < \frac{c_p\alpha}{R^2}$. Under these conditions, $\frac{R-c_p\alpha}{\alpha} < \frac{R-c_p\alpha}{2R}$ (which is the limit of p^*_{IV} as $\delta \to 0$) so that p^*_{IV} (in (B.46)) is above the upper bound of the range of p that defines this case.

Next, we find conditions under which p^*_{VI} (defined in (B.57)) lies inside the range of p that defines it. Taking the limit as $\delta \to 0$ of the range of p defining this case, we need conditions for which $\frac{R-c_p\alpha}{\alpha} < p^*_{VI} < 1-c_p-\frac{c_p}{R\pi_r}$. The limit of p^*_{VI} as $\delta \to 0$ is $\frac{1-c_p}{2}$, and since $\frac{1-c_p}{2} > \frac{R-c_p\alpha}{\alpha}$ holds under $c_p\alpha < R < \alpha$ and $\frac{c_p\alpha}{R(-R+\alpha)} < \pi_r < \frac{c_p\alpha}{R^2}$, it follows that p^*_{VI} is above its lower bound for sufficiently small δ. Comparing it to its upper bound, for sufficiently small δ, we need $\frac{1-c_p}{2} < 1-c_p-\frac{c_p}{R\pi_r}$. This becomes $R > \frac{2c_p}{\pi_r(1-c_p)}$. Then this is the only condition needed for p^*_{VI} to be interior to the range of p defining it in this region.

Similarly, we find conditions under which p^*_V (given in (B.51)) lies inside the range of p that defines it. Taking the limit as $\delta \to 0$, we have $p^*_V \to \frac{1}{2}$. Also taking the limit of the lower bound of p that defines this case in this region, we have that the condition for p^*_V to be interior to the range of p that defines if for sufficiently small δ is $1-c_p-\frac{c_p}{R\pi_r}<p_V^*<1$. This simplifies to $R<\frac{2c_p}{\pi_r(1-2c_p)}$ (noting that $c_p<\frac{1}{2}$ from the focal region conditions on c_p). Since $\frac{1}{2}<1$ holds, p_v^* will be less than the upper bound of the range of p that defines this case for sufficiently small δ. Hence, the only condition needed for p to be interior is $R<\frac{2c_p}{\pi_r(1-2c_p)}$.

Then note that $\frac{2c_p}{\pi_r(1-2c_p)}>\frac{2c_p}{\pi_r(1-c_p)}$, so there is a region over which both prices are interior to the range of p defining their respective cases. This means that we have to compare the interior optimal profits within each of the cases to find conditions under which one case dominates the other when $\frac{2c_p}{\pi_r(1-2c_p)}>\frac{2c_p}{\pi_r(1-c_p)}$ holds. The profit at p_V^* is given in (B.53) while the profit at p_{VI}^* is given in (B.59). The limit of Π_V^* as $\delta\to 0$ is $\frac{1}{4(1+R\pi_r)}$ while the limit of Π_{VI}^* is $\frac{1}{4}(1-c_p)^2$. Since $\frac{1}{4(1+R\pi_r)}\geq\frac{1}{4}(1-c_p)^2$ iff $R\leq\frac{(2-c_p)c_p}{(1-c_p)^2\pi_r}$, this implies that for sufficiently small δ, there is a threshold R_1 such that $\Pi_V^*\geq\Pi_{VI}^*$ iff $R\leq R_1$, where $R_1\to\frac{(2-c_p)c_p}{(1-c_p)^2\pi_r}$ as $\delta\to 0$. Lastly, note that $\frac{2c_p}{\pi_r(1-c_p)}<\frac{(2-c_p)c_p}{(1-c_p)^2\pi_r}<\frac{2c_p}{\pi_r(1-2c_p)}$ holds from $\pi_r\in(0,1)$ and the focal region assumption on c_p.

Altogether, this means that when the conditions of this region arise, if $R\leq R_1$ (where $R_1\to\frac{(2-c_p)c_p}{(1-c_p)^2\pi_r}$ as $\delta\to 0$), then the equilibrium outcome is $0<v_r<1$ and the vendor's price is p_V^*, which is given in (B.51). Otherwise, if $R>R_1$, then the equilibrium outcome is is $0<v_r<v_p<1$ and the vendor's price is p_{VI}^*, which is given in (B.57).

Next, we examine Region (viii) of Proposition B.1. Below are the conditions from Region (viii) of Proposition B.1 to help the reader.

(viii) $R\leq c_p\alpha(1-\delta)$ and $\pi_r\geq\frac{(1-\delta)\delta}{-R+c_p\alpha(1-\delta)}$ and $\pi_r>\frac{c_p(R+\alpha\delta(1-c_p))}{(1-c_p)(R+\alpha\delta)^2}$:

- $0\leq p<1-c_p-\frac{c_p(R+\alpha\delta(1-c_p))}{(R+\alpha\delta)^2\pi_r}:(0<v_r<v_p<1)$
- $1-c_p-\frac{c_p(R+\alpha\delta(1-c_p))}{(R+\alpha\delta)^2\pi_r}\leq p<1:(0<v_r<1)$

The analysis for the two cases of this region is nearly identical to that of the previous region and will be omitted for brevity. The difference here is that $0<v_{nr}<v_r<v_p<1$ is no longer feasible for any p in this region, which was shown in Proposition B.1. When the conditions of this region arise, if $R\leq R_1$ (where $R_1\to\frac{(2-c_p)c_p}{(1-c_p)^2\pi_r}$ as $\delta\to 0$), then the equilibrium outcome is $0<v_r<1$ and the vendor's price is p_V^*, which is given in (B.51). Otherwise, if $R>R_1$, then the equilibrium outcome is is $0<v_r<v_p<1$ and the vendor's price is p_{VI}^*, which is given in (B.57).

Lastly, we examine Region (ix) of Proposition B.1. Below are the conditions from Region (ix) of Proposition B.1 to help the reader.

(ix) $R<\alpha(1-\delta)$ and $\frac{1-\delta}{-R+\alpha(1-\delta)}\leq\pi_r\leq\frac{c_p(R+(1-c_p)\alpha\delta)}{(1-c_p)(R+\alpha\delta)^2}$:

- $0\leq p<1:(0<v_r<1)$

If the parameter conditions are such that this region arises, then the equilibrium outcome is $0<v_r<1$ and the vendor's price p_V^* is given in (B.51).

Altogether, across all the regions of Proposition B.1, we can define the following bounds on R and π_r for sufficiently small δ:

$$\bar{\pi}_r = \frac{(2-c_p)^2 c_p}{(1-c_p)^2 \alpha} + \sum_{k=1}^{\infty} a_k \delta^k, \tag{B.65}$$

$$R_1 = \frac{(2-c_p)c_p}{(1-c_p)^2 \pi_r} + \sum_{k=1}^{\infty} a_k \delta^k, \tag{B.66}$$

$$R_2 = \frac{\alpha}{2-c_p} + \sum_{k=1}^{\infty} a_k \delta^k, \tag{B.67}$$

$$R_3 = \frac{\alpha}{4} + \frac{\sqrt{\alpha(16c_p + \alpha\pi_r)}}{4\sqrt{\pi_r}} + \sum_{k=1}^{\infty} a_k \delta^k, \tag{B.68}$$

and

$$\omega = \{R : p_{boundary}(R) = p^*_{II}\}. \tag{B.69}$$

For $\pi_r > \bar{\pi}_r$,

(a) if $0 < R < R_1$, then the equilibrium consumer market structure is $0 < v_r < 1$;

(b) if $R_1 \leq R < R_2$, then the equilibrium consumer market structure is $0 < v_r < v_p < 1$;

(c) if $R_2 \leq R < R_3$, then the equilibrium consumer market structure is $0 < v_{nr} < v_r < v_p < 1$;

(d) if $R \geq R_3$, then the equilibrium consumer market structure is $0 < v_{nr} < v_p < 1$.

Furthermore, when $R_3 \leq R < \omega$ (defined in (B.63)), then the price is at $p_{boundary}$ (given in (B.62) the boundary price between $0 < v_{nr} < v_p < 1$ and $0 < v_{nr} < v_r < v_p < 1$. When $R \geq \omega$, then the equilibrium price is the interior optimal price of $0 < v_{nr} < v_p < 1$.

This concludes the proof of the lemma. ■

Lemma B.4. *There exist bounds $\tilde{\delta} > 0$ and $\hat{\omega} > R_2$ such that if $\delta < \tilde{\delta}$, then:*

(a) if $0 < R \leq \hat{R}_1$, then the equilibrium price is p^*_V and the equilibrium consumer market structure is $0 < v_r < 1$ for any π_r;

(b) if $\hat{R}_1 < R \leq \hat{R}_2$, then the equilibrium price is p^*_V and the equilibrium consumer market structure is $0 < v_r < 1$ for $\pi_r \in (0, \pi_1)$. For $\pi_r \in [\pi_1, 1]$, the equilibrium price is p^*_{VI} and the equilibrium outcome is $0 < v_r < v_p < 1$;

(c) if $\hat{R}_2 < R \leq R_2$, then the equilibrium price is p^*_{III} and the equilibrium consumer market structure is $0 < v_{nr} < v_r < 1$ for $\pi_r \in (0, \hat{\pi})$. For $\pi_r \in [\hat{\pi}, \pi_1)$, the equilibrium price is p^*_V and the equilibrium consumer market structure is $0 < v_r < 1$. For $\pi_r \in [\pi_1, 1]$, the equilibrium price is p^*_{VI} and the equilibrium consumer market structure is $0 < v_r < v_p < 1$;

(d) *if $R_2 < R \leq \hat{\omega}$, then the equilibrium price is p^*_{III} and the equilibrium consumer market structure is $0 < v_{nr} < v_r < 1$ for any $\pi_r \in (0, \hat{\pi})$. For $\pi_r \in [\hat{\pi}, \pi_2)$, the equilibrium price is p^*_V and the equilibrium consumer market structure is $0 < v_r < 1$. For $\pi_r \in [\pi_2, 1]$, the equilibrium price is p^*_{IV} and the equilibrium consumer market structure is $0 < v_{nr} < v_r < v_p < 1$,*

where the bounds on R and π_r are characterized in the proof below. The equilibrium prices p^*_V, p^*_{VI}, p^*_{III}, and p^*_{IV} are characterized in the proof of the previous lemma.

Proof of Lemma B.4: The proof of this lemma follows closely from the proof of Lemma B.3. Recall again that Lemma B.3 assumed that $\pi_r > \bar{\pi}_r$ (in (B.64)). In this lemma, we span across all π_r while focusing on $R \leq \hat{\omega}$, an R bound bigger than R_2 of Lemma B.3 that we show the existence of in the proof of this lemma. We first restate the results from Lemma B.3 in terms of π_r bounds instead of R bounds, focusing on the relevant boundaries that hold with $R \leq \hat{\omega}$. After restating the results from Lemma B.3 in terms of relevant π_r bounds, then we characterize what happens for $\pi_r \leq \bar{\pi}_r$ for $R \leq \hat{\omega}$. Combining the two scenarios ($\pi_r > \bar{\pi}_r$ and $\pi_r \leq \bar{\pi}_r$), we can characterize what happens across all π_r for $R \leq \hat{\omega}$.

The below comes directly from Lemma B.3, ignoring Regions (i) and (ii) since those regions cannot arise for sufficiently small δ under the conditions of this lemma. Specifically, Regions (i) and (ii) require the condition $R \geq \alpha(1-\delta)$ so that $R < \frac{\alpha}{2-c_p}$ cannot hold in these regions for sufficiently small δ. The bound $R < \frac{\alpha}{2-c_p}$ comes from $R < R_2$ from Lemma B.3 for sufficiently small δ. In what follows below, we first restate the results Lemma B.3 (with slight modifications due to the lemma's assumption $R \leq \hat{\omega}$ for some $\hat{\omega} > R_2$), re-expressing region boundaries in terms of π_r instead of R.

When $\pi_r > \bar{\pi}_r$ (Restating Outcomes from Lemma B.3): First, we examine Region (iii) of Proposition B.1.

When $\pi_r > \bar{\pi}_r$ and the Region (iii) conditions hold, then $0 < v_r < 1$ is the equilibrium outcome with p^*_V as the vendor's price.

Next, we examine Region (iv) of Proposition B.1.

When $\pi_r > \bar{\pi}_r$ and the Region (iv) conditions hold, then $0 < v_r < 1$ is the equilibrium outcome with p^*_V as the vendor's price.

Next, we examine Region (v) of Proposition B.1.

When $\pi_r > \bar{\pi}_r$ and the Region (v) conditions hold, then if $R < R_3$ where $R_3 \to \frac{\alpha}{4} + \frac{\sqrt{\alpha(16c_p+\pi_r\alpha)}}{4\sqrt{\pi_r}}$ as $\delta \to 0$, then $0 < v_{nr} < v_r < v_p < 1$ is the equilibrium outcome. Note $R_3 > \frac{\alpha}{2-c_p}$ for sufficiently small δ for all π_r due to the focal region assumptions. Since $\hat{\omega}$ can be chosen to be any R in $(R_2, \tilde{R}_2)$ (where $\tilde{R}_2$ will be defined later as the intersection of two region boundaries), then we can find $\hat{\omega}$ sufficiently close to R_2 so that when $\pi_r > \bar{\pi}_r$ and the Region (v) conditions hold, then $0 < v_{nr} < v_r < v_p < 1$ is the equilibrium outcome.

Next, we examine Region (vi) of Proposition B.1.

Altogether for this region when $\pi_r > \bar{\pi}_r$, if $R < R_1$, then $0 < v_r < 1$ is the equilibrium outcome. If $R_1 \leq R < R_2$, then $0 < v_r < v_p < 1$ is the equilibrium outcome. If $R_2 \leq R < R_3$, then $0 < v_{nr} < v_r < v_p < 1$ is the equilibrium outcome. Recall again that we can find $\hat{\omega}$ such that $R_2 < \hat{\omega} < R_3$, so we can focus on $R \leq \hat{\omega}$ in this last case.

Re-expressing the bounds in terms of π_r, recall R_1 from (B.66) has an asymptotic expansion in δ given by:

$$R_1 = \frac{(2-c_p)c_p}{(1-c_p)^2\pi_r} + \sum_{k=1}^{\infty} a_k\delta^k. \tag{B.70}$$

Taking the derivative with respect to π_r, the derivative is given by:

$$\frac{d}{d\pi_r}[R_1] = -\frac{(2-c_p)c_p}{(1-c_p^2)\pi_r^2} + \sum_{k=1}^{\infty} a_k\delta^k. \tag{B.71}$$

That $\frac{d}{d\pi_r}[R_1] < 0$ for sufficiently small δ means that this boundary (when viewing R as a function of π_r) is decreasing in π_r.

In terms of π_r, we can define $\pi^\dagger$ as the solution of (B.66):

$$\pi^\dagger = \frac{(2-c_p)c_p}{(1-c_p)^2 R} + \sum_{k=1}^{\infty} a_k\delta^k. \tag{B.72}$$

Then define:

$$\pi_1 = \min(\pi^\dagger, 1). \tag{B.73}$$

Since (B.72) is strictly decreasing in R for sufficiently small δ, the smallest R value at this boundary is when $\pi_r = 1$. We define $\hat{R}_1$ to be R_1 in (B.70) evaluated at $\pi_r = 1$:

$$\hat{R}_1 = \frac{(2-c_p)c_p}{(1-c_p)^2} + \sum_{k=1}^{\infty} a_k\delta^k. \tag{B.74}$$

Note that $\bar{\pi}_r < \pi_1$ from $R \in (\hat{R}_1, R_2)$ and the focal region assumptions on c_p and α. Then within this region for $\pi_r > \bar{\pi}_r$,

(a) if $0 < R \leq \hat{R}_1$, then the equilibrium outcome is $0 < v_r < 1$ for any $\pi_r > \bar{\pi}_r$;

(b) if $\hat{R}_1 < R \leq R_2$, then the equilibrium outcome is $0 < v_r < 1$ for $\pi_r \in (\bar{\pi}_r, \pi_1)$, and the equilibrium outcome is $0 < v_r < v_p < 1$ for $\pi_r \in [\pi_1, 1]$;

(c) if $R_2 < R \leq \hat{\omega}$, then the equilibrium outcome is $0 < v_{nr} < v_r < v_p < 1$ for $\pi_r \geq \bar{\pi}_r$,

Next, we examine Region (vii) of Proposition B.1.

When the conditions of Region (vii) arise along with $\pi_r > \bar{\pi}_r$, if $R \leq R_1$ (where $R_1 \to \frac{(2-c_p)c_p}{(1-c_p)^2\pi_r}$ as $\delta \to 0$), then the equilibrium outcome is $0 < v_r < 1$ and the vendor's price is p_V^*, which is given in (B.51). Otherwise, if $R > R_1$, then the equilibrium outcome is is $0 < v_r < v_p < 1$ and the vendor's price is p_{VI}^*, which is given in (B.57). In essence, this is the same as Region (vi), with the same π_r boundary between $0 < v_r < 1$ and $0 < v_r < v_p < 1$, given in (B.72).

Next, we examine Region (viii) of Proposition B.1.

When the conditions of this region arise and $\pi_r > \bar{\pi}_r$, if $R \leq R_1$ (where $R_1 \to \frac{(2-c_p)c_p}{(1-c_p)^2\pi_r}$ as $\delta \to 0$), then the equilibrium outcome is $0 < v_r < 1$ and the vendor's price is p_V^*, which is given in (B.51). Otherwise, if $R > R_1$, then the equilibrium outcome is is $0 < v_r < v_p < 1$ and the vendor's price is p_{VI}^*, which is given in (B.57). This gives the same boundary between $0 < v_r < 1$ and $0 < v_r < v_p < 1$ as in the previous two regions.

Next, we examine Region (ix) of Proposition B.1.

If the parameter conditions are such that this region arises, then the equilibrium outcome is $0 < v_r < 1$ and the vendor's price p_V^* is given in (B.51).

Now that we have characterized the part of this lemma for $\pi_r > \bar{\pi}_r$, we now characterize what happens for $\pi_r \leq \bar{\pi}_r$ within each of these relevant regions. Most of the analysis is the same as in Lemma B.3, and we will omit the analysis that is the same as in the proof of that lemma. The main difference between the outcomes in $\pi_r > \bar{\pi}_r$ and the outcomes in $\pi_r \leq \bar{\pi}_r$ is that now the interior-optimal price of $0 < v_{nr} < v_r < 1$ can indeed now be interior to the range of p defining that case. We will see how that impacts the characterization of the equilibrium outcomes below, and then we will combine the two sets of results (for $\pi_r > \bar{\pi}_r$ and $\pi_r \leq \bar{\pi}_r$) to get the statement of this lemma.

When $\pi_r \leq \bar{\pi}_r$: First, we examine Region (iii) of Proposition B.1.

In this region, we will show that there is an overlap in the region of the parameter space over which p_{III}^* and p_V^* are interior to their respective regions. Therefore the boundary between the two cases will be determined by the isoprofit curve (found by finding when the profits of the two cases are equal at their interior optimal prices).

First, note that for sufficiently small δ, we want to find conditions so that p_{III}^* is in $0 \leq p < \frac{R(1-\delta+(R-\alpha(1-\delta))\pi_r)}{\alpha(1-\delta)^2}$. The upper bound goes to $\frac{R(1-(\alpha-R)\pi_r)}{\alpha}$ as $\delta \to 0$, so we want to find conditions so that $p_{III}^* \in (0, \frac{R(1-(\alpha-R)\pi_r)}{\alpha})$ (where p_{III}^* is in (B.41)). That $p_{III}^* > 0$ follows from the focal region assumptions on c_p and α. The condition $p_{III}^* < \frac{R(1-(\alpha-R)\pi_r)}{\alpha}$ is equivalent to $R > \frac{-3+3\pi_r\alpha+\sqrt{5+\pi_r\alpha(-2+5\pi_r\alpha)+4\sqrt{1+\pi_r\alpha+(\pi_r\alpha)^3+(\pi_r\alpha)^4}}}{6\pi_r}$.

Next, recall p_V^* was given earlier in (B.51). The limit of (B.51) as $\delta \to 0$ is $\frac{1}{2}$, so for sufficiently small δ, $p_V^* < 1$. The condition $p_V^* \geq \frac{R(1-(\alpha-R)\pi_r)}{\alpha}$ is equivalent to $R \leq \frac{-1+\pi_r\alpha+\sqrt{1+(\pi_r\alpha)^2}}{2\pi_r}$. That $\frac{-1+\pi_r\alpha+\sqrt{1+(\pi_r\alpha)^2}}{2\pi_r} > \frac{-3+3\pi_r\alpha+\sqrt{5+\pi_r\alpha(-2+5\pi_r\alpha)+4\sqrt{1+\pi_r\alpha+(\pi_r\alpha)^3+(\pi_r\alpha)^4}}}{6\pi_r}$ follows from the focal region assumptions on c_p and α. Hence, for sufficiently small δ, there is an overlap in the regions of the parameter space over which p_{III}^* is interior to the range of p defining $0 < v_{nr} < v_r < 1$ and p_V^* is interior to the range of p defining $0 < v_r < 1$.

We can find the R boundary (or equivalently, the π_r boundary, as subsequently shown) between these two cases by equating their profits, (B.42) and (B.53). For sufficiently low δ, the boundary between these two cases can be expressed as

$$\tilde{R}_1 = A_0 + \sum_{k=1}^{\infty} a_k \delta^k, \text{ where} \tag{B.75}$$

$$\begin{aligned}
A_0 = \Bigg(& -6\alpha^3\pi_r^3 + 2\alpha\pi_r(9 + 3\sqrt{1+\alpha\pi_r+\alpha^3\pi_r^3+\alpha^4\pi_r^4} - \\
& 4\sqrt{5+\alpha\pi_r(-2+5\alpha\pi_r)+4\sqrt{1+\alpha\pi_r+\alpha^3\pi_r^3+\alpha^4\pi_r^4}}) - \\
& 2(-1+\sqrt{1+\alpha\pi_r+\alpha^3\pi_r^3+\alpha^4\pi_r^4})(-3+\sqrt{5+\alpha\pi_r(-2+5\alpha\pi_r)+4\sqrt{1+\alpha\pi_r+\alpha^3\pi_r^3+\alpha^4\pi_r^4}}) + \\
& \alpha^2\pi_r^2(-9+2\sqrt{5+\alpha\pi_r(-2+5\alpha\pi_r)+4\sqrt{1+\alpha\pi_r+\alpha^3\pi_r^3+\alpha^4\pi_r^4}})\Bigg) \times \\
& \Bigg(2\pi_r(-1+\alpha\pi_r(4-\alpha\pi_r)+\sqrt{1+\alpha\pi_r+\alpha^3\pi_r^3+\alpha^4\pi_r^4})(-3-3\alpha\pi_r + \\
& \sqrt{5+\alpha\pi_r(-2+5\alpha\pi_r)+4\sqrt{1+\alpha\pi_r+\alpha^3\pi_r^3+\alpha^4\pi_r^4}})\Bigg)^{-1}.
\end{aligned} \tag{B.76}$$

A_0 as a function of π_r is strictly increasing for $\pi_r > 0$, so this means that (viewing the boundary as a function of π_r so that $\tilde{R}_1 = f(\pi_r)$ is the boundary between the two regions), the function is invertible for sufficiently small δ. Define

$$\hat{\pi} \triangleq f^{-1}(\tilde{R}_1) \tag{B.77}$$

as the inverse of this R boundary above. Note $\hat{\pi} > 0$ for $R > R_2$ since $0 < A_0|_{\pi_r=0} < R_2$ and A_0 is increasing in π_r. Then for $\pi_r < \hat{\pi}$, the profit of $0 < v_{nr} < v_r < 1$ dominates the profit under $0 < v_r < 1$, while $0 < v_r < 1$ dominates $0 < v_{nr} < v_r < 1$ for $\pi_r > \hat{\pi}$. Note that $\hat{\pi} < \bar{\pi}$, since $0 < v_{nr} < v_r < 1$ did not arise for $\pi_r > \bar{\pi}_r$.

Since the boundary between $0 < v_{nr} < v_r < 1$ and $0 < v_r < 1$ (given in (B.77)) is strictly increasing in π_r, it follows that the smallest R value at this boundary is when $\pi_r = 0$. Evaluating (B.76) at $\pi_r = 0$, we define $\hat{R}_2$ as

$$\hat{R}_2 = \frac{\alpha}{2} + \sum_{k=1}^{\infty} a_k \delta^k. \tag{B.78}$$

Altogether within this region for $\pi_r \leq \bar{\pi}_r$,

(a) if $0 < R \leq \hat{R}_2$, then the equilibrium outcome is $0 < v_r < 1$ for any $\pi_r \leq \bar{\pi}_r$;

(b) if $R > \hat{R}_2$, then the equilibrium outcome is $0 < v_{nr} < v_r < 1$ for $\pi_r \in (0, \hat{\pi})$, and the equilibrium outcome is $0 < v_r < 1$ for $\pi_r \in [\hat{\pi}, \bar{\pi}_r]$.

Next, we examine Region (iv) of Proposition B.1.

Similar to the Proof of Lemma B.3, the interior optimal price for $0 < v_{nr} < v_r < v_p < 1$ will be outside of the range of p that defines this case. However, in contrast to when $\pi_r > \bar{\pi}_r$, when $\pi_r \leq \bar{\pi}_r$, then the interior optimal price of $0 < v_{nr} < v_r < 1$ can indeed be inside of the range of p that defines the case.

The analysis is the same as in the previous region and will be omitted for brevity. In particular, the boundary between $0 < v_{nr} < v_r < 1$ and $0 < v_r < 1$ is given in (B.77).

Next, we examine Region (v) of Proposition B.1.

Similar to the Proof of Lemma B.3, $0 < v_{nr} < v_p < 1$ will only arise for $R > R_3$ (from (B.68)). However, for any π_r, a bound $\hat{\omega} < R_3$ can be chosen such that $\hat{\omega} > R_2$ since $R_2 < R_3$ under the focal region assumptions. Since R_3 is decreasing in π_r for sufficiently low δ, we can choose $\hat{\omega}$ such that $\hat{\omega} < \bar{R}_3$ where $\bar{R}_3 = R_3|_{\pi_r=1}$ so that $0 < v_{nr} < v_p < 1$ does not arise in this region in this lemma.

In contrast to when $\pi_r > \bar{\pi}_r$, when $\pi_r \leq \bar{\pi}_r$, then the interior optimal price of $0 < v_{nr} < v_r < 1$ can indeed be inside of the range of p that defines the case. The analysis between $0 < v_{nr} < v_r < 1$ and $0 < v_r < 1$ is the same as in Region (iii). In particular, the boundary between $0 < v_{nr} < v_r < 1$ and $0 < v_r < 1$ is given in (B.77).

Next, we examine the boundary between $0 < v_r < 1$ and $0 < v_{nr} < v_r < v_p < 1$. The interior optimal price of $0 < v_r < 1$ is p_V^* given in (B.51). As $\delta \to 0$, this approaches $\frac{1}{2}$, so for sufficiently small δ, we want conditions so that this limit is in the price range defined in Region (v) for this case: $\frac{R(1-\delta+(R-\alpha(1-\delta))\pi_r)}{\alpha(1-\delta)^2} \leq p < 1$. Taking the limit as $\delta \to 0$ of the lower bound gives $\frac{R(1-(\alpha-R)\pi_r)}{\alpha} \leq p$ as a condition p_V^* needs to satisfy to be interior. This is equivalent to $\pi_r \geq \frac{2R-\alpha}{2R(\alpha-R)}$. On the other hand, for p_{IV}^* to be interior, we need $\frac{2c_p}{R} < \pi_r < \frac{2c_p\alpha}{2R^2-R\alpha}$ (doing the same algebra with p_{IV}^* in (B.46) being in the range of p for this case). For there to be an overlap in the parameter region over which both p_V^* and p_{IV}^* can be interior solutions, we would need $\frac{2R-\alpha}{2R(\alpha-R)} < \frac{2c_p\alpha}{2R^2-R\alpha}$ to hold. This is equivalent to $\frac{1}{2}\left(\alpha(1-c_p) - \sqrt{c_p(2+c_p)\alpha^2}\right) < R < \frac{1}{2}\left(\alpha(1-c_p) + \sqrt{c_p(2+c_p)\alpha^2}\right)$. Note that $\frac{1}{2}\left(\alpha(1-c_p) - \sqrt{c_p(2+c_p)\alpha^2}\right) < \alpha$ and $\frac{1}{2}\left(\alpha(1-c_p) + \sqrt{c_p(2+c_p)\alpha^2}\right) > c_p\alpha$ under the focal region assumptions, so there is a region of Region (v) over which p_V^* and p_{IV}^* can both be interior to their respective ranges of p.

Consequently in this region, we have to compare the profits at the interior-optimal solutions of the cases to find the boundary between these cases. Equating the profits given in (B.48) and (B.53), the boundary between $0 < v_r < 1$ and $0 < v_{nr} < v_r < v_p < 1$ is given by:

$$\pi_2 = \frac{c_p\alpha}{R^2 - c_pR\alpha} + \sum_{k=1}^{\infty} a_k\delta^k. \tag{B.79}$$

Then the profit of $0 < v_r < 1$ dominates the profit under $0 < v_{nr} < v_r < v_p < 1$ if and only if $\pi_r < \pi_2$.

We want to show that $\pi_2 > \hat{\pi}$ (where $\hat{\pi}$ comes from (B.77)). It suffices to show that sufficiently close to the boundary $R = R_2$, we have $\pi_2 < 1$ and $\Pi_V^* - \Pi_{III}^* > 0$ at $\pi_r = \pi_2$. That $\pi_2 < 1$ for sufficiently low δ follows from conditions on c_p and α from the focal region assumptions and $R \geq R_2$. Similarly, taking the difference between the profits and using the focal region assumptions, we examine the limit as $\delta \to 0$ of $\Pi_V^* - \Pi_{III}^*$ for π_r sufficiently close to $\pi_r = \pi_2$ when $R = R_2$. The limit as $\delta \to 0$ of the difference between these two profits at $\pi_r = \pi_2$ and $R = R_2$ is given below:

$$\lim_{\delta \to 0} \left(\Pi_V^* - \Pi_{III}^* \right) \Big|_{R=R_2, \pi_r=\pi_2} = \Bigg(\Bigg((c_p-1)^4 + (c_p-2)^2 c_p ((c_p-6)(c_p-2)c_p - 4) - \\ ((c_p-2)(c_p-1)c_p+1)\sqrt{(c_p-2)(c_p-1)c_p(c_p((c_p-6)c_p+11)-4)+1} \Bigg) \Bigg(-(c_p-1)^2 \\ \Bigg(\frac{5c_p^2(c_p-2)^4}{(c_p-1)^4} - \frac{2c_p(c_p-2)^2}{(c_p-1)^2} + \\ \frac{4((c_p-2)(c_p-1)c_p+1)\sqrt{(c_p-2)(c_p-1)c_p(c_p((c_p-6)c_p+11)-4)+1}}{(c_p-1)^4} + 5 \Bigg)^{\frac{1}{2}} + \\ 3c_p(c_p-2)^2 + 3(c_p-1)^2 \Bigg) \Bigg) \left(54(c_p-2)^4(c_p-1)^2 c_p^2 \right)^{-1} + \frac{1}{4}(c_p-1)^2. \quad \text{(B.80)}$$

The limit of this difference is positive from the focal region assumption on c_p. This implies that, for sufficiently small δ, $\pi_2 > \hat{\pi}$ for R sufficiently close to R_2. We also have π_2 being a decreasing function of R (under the focal region assumptions and $R \geq R_2$) and $\hat{\pi}$ increasing in R (from the analysis in Region (iii)). Lastly, note that R_3 (in (B.68)) is strictly decreasing in π_r for sufficiently small δ, so the smallest R_3 can be is when $\pi_r = 1$. Altogether, we can define $\hat{\omega}$ as

$$\hat{\omega} = \sup_R \{R : \hat{\pi}(R) \leq \pi_2(R) \text{ and } R < \bar{R}_3\}, \text{ where} \quad \text{(B.81)}$$

$$\bar{R}_3 = R_3|_{\pi_r=1}. \quad \text{(B.82)}$$

Note that $\hat{\omega} > R_2$, since $\hat{\pi} < \pi_2$ for R sufficiently close to R_2. The comparison of the profits Π_{IV}^* and Π_{III}^* only happens for $R > \hat{\omega}$ due to the ordering of the profits Π_{III}^*, Π_{IV}^* and Π_V^* when $R \leq \hat{\omega}$, so the boundary between those two cases does not appear for $R \leq \hat{\omega}$.

Altogether in this region when $\pi_r \leq \bar{\pi}_r$ and $R < \hat{\omega}$, then the equilibrium outcome is $0 < v_{nr} < v_r < 1$ for any $\pi_r \in (0, \hat{\pi})$, the equilibrium outcome is $0 < v_r < 1$ for $\pi_r \in [\hat{\pi}, \pi_2)$, and the equilibrium outcome is $0 < v_{nr} < v_r < v_p < 1$ for $\pi_r \in [\pi_2, \bar{\pi}_r]$.

Next, we examine Region (vi) of Proposition B.1.

We can follow the proof of Lemma B.3 to rule out $0 < v_{nr} < v_p < 1$ from appearing in equilibrium in this region. Also from the proof of Lemma B.3, a condition needed for $R_1 < R_2$ was $\pi_r > \bar{\pi}_r$. In other words, with $\pi_r \leq \bar{\pi}_r$ in this section, we do not have $R_1 < R_2$. As a result, $0 < v_r < v_p < 1$ can be ruled out. Consequently, the only two cases to consider in this region for $\pi_r < \bar{\pi}_r$ and $R \leq \hat{\omega}$ are $0 < v_{nr} < v_r < v_p < 1$ and $0 < v_r < 1$. The analysis is the same as in the previous region, with the same boundary between the two outcomes.

Altogether within this region when $\pi_r \leq \bar{\pi}_r$ and $R < \hat{\omega}$, then the equilibrium outcome is $0 < v_r < 1$ for $\pi_r < \pi_2$, and the equilibrium outcome is $0 < v_{nr} < v_r < v_p < 1$ for $\pi_r \in [\pi_2, \bar{\pi}_r]$.

Next, we examine Region (vii) of Proposition B.1.

When the conditions of Region (vii) arise along with $\pi_r \leq \bar{\pi}_r$, then the boundaries between regions are the same as for $\pi_r > \bar{\pi}_r$. In particular, if $R \leq R_1$ (where $R_1 \to \frac{(2-c_p)c_p}{(1-c_p)^2\pi_r}$ as $\delta \to 0$), then the equilibrium outcome is $0 < v_r < 1$ and the vendor's price is p_V^*, which is given in (B.51). Otherwise, if $R > R_1$, then the equilibrium outcome is is $0 < v_r < v_p < 1$ and the vendor's price is p_{VI}^*, which is given in (B.57). The π_r boundary between $0 < v_r < 1$ and $0 < v_r < v_p < 1$ is given in (B.72).

Next, we examine Region (viii) of Proposition B.1.

When the conditions of this region arise and $\pi_r \leq \bar{\pi}_r$, then the boundaries between regions are the same as for $\pi_r > \bar{\pi}_r$. In particular, if $R \leq R_1$, then the equilibrium outcome is $0 < v_r < 1$ and the vendor's price is p_V^*, which is given in (B.51). Otherwise, if $R > R_1$, then the equilibrium outcome is is $0 < v_r < v_p < 1$ and the vendor's price is p_{VI}^*, which is given in (B.57). This gives the same boundary between $0 < v_r < 1$ and $0 < v_r < v_p < 1$ as in the previous region.

Next, we examine Region (ix) of Proposition B.1.

If the parameter conditions are such that this region arises, then the equilibrium outcome is $0 < v_r < 1$ and the vendor's price p_V^* is given in (B.51).

Now that we have characterized the equilibrium outcomes across all the relevant regions that arise when $R \leq \hat{\omega}$ across both $\pi_r > \bar{\pi}_r$ and $\pi_r \leq \bar{\pi}_r$, we can organize all of the above in the following way. First, we list the relevant bounds on π_r and R again:

$$\pi^\dagger = \frac{(2-c_p)c_p}{(1-c_p)^2 R} + \sum_{k=1}^{\infty} a_k \delta^k, \tag{B.83}$$

$$\pi_1 = \min(\pi^\dagger, 1), \tag{B.84}$$

$$\hat{R}_1 = \frac{(2-c_p)c_p}{(1-c_p)^2} + \sum_{k=1}^{\infty} a_k \delta^k, \tag{B.85}$$

$$\tilde{R}_1 = A_0 + \sum_{k=1}^{\infty} a_k \delta^k, \text{ where} \tag{B.86}$$

$$A_0 = \Bigg(-6\alpha^3\pi_r^3 + 2\alpha\pi_r(9 + 3\sqrt{1+\alpha\pi_r+\alpha^3\pi_r^3+\alpha^4\pi_r^4} - 4\sqrt{5+\alpha\pi_r(-2+5\alpha\pi_r)+4\sqrt{1+\alpha\pi_r+\alpha^3\pi_r^3+\alpha^4\pi_r^4}}) - 2(-1+\sqrt{1+\alpha\pi_r+\alpha^3\pi_r^3+\alpha^4\pi_r^4})(-3+\sqrt{5+\alpha\pi_r(-2+5\alpha\pi_r)+4\sqrt{1+\alpha\pi_r+\alpha^3\pi_r^3+\alpha^4\pi_r^4}}) + \alpha^2\pi_r^2(-9+2\sqrt{5+\alpha\pi_r(-2+5\alpha\pi_r)+4\sqrt{1+\alpha\pi_r+\alpha^3\pi_r^3+\alpha^4\pi_r^4}})\Bigg) \times \Bigg(2\pi_r(-1+\alpha\pi_r(4-\alpha\pi_r)+\sqrt{1+\alpha\pi_r+\alpha^3\pi_r^3+\alpha^4\pi_r^4})(-3-3\alpha\pi_r + \sqrt{5+\alpha\pi_r(-2+5\alpha\pi_r)+4\sqrt{1+\alpha\pi_r+\alpha^3\pi_r^3+\alpha^4\pi_r^4}})\Bigg)^{-1}. \tag{B.87}$$

$$\hat{\pi} \triangleq f^{-1}(\tilde{R}_1), \tag{B.88}$$

$$\pi_2 = \frac{c_p\alpha}{R^2 - c_pR\alpha} + \sum_{k=1}^{\infty} a_k\delta^k, \tag{B.89}$$

$$\hat{R}_2 = \frac{\alpha}{2} + \sum_{k=1}^{\infty} a_k\delta^k, \tag{B.90}$$

$$R_2 = \frac{\alpha}{2-c_p} + \sum_{k=1}^{\infty} a_k\delta^k, \tag{B.91}$$

$$R_3 = \frac{\alpha}{4} + \frac{\sqrt{\alpha(16c_p+\alpha\pi_r)}}{4\sqrt{\pi_r}} + \sum_{k=1}^{\infty} a_k\delta^k, \tag{B.92}$$

$$\bar{R}_3 = R_3|_{\pi_r=1}, \text{ and} \tag{B.93}$$

$$\hat{\omega} = \sup_R\{R : \hat{\pi}(R) \leq \pi_2(R) \text{ and } R < \bar{R}_3\}. \tag{B.94}$$

Note that $\hat{R}_1 < \hat{R}_2$ under the conditions of the focal region, with $0 < c_p < 2-\sqrt{3}$ and $\frac{2}{(1-c_p)^2} - 2 < \alpha < 2(2-c_p)^2$ for sufficiently small δ (namely, $\frac{(2-c_p)c_p}{(1-c_p)^2} < \frac{\alpha}{2}$ under the focal region assumptions on c_p and α). Also, $\hat{R}_2 < R_2$ for sufficiently low δ (from $\frac{\alpha}{2} < \frac{\alpha}{2-c_p}$). That $\pi_1 > \hat{\pi}$ follows from $\pi_1 > \bar{\pi}$ (given in (B.64)) for $R < R_2$ and $\hat{\pi} < \bar{\pi}$ for $R < R_2$ (shown in the analysis of Region (iii) when $\pi_r < \bar{\pi}_r$).

Altogether, for sufficiently small δ, we can find $\hat{\omega} > R_2$ such that:

(a) if $0 < R \leq \hat{R}_1$, then the equilibrium outcome is $0 < v_r < 1$ for any π_r;

(b) if $\hat{R}_1 < R \leq \hat{R}_2$, then the equilibrium outcome is $0 < v_r < 1$ for $\pi_r \in (0, \pi_1)$, and the equilibrium outcome is $0 < v_r < v_p < 1$ for $\pi_r \in [\pi_1, 1]$;

(c) if $\hat{R}_2 < R \leq R_2$, then the equilibrium outcome is $0 < v_{nr} < v_r < 1$ for $\pi_r \in (0, \hat{\pi})$, the equilibrium outcome is $0 < v_r < 1$ for $\pi_r \in [\hat{\pi}, \pi_1)$, and the equilibrium outcome is $0 < v_r < v_p < 1$ for $\pi_r \in [\pi_1, 1]$;

(d) if $R_2 < R \leq \hat{\omega}$, then the equilibrium outcome is $0 < v_{nr} < v_r < 1$ for any $\pi_r \in (0, \hat{\pi})$, the equilibrium outcome is $0 < v_r < 1$ for $\pi_r \in [\hat{\pi}, \pi_2)$, and the equilibrium outcome is $0 < v_{nr} < v_r < v_p < 1$ for $\pi_r \in [\pi_2, 1]$.

■

B.1.3 v^*_{nr} Bounds for the Pricing Subgame

In this section, we provide supporting results for the pricing subgame analysis. In particular, we derive bounds on the v_{nr} threshold under optimal pricing when $0 < v_{nr} < v_p < 1$ arises. This, in turn, produces bounds on the interior-optimal price that are used in the equilibrium characterization of the pricing subgame.

Lemma B.5. *Under the conditions of the focal region* $\Big($*specifically,* $c_p \in (0, 2 - \sqrt{3})$ *and* $\alpha \in \left(\frac{2}{(1-c_p)^2} - 2, 2(2 - c_p)^2\right)\Big)$, *if* $0 < v_{nr} < v_p < 1$ *arises in equilibrium under optimal pricing of the benchmark case, then* $v^*_{nr} \geq \frac{1}{2}$ *in equilibrium. Furthermore, if* $\pi_r \geq \frac{1}{c_p \alpha}$, *then if* $0 < v_{nr} < v_p < 1$ *arises in equilibrium under optimal pricing of the benchmark case, then* $v^*_{nr} \geq \frac{1+c_p^2}{2}$.

Proof of Lemma B.5: Suppose that $0 < v_{nr} < v_p < 1$ is induced. From (B.13), we have that v_{nr} is the largest root of the cubic:

$$f_1(x) \triangleq \pi_r \alpha x^3 + (1 - \pi_r \alpha(c_p + p))x^2 - 2px + p^2. \tag{B.95}$$

Then in equilibrium, p^*_{IV} and v_{nr} must solve $\pi_r \alpha v_{nr}^3 + (1 - \pi_r \alpha(c_p + p))v_{nr}^2 - 2pv_{nr} + p^2 = 0$, where p^*_{II} is the equilibrium price of this case. From this, we have that

$$p^*_{II} = \frac{1}{2} v_{nr} \left(2 + \pi_r \alpha v_{nr} \pm \sqrt{\pi_r \alpha(4c_p + \pi_r \alpha v_{nr}^2)}\right). \tag{B.96}$$

Can $p^*_{II} = \frac{1}{2} v_{nr} \left(2 + \pi_r \alpha v_{nr} + \sqrt{\pi_r \alpha(4c_p + \pi_r \alpha v_{nr}^2)}\right)$? Suppose it were. Then it follows that $p^*_{II} > \frac{1}{2} v_{nr} (2 + \pi_r \alpha v_{nr} + \pi_r \alpha v_{nr}) = v_{nr}(1 + \pi_r \alpha v_{nr})$. This is a contradiction, since $v_{nr} > p^*_{II}$ in equilibrium (otherwise, some purchasing consumers would derive negative utility upon purchasing). Therefore, we have in equilibrium that

$$p^*_{II} = \frac{1}{2} v_{nr} \left(2 + \pi_r \alpha v_{nr} - \sqrt{\pi_r \alpha(4c_p + \pi_r \alpha v_{nr}^2)}\right). \tag{B.97}$$

From (B.97), we have that an expression of the vendor's equilibrium price as a function of v_{nr} when $0 < v_{nr} < v_p < 1$ is induced in equilibrium.

We will first show that as a function of v_{nr}, this price $p(v_{nr})$ increases in v_{nr} to prove monotonicity for any $v_{nr} < 1$ for which $p(v_{nr}) > 0$. Then we will show that $\Pi_{II}(v_{nr})$ increases in v_{nr} for all $v_{nr} < \frac{1}{2}$. By the chain rule, this proves that $v_{nr} \geq \frac{1}{2}$ in equilibrium whenever the interior optimal price of $0 < v_{nr} < v_p < 1$ can be achieved. Then we will show that a tighter bound can be used when $\pi_r \geq \frac{1}{c_p \alpha}$. Note that $p(v_{nr}) > 0$ only for $v_{nr} > \max(0, \frac{-1+c_p\alpha\pi_r}{\alpha\pi_r})$, so consider $v_{nr} \in (\max(0, \frac{-1+c_p\alpha\pi_r}{\alpha\pi_r}), 1)$. The derivative of the price in this case, we have:

$$\frac{d}{dv_{nr}}\left[p^*_{II}(v_{nr})\right] = 1 + \pi_r\alpha v_{nr} + \frac{2c_p\alpha\pi_r}{\sqrt{\pi_r\alpha(4c_p + v_{nr}^2\pi_r\alpha)}} - \sqrt{\pi_r\alpha(4c_p + \pi_r\alpha v_{nr}^2)}. \tag{B.98}$$

The derivative is positive when $1 + \pi_r\alpha v_{nr} + \frac{2c_p\alpha\pi_r}{\sqrt{\pi_r\alpha(4c_p+v_{nr}^2\pi_r\alpha)}} - \sqrt{\pi_r\alpha(4c_p + \pi_r\alpha v_{nr}^2)} > 0$. Multiplying both sides by $\sqrt{\pi_r\alpha(4c_p + \pi_r\alpha v_{nr}^2)}$ and simplifying, this is equivalent to $(1 + \pi_r\alpha v_{nr})\sqrt{\pi_r\alpha(4c_p + \pi_r\alpha v_{nr}^2)} > \pi_r\alpha(2c_p + v_{nr}^2\pi_r\alpha)$. Squaring both sides and simplifying under the assumptions of the focal region, this simplifies to $(1 + 2v_{nr}\pi_r\alpha)(4c_p + v_{nr}^2\pi_r\alpha) > 4c_p^2\pi_r\alpha$. This holds under Assumption 1 of the focal region ($0 < c_p < 2 - \sqrt{3}$) and $v_{nr} \in (\max(0, \frac{-1+c_p\alpha\pi_r}{\alpha\pi_r}), 1)$. Therefore, $\frac{d}{dv_{nr}} p^*_{II}(v_{nr}) > 0$ for all $v_{nr} \in (\max(0, \frac{-1+c_p\alpha\pi_r}{\alpha\pi_r}), 1)$. Hence, there is a one-to-one relationship between price and v_{nr} on this range.

Next, we show that the vendor's profit as a function of v_{nr} is increasing in $v_{nr} \in (\max(0, \frac{-1+c_p\alpha\pi_r}{\alpha\pi_r}), \frac{1}{2})$. Note that $\frac{1}{2} > \frac{-1+c_p\alpha\pi_r}{\alpha\pi_r}$ under the c_p and α assumptions of the focal region. The vendor's profit function in this case can be written as

$$\Pi_{II}(v_{nr}) = p^*_{II}(1 - v_{nr}), \tag{B.99}$$

where p^*_{II} comes from (B.97). Taking the derivative of (B.99) with respect to v_{nr},

$$\frac{d}{dv_{nr}}\left[\Pi^*_{II}(v_{nr})\right] = 1 - 2v_{nr} - \frac{\alpha\pi_r(3v_{nr} - 2)v_{nr}\left(\sqrt{4c_p + \alpha\pi_r v_{nr}^2} - v_{nr}\sqrt{\alpha\pi_r}\right)}{2\sqrt{4c_p + \alpha\pi_r v_{nr}^2}} + 2c_p(2v_{nr} - 1)\sqrt{\frac{\alpha\pi_r}{4c_p + \alpha\pi_r v_{nr}^2}}. \tag{B.100}$$

To show $\frac{d}{dv_{nr}}\left[\Pi_{II}(v_{nr})\right] > 0$, we need:

$$\sqrt{4c_p + \alpha\pi_r v_{nr}^2}\left(-(\alpha\pi_r(3v_{nr} - 2)v_{nr}) - 4v_{nr} + 2\right) > - \sqrt{\alpha\pi_r}\left(c_p(8v_{nr} - 4) + \alpha\pi_r(3v_{nr} - 2)v_{nr}^2\right) \tag{B.101}$$

Both the left- and right-hand sides are positive due to the focal region assumptions and the assumption that $\max(0, \frac{-1+c_p\pi_r\alpha}{\pi_r\alpha}) < v_{nr} < \frac{1}{2}$. Then squaring both sides, we want to have that

$$(\alpha\pi_r(3v_{nr}-2)v_{nr}+4v_{nr}-2)^2\left(4c_p+\alpha\pi_r v_{nr}^2\right) > \alpha\pi_r\left(c_p(8v_{nr}-4)+\alpha\pi_r(3v_{nr}-2)v_{nr}^2\right)^2 \tag{B.102}$$

This inequality is satisfied for $\max(0, \frac{-1+c_p\pi_r\alpha}{\pi_r\alpha}) < v_{nr} < \frac{1}{2}$. Hence, $\frac{d}{dv_{nr}}\left[\Pi_{II}(v_{nr})\right] > 0$ for all $v_{nr} \in (\max(0, \frac{-1+c_p\alpha\pi_r}{\alpha\pi_r}), \frac{1}{2})$. This proves that the profit function of this case is increasing for all $v_{nr} < \frac{1}{2}$ for which $p^*_{II}(v_{nr}) > 0$.

Furthermore, we can have a tighter bound on v_{nr} when $\pi_r \geq \frac{1}{c_p\alpha}$. The left- and right-sides of (B.101) are still positive for all $\max(0, \frac{-1+c_p\pi_r\alpha}{\pi_r\alpha}) < v_{nr} < \frac{1+c_p^2}{2}$ when $\pi_r \geq \frac{1}{c_p\alpha}$. Squaring both sides gets to (B.102). Doing a substitution $v_{nr} = Q \times \frac{-1+c_p\pi_r\alpha}{\pi_r\alpha}$, this inequality becomes

$$\begin{aligned}&\left(4\alpha c_p\pi_r + Q^2(\alpha c_p\pi_r-1)^2\right)\left(Q(\alpha c_p\pi_r-1)(\alpha\pi_r(3c_pQ-2)-3Q+4)-2\alpha\pi_r\right)^2-\\&\left(4\alpha^2c_p\pi_r^2-3Q^3(\alpha c_p\pi_r-1)^3+2\alpha\pi_rQ^2(\alpha c_p\pi_r-1)^2-8\alpha c_p\pi_rQ(\alpha c_p\pi_r-1)\right)^2 > 0\end{aligned} \tag{B.103}$$

This needs to hold for all Q such that v_{nr} goes from $\frac{-1+c_p\pi_r\alpha}{\pi_r\alpha}$ to $\frac{1+c_p^2}{2}$. This means Q needs to range from $1 \leq Q \leq \frac{(1+c_p^2)\pi_r\alpha}{-2+2c_p\pi_r\alpha}$. This inequality does indeed hold for this range of Q under the focal region assumptions on c_p and α. Hence, $\frac{d}{dv_{nr}}\left[\Pi^*(v_{nr})\right]$ is positive for all $v_{nr} \in (\frac{-1+c_p\pi_r\alpha}{\pi_r\alpha}, \frac{1+c_p^2}{2})$ if $\pi_r \geq \frac{1}{c_p\alpha}$. Under this higher π_r condition, we have a tighter lower bound on v_{nr} than $v^*_{nr} \geq \frac{1}{2}$ (which applies for all π_r under the focal region assumptions).

In either case, by the chain rule, since $p^*_{II}(v_{nr})$ is an increasing function of v_{nr}, it also follows that the profit function is increasing in p for all p until at least $v_{nr} = \frac{1}{2}$ (and if $\pi_r \geq \frac{1}{c_p\alpha}$, then the profit function is increasing in p for all p until at least $v_{nr} = \frac{1+c_p^2}{2}$). Therefore, if this case is induced in equilibrium under optimal pricing, then $v^*_{nr} \geq \frac{1}{2}$ in equilibrium (and if $\pi_r \geq \frac{1}{c_p\alpha}$, then $v^*_{nr} \geq \frac{1+c_p^2}{2}$ in equilibrium). ■

Lemma B.6. *Under the conditions of the focal region, if $0 < v_{nr} < v_p < 1$ arises in equilibrium under optimal pricing of the benchmark case, then $v_{nr} \leq \frac{1+c_p}{2}$.*

Proof of Lemma B.6: Again, from (B.97), we have that an expression of the vendor's optimal price as a function of v_{nr} when $0 < v_{nr} < v_p < 1$ is induced in equilibrium is given by

$$p^*_{II} = \frac{1}{2}v_{nr}\left(2+\pi_r\alpha v_{nr}-\sqrt{\pi_r\alpha(4c_p+\pi_r\alpha v_{nr}^2)}\right). \tag{B.104}$$

The vendor's profit as a function of v_{nr} is given by $\Pi_{II}(v_{nr}) = p^*_{II}(v_{nr})(1-v_{nr})$. To prove the lemma, we will show that $\frac{d}{dv_{nr}}\Pi_{II}(v_{nr}) < 0$ for $v_{nr} \in (\frac{1+c_p}{2}, 1)$.

Using the above expression of $p^*(v_{nr})$, we have that

$$\frac{d}{dv_{nr}}\Pi_{II}(v_{nr}) = \frac{1}{2}\Bigg((-1+v_{nr})\pi_r\alpha v_{nr}\Bigg(-1+v_{nr}\sqrt{\frac{\pi_r\alpha}{4c_p+\pi_r\alpha v_{nr}^2}}\Bigg)+ (-1+2v_{nr})\Bigg(-2+\pi_r\alpha v_{nr}+\sqrt{\pi_r\alpha(4c_p+\pi_r\alpha v_{nr}^2)}\Bigg)\Bigg). \quad \text{(B.105)}$$

Now to show that this negative for all $v_{nr} \in (\frac{1+c_p}{2}, 1)$, we will show that it is negative when v_{nr} is a convex combination of $\frac{1+c_p}{2}$ and 1. In particular, substituting $v_{nr} = w + (1-w)\frac{1+c_p}{2}$ into (B.105), we will show that this expression is negative for all $w \in (0,1)$.

In particular, $\frac{d}{dv_{nr}}\Pi_{II}(v_{nr})|_{v_{nr}=w+(1-w)\frac{1+c_p}{2}} < 0$ is equivalent to

$$\pi_r\alpha(32c_p(w+c_p-wc_p)-(1+3w(-1+c_p)-3c_p)(1+w+c_p-wc_p)^2\pi_r\alpha) < \sqrt{\pi_r\alpha\,(16c_p+(1+w+c_p-wc_p)^2\pi_r\alpha)}\Bigg(8w(1-c_p)+8c_p-\pi_r\alpha+ (-2+3w(-1+c_p)-3c_p)(w(-1+c_p)-c_p)\pi_r\alpha\Bigg). \quad \text{(B.106)}$$

Now we examine several subcases.

Subcase 1: $w \geq \frac{1}{3}$. First, suppose that $w \geq \frac{1}{3}$. This implies that $\pi_r\alpha(32c_p(w+c_p-wc_p)-(1+3w(-1+c_p)-3c_p)(1+w+c_p-wc_p)^2\pi_r\alpha) > 0$ and $\Bigg(8w(1-c_p)+8c_p-\pi_r\alpha+(-2+3w(-1+c_p)-3c_p)(w(-1+c_p)-c_p)\pi_r\alpha\Bigg) > 0$ as well, for any π_r, α, and c_p in the focal region. So both the left and right side of the inequailty in (B.106) are positive. We isolate the radicand and square both sides. Simplifying and omitting the algebra, this is equivalent to

$$64c_p(w+c_p-wc_p)^2 > -(4(w(-1+c_p)-c_p)(w^3(-1+c_p)^3+(-3+c_p)c_p(-1+3c_p)+ w(-1+c_p)(1+c_p)(1+11c_p)-w^2(1-c_p)^2(2+15c_p))\pi_r\alpha+ w(1+3w(-1+c_p)-3c_p)(-1+c_p)(1+w+c_p-wc_p)^3(\pi_r\alpha)^2) \quad \text{(B.107)}$$

Now viewing the left-hand side as a constant function in α and the right-hand side as a quadratic function in α, we want to show that the quadratic in α is smaller than a constant in α. With $w \geq \frac{1}{3}$, the coefficient on α^2 on the right-hand side is negative. Then it suffices to show that the maximum of that quadratic is less than $64c_p(w+c_p-wc_p)^2$. Differentiating the right-hand side of the inequality with respect to α and solving for the maximum, we find that the maximizing α is negative. Therefore, the right-hand side of (B.107) is maximized at $\alpha = 0$, which would the right-hand side of the inequailty 0. Then to show that the inequality above holds for all $\alpha > 0$, it suffices to show that $64c_p(w+c_p-wc_p)^2 > 0$, which is true since

$c_p > 0$. $\square$

Subcase 2a: $0 \leq w < \frac{1}{3}$ and $1 - \frac{2}{3(1-w)} \leq c_p < 1$ Now suppose that $0 \leq w < \frac{1}{3}$ and $1 - \frac{2}{3(1-w)} \leq c_p < 1$. Going back to the original inequality we want to show, to show that $\frac{d}{dv_{nr}}\Pi_{II}(v_{nr})\big|_{v_{nr}=w+(1-w)\frac{1+c_p}{2}} < 0$, we need to show (B.107).

When $0 \leq w < \frac{1}{3}$ and $1 - \frac{2}{3(1-w)} \leq c_p \leq 1$, then $32c_p(w + c_p - wc_p) - (1 + 3w(-1 + c_p) - 3c_p)(1 + w + c_p - wc_p)^2\pi_r\alpha > 0$ and $8w(1 - c_p) + 8c_p - \pi_r\alpha +$ $(-2 + 3w(-1 + c_p) - 3c_p)(w(-1 + c_p) - c_p)\pi_r\alpha > 0$. Then similar to Subcase 1, both the left-hand side and right-hand side of the inequality are positive. We isolate the radicand and square both sides. Omitting the algebra, we again want to show the inequality (B.107).

When $c_p > 1 - \frac{2}{3(1-w)}$, the coefficient on the quadratic α term of (B.107) is negative, and the same argument as Subcase 1 applies to show that the inequality holds for all $\alpha > 0$ when $0 \leq w < \frac{1}{3}$ and $1 - \frac{2}{3(1-w)} \leq c_p \leq 1$.

On the other hand, if $c_p = 1 - \frac{2}{3(1-w)}$, then (B.106) reduces to $\pi_r\alpha(1 - 3w) - (1 - w)\sqrt{\frac{\pi_r\alpha(3+\pi_r\alpha-w(9+\pi_r\alpha))}{1-w}} < 0$, which holds for $\pi_r\alpha > 0$ and $0 \leq w < \frac{1}{3}$. $\square$

Subcase 2b: $0 \leq w < \frac{1}{3}$ and $0 < c_p < 1 - \frac{2}{3(1-w)}$. Lastly, consider when $0 \leq w < \frac{1}{3}$ and $0 < c_p < 1 - \frac{2}{3(1-w)}$.

Firstly, if $32c_p(w + c_p - wc_p) - (1 + 3w(-1 + c_p) - 3c_p)(1 + w + c_p - wc_p)^2\pi_r\alpha \geq 0$, then $8w(1 - c_p) + 8c_p - \pi_r\alpha + (-2 + 3w(-1 + c_p) - 3c_p)(w(-1 + c_p) - c_p)\pi_r\alpha > 0$ holds as well, for any $\pi_r > 0$, $\alpha > 0$, $w \in [0, 1]$, and $c_p \in (0, 1)$. In that case, again the inequality (B.106) would reduce to (B.107), and the same argument from Subcase 1 would apply to show that (B.106) holds.

On the other hand, consider if $32c_p(w + c_p - wc_p) - (1 + 3w(-1 + c_p) - 3c_p)(1 + w + c_p - wc_p)^2\pi_r\alpha < 0$. If $8w(1 - c_p) + 8c_p - \pi_r\alpha + (-2 + 3w(-1 + c_p) - 3c_p)(w(-1 + c_p) - c_p)\pi_r\alpha \geq 0$, then (B.106) holds without further conditions since the left-hand side would be negative while the right-hand side would be non-negative.

However, if $8w(1-c_p)+8c_p-\pi_r\alpha+(-2+3w(-1+c_p)-3c_p)(w(-1+c_p)-c_p)\pi_r\alpha < 0$, then we can divide by it and isolate the radicand of (B.106). Squaring both sides and omitting the algebra, this is equivalent to

$$\begin{aligned}64c_p(w + c_p - wc_p)^2 < -(4(w(-1 + c_p) - c_p)(w^3(-1 + c_p)^3 + (-3 + c_p)c_p(-1 + 3c_p)+\\ w(-1 + c_p)(1 + c_p)(1 + 11c_p) - w^2(1 - c_p)^2(2 + 15c_p))\pi_r\alpha+\\ w(1 + 3w(-1 + c_p) - 3c_p)(-1 + c_p)(1 + w + c_p - wc_p)^3(\pi_r\alpha)^2) \quad \text{(B.108)}\end{aligned}$$

Since $0 < c_p < 1 - \frac{2}{3(1-w)}$, the coefficient on the quadratic α term in (B.108) is positive. So to prove the inequality for all α, it suffices to show that the minimum of this quadratic in α is larger than $64c_p(w+c_p-wc_p)^2$. Finding the minimizer of the quadratic in α and comparing it to the lower bound on α given by $8w(1-c_p)+8c_p-\pi_r\alpha+(-2+3w(-1+c_p)-3c_p)(w(-1+c_p)-$

$c_p)\pi_r\alpha < 0$, we find that the quadratic is minimized at $\alpha = \frac{8(w(1-c_p)+c_p)}{(1+3w(-1+c_p)-3c_p)(1+w+c_p-wc_p)\pi_r}$. The right-hand side of (B.108) evaluated at this α is indeed larger than $64c_p(w+c_p-wc_p)^2$ when $0 < c_p < 1 - \frac{2}{3(1-w)}$ and $0 \leq w < \frac{1}{3}$. $\square$

Then exhausting all sub-cases, it follows that (B.106) holds for all $w \in [0,1]$. In particular, this means that $\frac{d}{dv_{nr}}\Pi_{II}(v_{nr}) < 0$ for any $v_{nr} > \frac{1+c_p}{2}$. Therefore, $v_{nr} \leq \frac{1+c_p}{2}$ whenever $0 < v_{nr} < v_p < 1$ is induced in equilibrium. $\blacksquare$

B.2 Other Ransomware Attack Vectors, *RW-OV*

This section contains the solutions of the consumption subgame and pricing subgame under *RW-OV*.

B.2.1 Consumption Subgame

Lemma B.7. [*Consumption Subgame*] *Under RW-OV, for a given price p, the complete threshold characterization of the consumption subgame of the model without patching is as follows:*

(I) $(0 < v_{nr} < 1)$, *where* $v_{nr} = \frac{\pi_r\alpha - 1 + \sqrt{1+\pi_r\alpha(-2+4p+\pi_r\alpha)}}{2\pi_r\alpha}$, *if the following conditions hold:*

- *(A)* $p < 1$, *and*
- *(B)* $R \geq \alpha(1-\delta)$

(II) $(0 < v_{nr} < v_r < 1)$, *where* $v_{nr} = \frac{-1+\alpha\pi_r+\sqrt{1+\alpha\pi_r(-2+4p+\alpha\pi_r)}}{2\alpha\pi_r}$ *and* $v_r = \frac{R}{\alpha(1-\delta)}$, *if the following conditions hold:*

- *(A)* $R < \alpha(1-\delta)$, *and*
- *(B)* $(\alpha - \alpha\delta)(-1+\alpha\pi_r+\sqrt{1+\alpha\pi_r(-2+4p+\alpha\pi_r)}) < 2R\alpha\pi_r$

(III) $(0 < v_r < 1)$, *where* $v_r = \frac{-1-R\pi_r+\delta\pi_r\alpha+\sqrt{4\delta\pi_r\alpha(p+R\pi_r)+(1+R\pi_r-\delta\pi_r\alpha)^2}}{2\delta\pi_r\alpha}$, *if the following conditions hold:*

- *(A)* $p < 1$, *and*
- *(B) Either* $\left(2\delta + \sqrt{4\pi_r\alpha\delta(p+\pi_rR)+(-\pi_r\alpha\delta+\pi_rR+1)^2} \leq \pi_r\alpha\delta + \pi_rR + 1\right)$, *or* $\left(2\delta + \sqrt{4\pi_r\alpha\delta(p+\pi_rR)+(-\pi_r\alpha\delta+\pi_rR+1)^2} > \pi_r\alpha\delta+\pi_rR+1 \text{ and } \frac{\pi_r\alpha\delta+\sqrt{4\pi_r\alpha\delta(p+\pi_rR)+(-\pi_r\alpha\delta+\pi_rR+1)^2}-R\pi_r-1}{2\pi_r\alpha\delta} \leq -\frac{2\delta p}{\pi_r\alpha\delta-2\delta-\sqrt{4\pi_r\alpha\delta(p+\pi_rR)+(-\pi_r\alpha\delta+\pi_rR+1)^2}+\pi_rR+1}\right)$

(IV) $(0 < 1)$ *(in which no one purchases) if the following conditions hold:*

- *(A)* $p = 1$

Proof of Lemma B.7: From the same argument as the proof of Lemma B.1, we have threshold-type equilibrium structure.

Next, we characterize in more detail each outcome that can arise in the consumption subgame, as well as the corresponding parameter regions. For the case of $0 < v_{nr} < 1$, based on the threshold-type equilibrium structure, we have $u(\sigma) = 1 - v_{nr}$. We prove the following claim related to the corresponding parameter region in which this case arises.

Claim 1. *The subgame outcome that corresponds to case $0 < v_{nr} < 1$ arises if and only if the following conditions are satisfied:*

$$p < 1 \text{ and } R \geq \alpha(1-\delta). \tag{B.109}$$

The consumer indifferent between not purchasing at all and purchasing and remaining unpatched, v_{nr}, satisfies $v_{nr} - p - \pi_r \alpha u(\sigma) v_{nr} = 0$. To solve for the threshold v_{nr}, using $u(\sigma) = 1 - v_{nr}$, we solve

$$v_{nr} = \frac{p}{1 - \pi_r \alpha u(\sigma)} = \frac{p}{1 - \pi_r \alpha (1 - v_{nr})}. \tag{B.110}$$

For this to be an equilibrium, we have that $v_{nr} \geq 0$. This rules out the smaller root of the quadratic as a solution. Given the underlying model assumptions, the other root is strictly positive, so the root characterizing v_{nr} is

$$v_{nr} = \frac{\pi_r \alpha - 1 + \sqrt{1 + \pi_r \alpha(-2 + 4p + \pi_r \alpha)}}{2\pi_r \alpha} \tag{B.111}$$

For this to be an equilibrium, the necessary and sufficient conditions are that $0 < v_{nr} < 1$, type $v = 1$ weakly prefers (B, NR) to both (B, R).

For $v_{nr} < 1$, it is equivalent to have $p < 1$.

For $v = 1$ to prefer (B, NR) over (B, R), we need $1 \leq \frac{R}{\alpha(1-\delta)}$. Simplifying, this becomes $R \geq \alpha(1 - \delta)$. The conditions above are given in the lemma. $\square$

Next, for the case of $0 < v_{nr} < v_r < 1$, we have $u = 1 - v_{nr}$. Following the same steps as before, we prove the following claim related to the corresponding parameter region in which this case arises.

Claim 2. *The subgame outcome that corresponds to case $0 < v_{nr} < v_r < 1$ arises if and only if the following conditions are satisfied:*

$$p > 0 \text{ and } \alpha\big(-2R\pi_r + (1-\delta)(-1 + \pi_r\alpha + \sqrt{1 + \pi_r\alpha(-2 + 4p + \pi_r\alpha)})\big) < 0. \tag{B.112}$$

To solve for the thresholds v_{nr} and v_r, using $u = 1 - v_{nr}$, note that they solve

$$v_{nr} = \frac{p}{1 - \pi_r \alpha (1 - v_{nr})}, \text{ and} \tag{B.113}$$

$$v_r = \frac{R}{\alpha(1-\delta)}, \tag{B.114}$$

where the expression in (B.114) comes from (B.5).

Solving for v_{nr} in (B.16), we have

$$v_{nr} = \frac{-1 + \pi_r\alpha \pm \sqrt{1 + \pi_r\alpha(-2 + 4p + \pi_r\alpha)}}{2\pi_r\alpha}. \tag{B.115}$$

Note that $\frac{-1+\pi_r\alpha-\sqrt{1+\pi_r\alpha(-2+4p+\pi_r\alpha)}}{2\pi_r\alpha} < p$ while $\frac{-1+\pi_r\alpha+\sqrt{1+\pi_r\alpha(-2+4p+\pi_r\alpha)}}{2\pi_r\alpha} > p$, and since $v_{nr} > p$ in equilibrium, it follows that

$$v_{nr} = \frac{-1 + \pi_r\alpha + \sqrt{1 + \pi_r\alpha(-2 + 4p + \pi_r\alpha)}}{2\pi_r\alpha}. \tag{B.116}$$

For this to be an equilibrium, the necessary and sufficient conditions are $0 < v_{nr} < v_r < 1$. Type $v = v_r$ is indifferent between (B, R) and (B, NR), so this implies that $v = v_r$ strictly prefers (B, NR) over (B, P).

Note $v_{nr} > 0$ is satisfied if $p > 0$ since $v_{nr} > p$ under the preliminary model assumptions.

For $v_{nr} < v_r$, from (B.19) and (B.17), this simplifies to $\alpha\big(-2R\pi_r + (1-\delta)(-1 + \pi_r\alpha + \sqrt{1 + \pi_r\alpha(-2 + 4p + \pi_r\alpha)})\big) < 0$.

For $v_r < 1$, from (B.17), this simplifies to $R < \alpha(1-\delta)$. The conditions above are summarized in the lemma. □

Next, for case the case of $0 < v_r < 1$, we have $u = 1 - v_r$. Following the same steps as before, we prove the following claim related to the corresponding parameter region in which this case arises.

Claim 3. *The subgame outcome that corresponds to case $0 < v_r < 1$ arises if and only if the following conditions are satisfied:*

$$\begin{aligned}
&p < 1 \text{ and} \\
&\quad \text{and either } 2\delta + \sqrt{4\pi_r\alpha\delta(p + \pi_r R) + (-\pi_r\alpha\delta + \pi_r R + 1)^2} \le \pi_r\alpha\delta + \pi_r R + 1, \text{ or} \\
&\qquad 2\delta + \sqrt{4\pi_r\alpha\delta(p + \pi_r R) + (-\pi_r\alpha\delta + \pi_r R + 1)^2} > \pi_r\alpha\delta + \pi_r R + 1 \text{ and} \\
&\qquad\quad \frac{\pi_r\alpha\delta + \sqrt{4\pi_r\alpha\delta(p + \pi_r R) + (-\pi_r\alpha\delta + \pi_r R + 1)^2} - R\pi_r - 1}{2\pi_r\alpha\delta} \le \\
&\qquad\quad -\frac{2\delta p}{\pi_r\alpha\delta - 2\delta - \sqrt{4\pi_r\alpha\delta(p + \pi_r R) + (-\pi_r\alpha\delta + \pi_r R + 1)^2} + \pi_r R + 1}.
\end{aligned} \tag{B.117}$$

To solve for the thresholds v_r, using $u = 1 - v_r$, note it solves

$$v_r = \frac{p + R\pi_r(1 - v_r)}{1 - \delta\pi_r\alpha(1 - v_r)} \tag{B.118}$$

Then v_r is one of the two roots of the equation above, $\frac{-1-R\pi_r+\delta\pi_r\alpha\pm\sqrt{4\delta\pi_r\alpha(p+R\pi_r)+(1+R\pi_r-\delta\pi_r\alpha)^2}}{2\delta\pi_r\alpha}$. However, the smaller of the two roots is negative, so v_r must be the larger of the two roots in equilibrium. Hence, we have

$$v_r = \frac{-1-R\pi_r+\delta\pi_r\alpha+\sqrt{4\delta\pi_r\alpha(p+R\pi_r)+(1+R\pi_r-\delta\pi_r\alpha)^2}}{2\delta\pi_r\alpha}. \tag{B.119}$$

For this to be an equilibrium, the necessary and sufficient conditions are $p < v_r < 1$, and no consumer prefers to patch or not pay ransom over paying ransom.

For $v_r > p$, using (B.28), this simplifies to $p < 1$. For $v_r > p$, using (B.119), this also simplifies to $p < 1$. Similarly, $v_r < 1$ also simplifies to $p < 1$.

For no consumer to strictly prefer not paying ransom over paying ransom, it suffices to have $v = v_r$ weakly prefer not to buy over buying and not paying ransom (since type $v = v_r$ is indifferent between the option of not purchasing and the option of purchasing, remaining unpatched, and paying ransom). Now if $1 - \pi_r\alpha u[\sigma] \leq 0$, then $v(1 - \pi_r\alpha u[\sigma]) - p < 0$, so that everyone would prefer (NB, NP) over (B, NP, NR). In this case, no further conditions are needed. On the other hand, if $1 - \pi_r\alpha u[\sigma] > 0$, then we will need the condition $v_r \leq \frac{p}{1-\pi_r\alpha(1-v_r)}$ for $v = v_r$ to weakly prefer not buying over buying but not paying ransom.

In the first sub-case, the condition $v(1 - \pi_r\alpha u[\sigma]) - p < 0$ simplifies to $2\delta + \sqrt{4\pi_r\alpha\delta(p + \pi_r R) + (-\pi_r\alpha\delta + \pi_r R + 1)^2} \leq \pi_r\alpha\delta + \pi_r R + 1$, using (B.28).

In the second sub-case, the conditions $1 - \pi_r\alpha u[\sigma] > 0$ and $v_r \leq \frac{p}{1-\pi_r\alpha(1-v_r)}$ simplify to $2\delta + \sqrt{4\pi_r\alpha\delta(p + \pi_r R) + (-\pi_r\alpha\delta + \pi_r R + 1)^2} > \pi_r\alpha\delta + \pi_r R + 1$ and $\frac{\pi_r\alpha\delta+\sqrt{4\pi_r\alpha\delta(p+\pi_r R)+(-\pi_r\alpha\delta+\pi_r R+1)^2}-R\pi_r-1}{2\pi_r\alpha\delta} \leq -\frac{2\delta p}{\pi_r\alpha\delta-2\delta-\sqrt{4\pi_r\alpha\delta(p+\pi_r R)+(-\pi_r\alpha\delta+\pi_r R+1)^2}+\pi_r R+1}$. The conditions above are summarized in (B.26). □

Lastly, for the case of $0 < 1$ (in which no one purchases), we have the following.

Claim 4. *The subgame outcome that corresponds to case* $0 < 1$ *(in which no one purchases) arises if and only if the following conditions are satisfied:*

$$p = 1 \tag{B.120}$$

When $p = 1$, then NB dominates every other option for all consumers $v \in [0, 1]$. Consequently, no one purchases when $p = 1$.

This completes the proof of the equilibrium characterization in the consumption subgame of the model in Section 4 of the paper. ■

Lemma B.8. [*Consumption Subgame*] *Under RW-OV, there exist sets of mutually exclusive conditions on* R, α, δ, *and* π_r *that cover the parameter space and cleanly organize the equilibrium outcome by price. Under each of these parameter sets, the feasible space for price* p *can be split into adjacent intervals, each of them with a single structure that characterizes the unique outcome in the consumption subgame.*

(i) $R \geq \alpha(1-\delta)$*:*

- $0 \leq p < 1 : (0 < v_{nr} < 1)$

(ii) $R < \alpha(1-\delta)$ *and* $0 < \pi_r < \frac{1-\delta}{-R+\alpha(1-\delta)}$*:*

- $0 \leq p < \frac{R(1-\delta+(R-\alpha(1-\delta))\pi_r)}{\alpha(1-\delta)^2} : (0 < v_{nr} < v_r < 1)$
- $\frac{R(1-\delta+(R-\alpha(1-\delta))\pi_r)}{\alpha(1-\delta)^2} \leq p < 1 : (0 < v_r < 1)$

(iii) $R < \alpha(1-\delta)$ *and* $\pi_r \geq \frac{1-\delta}{-R+\alpha(1-\delta)}$*:*

- $0 \leq p < 1 : (0 < v_r < 1)$

Under each of these parameter sets, setting $p = 1$ *leads to the trivial outcome of no one purchasing.*

Proof of Lemma B.8: For sufficiently large c_p, this comes directly from Proposition B.1. ■

B.2.2 Pricing Subgame

Lemma B.9. *Under RW-OV, there exists a bound* $\tilde{\delta} > 0$ *such that if* $\delta < \tilde{\delta}$*, then:*

(a) if $0 < R \leq \tilde{R}_1$ *(where* $\tilde{R}_1$ *is defined in the proof below), then the equilibrium consumer market structure is* $0 < v_r < 1$*;*

(b) if $\tilde{R}_1 < R < \alpha(1-\delta)$*, then the equilibrium consumer market structure is* $0 < v_{nr} < v_r < 1$*;*

(c) if $R \geq \alpha(1-\delta)$*, then the equilibrium consumer market structure is* $0 < v_{nr} < 1$*.*

Proof of Lemma B.9: From Lemmas B.7 and B.8, we express the consumption subgame outcomes across the parameter space in terms of intervals of price. We use this to specify the interior optimal price and vendor's profit at that interior optimal price for each of these potential market outcomes.

The following expressions come from Lemma B.3 and are repeated below for the reader to follow along the proof more easily in this lemma.

Given a price p, the region of the parameter space defining $0 < v_{nr} < 1$ is given in part (I) of Lemma B.2. For this case, we have

$$v_{nr} = \frac{-1 + \pi_r\alpha + \sqrt{1 + \pi_r\alpha(-2 + 4p + \pi_r\alpha)}}{2\pi_r\alpha}. \tag{B.121}$$

The interior optimal price of this case is given by:

$$p_I^* = \frac{1}{9}\left(4 - \frac{1}{\pi_r\alpha} - \pi_r\alpha + \frac{\sqrt{1+\pi_r\alpha+(\pi_r\alpha)^3+(\pi_r\alpha)^4}}{\pi_r\alpha}\right). \tag{B.122}$$

Given a price p, the region of the parameter space defining $0 < v_{nr} < v_r < 1$ is given in Lemma B.7. This also a special case of part (III) of Lemma B.2. For this case, we have

$$v_{nr} = \frac{-1+\pi_r\alpha+\sqrt{1+\pi_r\alpha(-2+4p+\pi_r\alpha)}}{2\pi_r\alpha}. \tag{B.123}$$

The interior optimal price of this case is given by:

$$p_{III}^* = \frac{1}{9}\left(4 - \frac{1}{\pi_r\alpha} - \pi_r\alpha + \frac{\sqrt{1+\pi_r\alpha+(\pi_r\alpha)^3+(\pi_r\alpha)^4}}{\pi_r\alpha}\right). \tag{B.124}$$

The profit corresponding to this price for this case is given by:

$$\Pi_{III}^* = \frac{\left(3+3\alpha\pi_r - \sqrt{5+\pi_r\alpha(-2+5\pi_r\alpha)+4\sqrt{1+\pi_r\alpha+(\pi_r\alpha)^3+(\pi_r\alpha)^4}}\right)}{54(\pi_r\alpha)^2}\times \\ \left(-1+\pi_r\alpha(4-\pi_r\alpha)+\sqrt{1+\pi_r\alpha+(\pi_r\alpha)^3+(\pi_r\alpha)^4}\right). \tag{B.125}$$

Given a price p, the region of the parameter space defining $0 < v_r < 1$ is given in Lemma B.7, which is a special case of part (V) of Lemma B.2.

From (B.28), we have that

$$v_r = \frac{-1-R\pi_r+\delta\pi_r\alpha+\sqrt{4\delta\pi_r\alpha(p+R\pi_r)+(1+R\pi_r-\delta\pi_r\alpha)^2}}{2\delta\pi_r\alpha}. \tag{B.126}$$

The interior optimal price of this case is given by:

$$p_V^* = \Big(-1-2R\pi_r+4\delta\pi_r\alpha-R^2\pi_r^2-2R\delta\alpha\pi_r^2-(\delta\pi_r\alpha)^2+ \\ (1+R\pi_r+\delta\pi_r\alpha)\sqrt{1+\pi_r(2R-\delta\alpha+(R+\delta\alpha)^2\pi_r)}\Big)\Big(9\delta\pi_r\alpha\Big)^{-1}. \tag{B.127}$$

Substituting (B.127) into the profit function of this case associated maximal profit of this case. Characterizing this profit expression in terms of a Taylor Series expansion in δ, we have:

$$\Pi_V^* = \frac{1}{4(1+R\pi_r)} + \sum_{k=1}^{\infty} a_k\delta^k. \tag{B.128}$$

That the second-order conditions are satisfied at each interior-optimal solution for each of the cases above is given in Lemma B.3.

Now that we have found the interior optimal prices for these regions, we use Lemma B.8 to find conditions under which the interior optimal price for a case lies within the set of conditions defining that case. When more than one interior optimal price is within the price range defining that case, then we have to compare the profits of those cases. In what follows below, we will go through each relevant region in Lemma B.8.

First, we examine Region (i) of Lemma B.8.

(i) $R \geq \alpha(1-\delta)$:

- $0 \leq p < 1 : (0 < v_{nr} < 1)$

There is only one market outcome in this region. When the condition under which this region arises holds, then the equilibrium outcome will be $0 < v_{nr} < 1$ with the price being given as p_I^* in (B.122) (which is in the interval $(0,1)$ by the modified focal region assumptions on α, namely $\alpha \in (\sqrt{3}, 6)$).

Next, we examine Region (ii) of Lemma B.8.

(ii) $R < \alpha(1-\delta)$ and $0 < \pi_r < \frac{1-\delta}{-R+\alpha(1-\delta)}$:

- $0 \leq p < \frac{R(1-\delta+(R-\alpha(1-\delta))\pi_r)}{\alpha(1-\delta)^2} : (0 < v_{nr} < v_r < 1)$
- $\frac{R(1-\delta+(R-\alpha(1-\delta))\pi_r)}{\alpha(1-\delta)^2} \leq p < 1 : (0 < v_r < 1)$

In this region, we will show that there is an overlap in the region of the parameter space over which p_{III}^* and p_V^* are interior to their respective regions. Therefore the boundary between the two cases will be determined by the isoprofit curve (found by finding when the profits of the two cases are equal at their interior optimal prices).

First, note that for sufficiently small δ, we want to find conditions so that p_{III}^* is in $0 \leq p < \frac{R(1-\delta+(R-\alpha(1-\delta))\pi_r)}{\alpha(1-\delta)^2}$. The upper bound goes to $\frac{R(1-(\alpha-R)\pi_r)}{\alpha}$ as $\delta \to 0$, so we want to find conditions so that $p_{III}^* \in (0, \frac{R(1-(\alpha-R)\pi_r)}{\alpha})$ (where p_{III}^* is in (B.124)). That $p_{III}^* > 0$ follows from the focal region assumptions on c_p and α. The condition $p_{III}^* < \frac{R(1-(\alpha-R)\pi_r)}{\alpha}$ is equivalent to $R > \frac{-3+3\pi_r\alpha+\sqrt{5+\pi_r\alpha(-2+5\pi_r\alpha)+4\sqrt{1+\pi_r\alpha+(\pi_r\alpha)^3+(\pi_r\alpha)^4}}}{6\pi_r}$.

Next, recall p_V^* was given earlier in (B.127). The limit of (B.127) as $\delta \to 0$ is $\frac{1}{2}$, so for sufficiently small δ, $p_V^* < 1$. The condition $p_V^* \geq \frac{R(1-(\alpha-R)\pi_r)}{\alpha}$ is equivalent to $R \leq \frac{-1+\pi_r\alpha+\sqrt{1+(\pi_r\alpha)^2}}{2\pi_r}$. That $\frac{-1+\pi_r\alpha+\sqrt{1+(\pi_r\alpha)^2}}{2\pi_r} > \frac{-3+3\pi_r\alpha+\sqrt{5+\pi_r\alpha(-2+5\pi_r\alpha)+4\sqrt{1+\pi_r\alpha+(\pi_r\alpha)^3+(\pi_r\alpha)^4}}}{6\pi_r}$ follows from the modified focal region assumptions on α ($\alpha \in (\sqrt{3}, 6)$). Hence, for sufficiently small δ, there is an overlap in the regions of the parameter space over which p_{III}^* is interior to the range of p defining $0 < v_{nr} < v_r < 1$ and p_V^* is interior to the range of p defining $0 < v_r < 1$.

We can find the R boundary (or equivalently, the π_r boundary, as shown later in Lemma B.10) between these two cases by equating their profits, (B.125) and (B.128). For sufficiently low δ, the boundary between these two cases can be expressed as

$$\tilde{R}_1 = A_0 + \sum_{k=1}^{\infty} a_k \delta^k, \text{ where} \tag{B.129}$$

$$\begin{multline} A_0 = \Bigg(-6\alpha^3\pi_r^3 + 2\alpha\pi_r(9 + 3\sqrt{1+\alpha\pi_r+\alpha^3\pi_r^3+\alpha^4\pi_r^4} - \\ 4\sqrt{5+\alpha\pi_r(-2+5\alpha\pi_r)+4\sqrt{1+\alpha\pi_r+\alpha^3\pi_r^3+\alpha^4\pi_r^4}}) - \\ 2(-1+\sqrt{1+\alpha\pi_r+\alpha^3\pi_r^3+\alpha^4\pi_r^4})(-3+\sqrt{5+\alpha\pi_r(-2+5\alpha\pi_r)+4\sqrt{1+\alpha\pi_r+\alpha^3\pi_r^3+\alpha^4\pi_r^4}}) + \\ \alpha^2\pi_r^2(-9+2\sqrt{5+\alpha\pi_r(-2+5\alpha\pi_r)+4\sqrt{1+\alpha\pi_r+\alpha^3\pi_r^3+\alpha^4\pi_r^4}})\Bigg) \times \\ \Bigg(2\pi_r(-1+\alpha\pi_r(4-\alpha\pi_r)+\sqrt{1+\alpha\pi_r+\alpha^3\pi_r^3+\alpha^4\pi_r^4})(-3-3\alpha\pi_r+ \\ \sqrt{5+\alpha\pi_r(-2+5\alpha\pi_r)+4\sqrt{1+\alpha\pi_r+\alpha^3\pi_r^3+\alpha^4\pi_r^4}})\Bigg)^{-1}. \tag{B.130} \end{multline}$$

A_0 as a function of π_r is strictly increasing for $\pi_r > 0$, and since $A_0 < \alpha$ both when $\pi_r = 0$ and when $\pi_r = 1$, it follows that $A_0 < \alpha$ for all $\pi_r \in (0,1)$. Then $\tilde{R}_1 < \alpha(1-\delta)$ for sufficiently small δ. Also, $A_0 \in \left(\frac{-3+3\pi_r\alpha+\sqrt{5+\pi_r\alpha(-2+5\pi_r\alpha)+4\sqrt{1+\pi_r\alpha+(\pi_r\alpha)^3+(\pi_r\alpha)^4}}}{6\pi_r}, \frac{-1+\pi_r\alpha+\sqrt{1+(\pi_r\alpha)^2}}{2\pi_r}\right)$ follows from the modified focal region assumptions on α and $\pi_r \in (0,1)$.

Altogether for this region, for $R > \tilde{R}_1$, the profit of $0 < v_{nr} < v_r < 1$ is the equilibrium outcome with p^*_{III} as the vendor's price. When $R \leq \tilde{R}_1$, then $0 < v_r < 1$ is the outcome and p^*_V is the vendor's price.

Next, we examine Region (iii) of Lemma B.8.

(iii) $R < \alpha(1-\delta)$ and $\pi_r \geq \frac{1-\delta}{-R+\alpha(1-\delta)}$:

- $0 \leq p < 1 : (0 < v_r < 1)$

There is only one market outcome in this region. When the condition under which this region arises holds, then the equilibrium outcome will be $0 < v_r < 1$ with the price being given as p^*_V in (B.127). The limit of (B.127) as $\delta \to 0$ is $\frac{1}{2}$, so for sufficiently small δ, p^*_V falls in the interval $(0,1)$.

Altogether across the three regions of Lemma B.8, for sufficiently small δ:

(a) if $0 < R \leq \tilde{R}_1$, then the equilibrium consumer market structure is $0 < v_r < 1$;

(b) if $\tilde{R}_1 < R < \alpha(1-\delta)$, then the equilibrium consumer market structure is $0 < v_{nr} < v_r < 1$;

(c) if $R \geq \alpha(1-\delta)$, then the equilibrium consumer market structure is $0 < v_{nr} < 1$.

■

Lemma B.10. *In the setting of non-patchable ransomware, there exists a bound $\tilde{\delta} > 0$ such that if $\tilde{R}_2 < R < \tilde{R}_3$:*

(a) if $0 < \pi_r < \pi'$ (where π' is defined in the proof below), then the equilibrium consumer market structure is $0 < v_{nr} < v_r < 1$;

(b) if $\pi' \leq \pi_r \leq 1$, then the equilibrium consumer market structure is $0 < v_r < 1$,

where $\tilde{R}_2$ and $\tilde{R}_3$ are defined in the proof below.

Proof of Lemma B.10: This follows directly from Lemma B.9, viewing the characterization in terms of π_r instead of R and focusing on a range of R values above $\frac{1}{2}\alpha(1-\delta)$. Specifically, define

$$\tilde{R}_2 = \tilde{R}_1|_{\pi_r=0} = \frac{1}{2}\alpha(1-\delta) \tag{B.131}$$

as the value of $\tilde{R}_1$ when $\pi_r = 0$, and define

$$\tilde{R}_3 = \tilde{R}_1|_{\pi_r=1} \tag{B.132}$$

as the value of $\tilde{R}_1$ when $\pi_r = 1$. Also, since A_0 is strictly increasing in π_r, this means that (viewing the boundary as a function of π_r), the function is invertible for sufficiently small δ. Define

$$\pi' \triangleq \max(0, \tilde{R}_1^{-1}(\pi_r)) \tag{B.133}$$

as the max of 0 and the inverse function $\tilde{R}_1(\pi_r)$. Then the profit of $0 < v_{nr} < v_r < 1$ dominates the profit under $0 < v_r < 1$ iff $\pi_r < \pi'$ (equivalent to $R > \tilde{R}_1$ in Lemma B.9). ■

B.3 Benchmark for Multiple Classes of Threats, *BM*

This section contains the characterization of the consumption subgame and pricing subgame under *BM*. The outcomes here can be thought of as the scenarios that would arise under the ransomware model with $\delta = 1$. The risk factor parameter is denoted by π_n in this scenario.

B.3.1 Consumption Subgame

Lemma B.11. [*Consumption Subgame*] *Under BM, for a given price p, the complete threshold characterization of the consumption subgame in Section 5 of the paper is as follows:*

(I) $(0 < v_{nr} < 1)$, *where* $v_{nr} = \frac{\pi_n\alpha - 1 + \sqrt{1+\pi_n\alpha(-2+4p+\pi_n\alpha)}}{2\pi_n\alpha}$ *if*

$$c_p \geq \frac{1}{2}\left(\pi_n\alpha - \sqrt{\pi_n\alpha(\pi_n\alpha + 4p - 2) + 1} + 1\right);$$

(II) $(0 < v_{nr} < v_p < 1)$, *where* v_{nr} *is the largest root of the cubic* $f(x) = \pi_n\alpha x^3 + (1 - (c_p + p)\pi_n\alpha)x^2 - 2px + p^2$ *and* $v_p = \frac{c_p v_{nr}}{v_{nr} - p}$ *if*

$$c_p < \frac{1}{2}\left(\pi_n\alpha - \sqrt{\pi_n\alpha(\pi_n\alpha + 4p - 2) + 1} + 1\right);$$

(III) $(0 < 1)$ *(in which no one purchases) if* $p = 1$.

Proof of Lemma B.11: This follows from the proof of Lemma B.2 with $\delta = 1$. ∎

B.3.2 Pricing Subgame

Lemma B.12. *The equilibrium outcome is given by* $0 < v_{nr} < 1$ *when patching costs are high* $(\pi_n\alpha < c_p)$ *and* $0 < v_{nr} < v_p < 1$ *when patching costs are low* $(\frac{c_p(2-3c_p)}{1-2c_p} < \pi_n\alpha)$.

Proof of Lemma B.12: We use the general model with ransomware to characterize outcomes in this benchmark model. Specifically, from the consumer utility function (B.1), if $\delta > 1 - \frac{R}{\alpha}$, then no consumer would prefer (B, NP, R) over (B, NP, NR). The equilibrium characterization of the consumption subgame is given in Lemma B.11.

In particular, if Case (I) of Lemma B.11 holds, then $0 < v_{nr} < 1$ would be the equilibrium outcome. From the first-order condition, the interior optimal price of this case is given in (B.37), repeated again for the reader here: $p_I^* = \frac{-1+\pi_n\alpha(4-\pi_n\alpha)+\sqrt{1+\pi_n\alpha+(\pi_n\alpha)^3+(\pi_n\alpha)^4}}{9\pi_n\alpha}$. If $\pi_n\alpha > \frac{(2-3c_p)c_p}{1-2c_p}$, then p_I^* will not satisfy the c_p condition in (I) of Lemma B.11. The condition $\frac{(2-3c_p)c_p}{1-2c_p} > 0$ holds under the conditions of the focal region. Hence, if $\pi_n\alpha > \frac{c_p(-2+3c_p)}{-1+2c_p}$, then $0 < v_{nr} < v_p < 1$ is the equilibrium outcome, and the equilibrium price is characterized in (B.97).

The condition of Case (II) of Lemma B.11 is $c_p < \frac{1}{2}\left(\pi_n\alpha - \sqrt{\pi_n\alpha(\pi_n\alpha + 4p - 2) + 1} + 1\right)$. However, $\frac{1}{2}\left(\pi_n\alpha - \sqrt{\pi_n\alpha(\pi_n\alpha + 4p - 2) + 1} + 1\right) < \pi_n\alpha$ from $\pi_n > 0$, $\alpha > 0$, and $p > 0$. Therefore, if $\pi_n\alpha < c_p$, then this condition cannot be met by any price $p > 0$.

The condition of Case (I) of Lemma B.11 is $c_p \geq \frac{1}{2}\left(\pi_n\alpha - \sqrt{\pi_n\alpha(\pi_n\alpha + 4p - 2) + 1} + 1\right)$. Evaluating $p = p_I^* = \frac{-1+\pi_n\alpha(4-\pi_n\alpha)+\sqrt{1+\pi_n\alpha+(\pi_n\alpha)^3+(\pi_n\alpha)^4}}{9\pi_n\alpha}$ into right-hand side of the inequality gives the expression $\frac{1}{6}\left(3+3\pi_n\alpha-\sqrt{5+\pi_n\alpha(-2+5\pi_n\alpha)+4\sqrt{1+\pi_n\alpha+(\pi_n\alpha)^3+(\pi_n\alpha)^4}}\right)$. That $\pi_n\alpha > \frac{1}{6}\left(3+3\pi_n\alpha-\sqrt{5+\pi_n\alpha(-2+5\pi_n\alpha)+4\sqrt{1+\pi_n\alpha+(\pi_n\alpha)^3+(\pi_n\alpha)^4}}\right)$ follows from $\alpha > 0$ and $\pi_n > 0$. This means that $c_p \geq \frac{1}{2}\left(\pi_n\alpha-\sqrt{\pi_n\alpha(\pi_n\alpha+4p-2)+1}+1\right)$ holds for $p = p_I^*$ under the condition $c_p > \pi_n\alpha$. In other words, under the condition $c_p > \pi_n\alpha$, p_I^* is interior to the range of p defining $0 < v_{nr} < 1$. Altogether, if $c_p > \pi_n\alpha$ in the benchmark scenario, then the equilibrium outcome will be $0 < v_{nr} < 1$. ■

B.4 Summary of Notation

To assist the reader, we have included a table summarizing the notation of boundaries between regions defined in the lemmas of Section B.1.2 and used throughout the paper.

Result	Sensitivity analysis in parameter	Bound	Description
Proposition 1	R	R_1	boundary between equilibrium regions (A) and (B) in Figure 1 (between $0 < v_r < 1$ and $0 < v_r < v_p < 1$ regions)
		R_2	boundary between equilibrium regions (B) and (C) in Figure 1 (between $0 < v_r < v_p < 1$ and $0 < v_{nr} < v_r < v_p < 1$ regions)
		R_3	boundary between equilibrium regions (C) and (D) in Figure 1 (between $0 < v_{nr} < v_r < v_p < 1$ and $0 < v_{nr} < v_p < 1$ regions)
Propositions 2 and 3	π_r	π_1	boundary between equilibrium regions (A) and (B) in Figure 1 (between $0 < v_r < 1$ and $0 < v_r < v_p < 1$ regions)
		π_2	boundary between equilibrium regions (A) and (C) in Figure 1 (between $0 < v_r < 1$ and $0 < v_{nr} < v_r < v_p < 1$ regions)
		$\hat{\pi}$	boundary between equilibrium regions (E) and (A) in Figure 1 (between $0 < v_{nr} < v_r < 1$ and $0 < v_r < 1$ regions)
		$\tilde{\pi}$	$\tilde{\pi} = min(\pi_1, \pi_2)$
Proposition 4	R	$\tilde{R}_1$	boundary between equilibrium regions (A) and (E) in Figure 4 (between $0 < v_r < 1$ and $0 < v_{nr} < v_r < 1$ regions)
Propositions 5 and 6	π_r	π'	boundary between equilibrium regions (E) and (A) in Figure 4 (between $0 < v_{nr} < v_r < 1$ and $0 < v_r < 1$ regions)

Table B.1: Description of bounds used in Propositions 1-6.

B.5 Proofs of Propositions 1 - 8

Proof of Proposition 1: From Lemma B.3, we have how the equilibrium market outcome changes in R. To complete the proof, we do comparative statics on the vendor's equilibrium price, profits, and market size with respect to R for each of the market outcomes.

For $R < R_1$, the equilibrium market outcome is $0 < v_r < 1$. The vendor's equilibrium price for sufficiently small δ has a Taylor series expansion of the form

$$p_V^* = \frac{1}{2} - \frac{\alpha \pi_r}{8(1+R\pi_r)^2}\delta + \sum_{k=2}^{\infty} a_k \delta^k. \tag{B.134}$$

From (B.53), we have the an asymptotic expression in δ of the vendor's equilibrium profit. Finally, the market size of this case is $1 - v_r^*$, and this has an asymptotic expansion given by

$$M_V^* = \frac{1}{2(1+R\pi_r)} + \sum_{k=1}^{\infty} a_k \delta^k. \tag{B.135}$$

Taking the derivatives of p_V^*, Π_V^*, and M_V^* with respect to R, we have:

$$\frac{d}{dR}\left[p_V^*\right] = \frac{\alpha \pi_r^2}{4(1+R\pi_r)^3}\delta + \sum_{k=2}^{\infty} a_k \delta^k, \tag{B.136}$$

$$\frac{d}{dR}\left[\Pi_V^*\right] = -\frac{\pi_r}{4(1+R\pi_r)^2} + \sum_{k=1}^{\infty} a_k \delta^k, \tag{B.137}$$

$$\frac{d}{dR}\left[M_V^*\right] = -\frac{\pi_r}{2(1+R\pi_r)^2} + \sum_{k=1}^{\infty} a_k \delta^k. \tag{B.138}$$

This show that for sufficiently low δ, the vendor's price in this case increases in R while the vendor's profit and market size decrease.

For R in $R_1 \leq R < R_2$, the equilibrium market outcome is $0 < v_r < v_p < 1$. The vendor's equilibrium price in this case is given by (B.57), and the vendor's profit at this price is given by (B.59). The market size of this case is given by $1 - v_r^*$, and this has an asymptotic expansion given by

$$M_{VI}^* = \frac{1-c_p}{2} + \frac{c_p^2 \alpha}{2R^2 \pi_r}\delta + \sum_{k=2}^{\infty} a_k \delta^k. \tag{B.139}$$

Taking the derivatives of p_{VI}^*, Π_{VI}^*, and M_{VI}^* with respect to R, we have:

$$\frac{d}{dR}\left[p_{VI}^*\right] = -\frac{c_p^2 \alpha}{R^3 \pi_r}\delta + \sum_{k=2}^{\infty} a_k \delta^k, \tag{B.140}$$

$$\frac{d}{dR}\left[\Pi_{VI}^{*}\right] = -\frac{(1-c_p)c_p^2\alpha}{R^3\pi_r}\delta + \sum_{k=2}^{\infty} a_k\delta^k, \tag{B.141}$$

$$\frac{d}{dR}\left[M_{VI}^{*}\right] = -\frac{c_p^2\alpha}{R^3\pi_r}\delta + \sum_{k=2}^{\infty} a_k\delta^k. \tag{B.142}$$

This shows that for sufficiently low δ, the vendor's price, profit, and market size all decrease in R.

For R in $R_2 \leq R < R_3$, the equilibrium market outcome is $0 < v_{nr} < v_r < v_p < 1$. The vendor's equilibrium price in this case is given by (B.46), and the vendor's profit at this price is given by (B.48). The market size of this case is given by $1 - v_{nr}^*$, and this has an asymptotic expansion given by

$$M_{IV}^* = \frac{1}{2} - \frac{c_p\alpha^2}{8(R(R-c_p\alpha))}\delta + \sum_{k=2}^{\infty} a_k\delta^k. \tag{B.143}$$

Taking the derivatives of p_{IV}^*, Π_{IV}^*, and M_{IV}^* with respect to R, we have:

$$\frac{d}{dR}\left[p_{IV}^{*}\right] = \frac{c_p\alpha}{2R^2} + \sum_{k=1}^{\infty} a_k\delta^k, \tag{B.144}$$

$$\frac{d}{dR}\left[\Pi_{IV}^{*}\right] = \frac{c_p\alpha}{4R^2} + \sum_{k=1}^{\infty} a_k\delta^k, \tag{B.145}$$

$$\frac{d}{dR}\left[M_{IV}^{*}\right] = -\frac{c_p\alpha^2(-2R+c_p\alpha)}{8R^2(R-c_p\alpha)^2}\delta + \sum_{k=2}^{\infty} a_k\delta^k. \tag{B.146}$$

Note that $R > R_2$ in this region, where $R_2 = \frac{\alpha}{2-c_p} + \sum_{k=1}^{\infty} a_k\delta^k$. Also note that $\frac{c_p\alpha}{2} < \frac{\alpha}{2-c_p}$ for all $\alpha > 0$ and $0 < c_p < 1$. This shows that for sufficiently low δ, the vendor's price, profit, and market size all increase in R in this region since $R > R_2$.

For $R \in [R_3, \omega)$, the equilibrium market outcome is $0 < v_{nr} < v_p < 1$, and the comparative statics with respect to R are still driven by movement of $p_{boundary}$ in (B.62). In that region of R, the vendor's price at the boundary increases in R, and the vendor's profit increases in R as $p_{boundary}$ moves toward the interior optimal p_{II}^* of this case (which does not change in R) instead of being constrained by a boundary condition with $0 < v_{nr} < v_r < 1$. In the proof of Lemma B.5, we show that $p_{II}^*(v_{nr})$ is a strictly increasing function of v_{nr}. Since the market size is $1 - v_{nr}$, it follows that the market size shrinks as R increases.

For $R \geq \omega$, the consumer market equilibrium is again $0 < v_{nr} < v_p < 1$, and the vendor can achieve this using its interior optimal price. At this point, the pricing is no longer driven by boundary pricing, and the vendor's price, market size, and profits remain constant in R.

Regarding continuity of the vendor's equilibrium profit as a function of R, this follows from Berge's Maximum Theorem. For any R, the set of feasible prices for the vendor is just the closed, compact set $[0, 1]$. Consequently, the mapping between the set of values R can be and the set of feasible prices is a constant, compact-valued correspondence. Because this correspondence is non-empty and constant (the set of feasible prices is always $[0, 1]$ regardless of R), it is continuous. Then by Berge's Maximum Theorem, the optimal value function is continuous in R. Regarding the optimal price as a function of R, the optimal price function is upper hemicontinuous in R from Berge's Maximum Theorem as well.

This completes the proof of Proposition 1. ■

Proof of Proposition 2: From Lemma B.4, we have how the equilibrium market outcome changes in π_r for all $R \leq \hat{\omega}$. The upper hemicontinuity of the optimal price as a function of π_r follows from the Berge Maximum Theorem in the same way that the optimal price function is upper hemicontinuous in R from Proposition 1. To complete the proof, we do comparative statics on the vendor's equilibrium price with respect to π_r for each of the market outcomes and compare the price values at the π_r values marking the regime switches.

For $R \leq \hat{R}_1$, the equilibrium outcome is $0 < v_r < 1$. The vendor's price of this case is given in (B.51). This has an asymptotic expression in δ given by

$$p_V^* = \frac{1}{2} - \frac{\pi_r \alpha}{8(1 + R\pi_r)^2}\delta + \sum_{k=2}^{\infty} a_k \delta^k. \tag{B.147}$$

In this low range of R, there is no strategic price jump or drop to induce another market outcome, and taking the derivative with respect to π_r, we have that:

$$\frac{d}{d\pi_r}\left[p_V^*\right] = \frac{\alpha(-1 + R\pi_r)}{8(1 + R\pi_r)^3}\delta + \sum_{k=2}^{\infty} a_k \delta^k. \tag{B.148}$$

This shows that $\frac{d}{d\pi_r}\left[p_V^*\right] < 0$ for all $\pi_r \in (0, \frac{1}{R})$. From (B.74), $\hat{R}_1 = \frac{1}{(1-c_p)^2} - 1 + \sum_{k=1}^{\infty} a_k \delta^k$. Note that $\frac{1}{(1-c_p)^2} - 1 < 1$ holds for the focal region assumptions on c_p ($0 < c_p < 2 - \sqrt{3}$), so this means that the price is decreasing in π_r for all $\pi_r < 1$ in this region of R for sufficiently low δ.

For $\hat{R}_1 < R \leq \hat{R}_2$, the equilibrium outcome is $0 < v_r < 1$ for $\pi_r < \pi_1$ by Lemma B.4. Here, π_1 is given in (B.72) and (B.73), which is given by $\pi_1 = \frac{(2-c_p)c_p}{(1-c_p)^2 R} + \sum_{k=1}^{\infty} a_k \delta^k$. That $\pi_1 < \frac{1}{R}$ for sufficiently low δ follows from $R > 0$ and $0 < c_p < 2 - \sqrt{3}$. The derivative of p_V^* with respect to π_r is given in (B.148), so that $\frac{d}{d\pi_r}\left[p_V^*\right] < 0$ for all $\pi_r \in (0, \frac{1}{R})$. Since $\pi_1 < \frac{1}{R}$, the price is decreasing for all $\pi_r < \pi_1$ in this region of R.

On the other hand, still for $\hat{R}_1 < R \leq \hat{R}_2$ but when for $\pi_r > \pi_1$, the equilibrium outcome is $0 < v_r < v_p < 1$ by Lemma B.4. The price of $0 < v_r < v_p < 1$ is given in (B.57) and written again here for clarity: $p^*_{VI} = \frac{1-c_p}{2} + \frac{c_p^2\alpha}{2R^2\pi_r}\delta + \sum_{k=2}^{\infty} a_k\delta^k$. Taking the derivative of this with respect to π_r, we have:

$$\frac{d}{d\pi_r}\left[p^*_{VI}\right] = -\frac{c_p^2\alpha}{2R^2\pi_r^2}\delta + \sum_{k=2}^{\infty} a_k\delta^k. \tag{B.149}$$

$\frac{d}{d\pi_r}\left[p^*_{VI}\right] < 0$ for sufficiently small δ, so this price is strictly decreasing for all π_r, regardless of R. Moreover, note that the price given in (B.57) is arbitrarily close to $\frac{1-c_p}{2}$ for sufficiently small δ while (B.147) is strictly larger than that (being arbitrarily close to $\frac{1}{2}$ for sufficiently small δ). Hence, there is a price drop at $\pi_r = \pi_1$.

For $\hat{R}_2 < R \leq \hat{\omega}$, there are two cases depending on whether $R > R_2$ from Lemma B.4. In either case, for $\pi_r < \hat{\pi}$ (where $\hat{\pi}$ is given in (B.77)), the consumer market equilibrium structure that arises under optimal pricing is $0 < v_{nr} < v_r < 1$. For $\hat{R}_2 < R \leq R_2$, then if $\pi_r \in (\hat{\pi}, \pi_1)$, the equilibrium outcome is $0 < v_r < 1$. For $R_2 < R \leq \hat{\omega}$, then if $\pi_r \in (\hat{\pi}, \pi_2)$ (where π_2 is given in (B.79) and given again here for clarity as $\pi_2 = \frac{c_p\alpha}{R^2 - c_pR\alpha} + \sum_{k=1}^{\infty} a_k\delta^k$), the equilibrium outcome is $0 < v_r < 1$. Furthermore, note that when $\hat{R}_2 < R \leq R_2$, then $\pi_1 \leq \pi_2$ under the conditions of the focal region. On the other hand, when $R_2 < R \leq \hat{\omega}$, then $\pi_2 \leq \pi_1$. Hence, when $\hat{R}_2 < R \leq \hat{\omega}$, we can define $\tilde{\pi} = \min(\pi_1, \pi_2)$ as the π_r cutoff above which $0 < v_r < 1$ no longer holds in equilibrium.

In the case of $0 < v_{nr} < v_r < 1$, the vendor's price, provided in (B.41), is again given as

$$p^*_{III} = \frac{1}{9}\left(4 - \frac{1}{\pi_r\alpha} - \pi_r\alpha + \frac{\sqrt{1 + \pi_r\alpha + (\pi_r\alpha)^3 + (\pi_r\alpha)^4}}{\pi_r\alpha}\right). \tag{B.150}$$

Note that $\lim_{\pi_r \to 0}\left[p^*_{III}\right] = \frac{1}{2}$. Furthermore, the derivative of this price is given by:

$$\frac{d}{d\pi_r}\left[p^*_{III}\right] = \frac{(\alpha^2\pi_r^2 - 1)\left(-2\sqrt{\alpha^4\pi_r^4 + \alpha^3\pi_r^3 + \alpha\pi_r + 1} + \alpha\pi_r(2\alpha\pi_r + 1) + 2\right)}{18\alpha\pi_r^2\sqrt{\alpha^4\pi_r^4 + \alpha^3\pi_r^3 + \alpha\pi_r + 1}} \tag{B.151}$$

$\frac{d}{d\pi_r}\left[p^*_{III}\right] < 0$ is equivalent to $\pi_r < \frac{1}{\alpha}$ (using $\pi_r > 0$ and $\alpha > 0$). We need to show that $\hat{\pi} < \frac{1}{\alpha}$ in this range of R. Recall $\hat{\pi}$ is the unique root in π_r to (B.76). To show that $\hat{\pi} < \frac{1}{\alpha}$, since the expression defining (B.76) was shown to be strictly increasing in π_r, it suffices to show that that expression evaluated at $\pi_r = \frac{1}{\alpha}$ is greater than R in this region of the parameter space. Evaluating (B.76) at $\pi_r = \frac{1}{\alpha}$ yields $\frac{11\alpha}{16}$. We want to show that $R < \frac{11\alpha}{16}$ under the conditions of this case. Using $R < R_2$ (and recalling R_2 is defined in (B.67) as $R_2 = \frac{\alpha}{2-c_p} + \sum_{k=1}^{\infty} a_k\delta^k$), it suffices to show that $\frac{11\alpha}{16} > \frac{\alpha}{2-c_p}$ to show that $\pi_r < \frac{1}{\alpha}$ for sufficiently small δ. This holds for $\alpha > 0$ and $0 < c_p < 2 - \sqrt{3}$. Therefore, p^*_{III} is decreasing in π_r

in this range of π_r when δ is sufficiently small. Moreover, combining $\lim_{\pi_r \to 0} [p^*_{III}] = \dfrac{1}{2}$ with $\frac{d}{d\pi_r}[p^*_{III}] < 0$, we have $p^*_{III} < \frac{1}{2}$ for $\pi_r > 0$ in this region.

On the other hand, when $\pi_r \in (\hat{\pi}, \tilde{\pi})$, the consumer market equilibrium structure that arises under optimal pricing is $0 < v_r < 1$. The asymptotic expression for the vendor's price was provided above in (B.147). Taking the derivative with respect to π_r (shown in (B.148)), using the same argument as before, we have that $\frac{d}{d\pi_r}[p^*_V] < 0$ for sufficiently small δ is equivalent to $\pi_r < \frac{1}{R}$. Using that R is close to $R = \frac{\alpha}{2-c_p}$ for sufficiently small δ and using $c_p < \frac{1}{2}\left(2 - \sqrt{2}\right)$ (which is implied by the conditions on c_p from the focal region assumptions since $c_p < 2 - \sqrt{3}$ and $2 - \sqrt{3} < \frac{1}{2}\left(2 - \sqrt{2}\right)$), we have that $R\pi_r < 1$ so that the price is decreasing in π_r over this region. Moreover, for sufficiently small δ, p^*_V is close to $\frac{1}{2}$. Combined with $p^*_{III} < \frac{1}{2}$ for $\pi_r > 0$ in this region, this implies that there is a price hike at $\pi_r = \hat{\pi}$.

Lastly, when $\pi_r > \min(\pi_1, \pi_2)$, the equilibrium outcome is either $0 < v_r < v_p < 1$ if $R < R_2$ or $0 < v_{nr} < v_r < v_p < 1$ if $R \geq R_2$. In the former case, the comparative statics and price drop at $\pi_r = \pi_1$ are the same as before when $\hat{R}_1 < R < \hat{R}_2$. The vendor's price in the latter case, provided in (B.46), is given below:

$$p^*_{IV} = \frac{R - c_p\alpha}{2R} + \frac{c_p\alpha^2(4c_p + 3R\pi_r)\delta}{8R^3\pi_r} + \sum_{k=2}^{\infty} a_k\delta^k. \tag{B.152}$$

Taking the derivative with respect to π_r, we have:

$$\frac{d}{d\pi_r}[p^*_{IV}] = -\frac{c_p^2\alpha^2}{2R^3\pi_r^2}\delta + \sum_{k=2}^{\infty} a_k\delta^k. \tag{B.153}$$

Therefore, $\frac{d}{d\pi_r}[p^*_{IV}] < 0$ for sufficiently small δ. Moreover, $\frac{R-c_p\alpha}{2R} < \frac{1}{2}$ for $c_p > 0$, $\alpha > 0$, and $R > 0$ so that for sufficiently small δ, $p^*_{IV} < p^*_V$ (i.e., there is a price drop at $\pi_r = \pi_2$). ■

Full Statement of Proposition 3: Suppose the conditions of Proposition 2 are satisfied.

(a) The vendor's profit and market size are piecewise decreasing in π_r.

(b) If $0<R<R_2$:

  (i) if $0<R \leq \hat{R}_1$, the size of the ransom-paying population decreases in π_r and the expected total ransom paid increases in π_r for all $\pi_r \in (0,1)$.

  (ii) if $\hat{R}_1 < R \leq \hat{R}_2$, the size of the ransom-paying population decreases in π_r and the expected total ransom paid increases in π_r for $\pi_r \in (0,\pi_1)$. On the other hand, for $\pi_r \in (\pi_1, 1)$, both the ransom-paying population size and the expected total ransom paid increase in π_r.

  (iii) If $\hat{R}_2 < R \leq R_2$, the size of the ransom-paying population is constant in π_r and the expected total ransom paid increases in π_r for $\pi_r \in (0,\hat{\pi})$. For $\pi_r \in (\hat{\pi},\pi_1)$, the ransom-paying population size decreases and the expected total ransom paid increases in π_r. For $\pi_r \in (\pi_1, 1)$, both the ransom-paying population size and the expected total ransom paid increase in π_r.

(c) If $R_2 \leq R < \hat{\omega}$, then the size of the ransom-paying segment is constant in π_r and the expected total ransom paid is increasing in π_r over $\pi_r \in (0,\hat{\pi})$. If $\pi_r \in (\hat{\pi},\pi_2)$, then the size of the ransom-paying segment is decreasing in π_r, and the expected total ransom paid is increasing in π_r. If $\pi_r \in (\pi_2, 1)$, then both the size of the ransom-paying segment and expected total ransom paid are decreasing in π_r,

where $\hat{R}_1 = \frac{1}{(1-c_p)^2} - 1 + \tilde{\kappa}_1(\delta)$, $\hat{R}_2 = \frac{\alpha}{2} + \tilde{\kappa}_2(\delta)$, and $R_2 = \frac{\alpha}{2-c_p} + \tilde{\kappa}_3(\delta)$. The vendor's profit and market size are decreasing in π_r on each of the specified intervals above. Finally, when $R_2 < R \leq \hat{\omega}$, there exists an open interval centered at $\tilde{\pi}$ such that the expected total ransom paid is discontinuously higher in the upper half of in the interval in comparison to its measure in the lower half.

Proof of Full Statement of Proposition 3: From Lemma B.4, we have how the equilibrium market outcome changes in π_r. To complete the proof, we do comparative statics on the vendor's equilibrium profit, market size, the ransom-paying population size, and the expected total ransom paid with respect to π_r for each of the market outcomes. We then compare the values for the ransom-paying population size and expected total ransom paid to the left and right of the boundary $\pi_r = \pi_2$ marking the regime switch between $0 < v_r < 1$ and $0 < v_{nr} < v_r < v_p < 1$.

Consider $0<R \leq \hat{R}_1$. By Lemma B.4, the equilibrium outcome is $0<v_r<1$. For $0 < v_r < 1$, the size of the consumer segment willing to pay ransom in equilibrium is given as $r(\sigma^*) \triangleq 1-v_r$. In this market structure, v_r was given by (B.49). We provide it again below.

$$v_r = \frac{-1 - R\pi_r + \delta\pi_r\alpha + \sqrt{4\delta\pi_r\alpha(p+R\pi_r) + (1+R\pi_r - \delta\pi_r\alpha)^2}}{2\delta\pi_r\alpha}. \tag{B.154}$$

The vendor's price p_V^* for sufficiently small δ was given by (B.51).

Again, p_V^* has an asymptotic expression in δ given by

$$p_V^* = \frac{1}{2} - \frac{\pi_r \alpha}{8(1+R\pi_r)^2}\delta + \sum_{k=2}^{\infty} a_k \delta^k. \tag{B.155}$$

Substituting this into the expression for v_r and simplifying $r(\sigma^*)$, we have the equilibrium size of the consumer segment willing to pay ransom is

$$r_V(\sigma^*) = \frac{1}{2(1+R\pi_r)} + \sum_{k=1}^{\infty} a_k \delta^k. \tag{B.156}$$

The derivative of this with respect to π_r is given by:

$$\frac{d}{d\pi_r}\left[r_V(\sigma^*)\right] = -\frac{R}{2(1+R\pi_r)^2} + \sum_{k=1}^{\infty} a_k \delta^k. \tag{B.157}$$

Then $\frac{d}{d\pi_r}\left[r_V(\sigma^*)\right] < 0$ for sufficiently small δ for $R \leq \hat{R}_1$. Since $1 - v_r$ is also the market size of this case, this implies that the market size is decreasing in π_r in this case.

The vendor's profit has an asymptotic expansion given in (B.53), written here for reference: $\Pi_V^* = \frac{1}{4(1+R\pi_r)} + \sum_{k=1}^{\infty} a_k \delta^k$. The derivative of the profit with respect to π_r is given as:

$$\frac{d}{d\pi_r}\left[\Pi_V^*\right] = -\frac{R}{4(1+R\pi_r)^2} + \sum_{k=1}^{\infty} a_k \delta^k. \tag{B.158}$$

That $\frac{d}{d\pi_r}\left[\Pi_V^*\right] < 0$ in this range of π_r for sufficiently small δ follows from $R > 0$ and $\pi_r > 0$.

The expected total ransom paid in the case of $0 < v_r < 1$ is given as

$$T_V(\sigma^*) = \frac{R\pi_r}{4(1+R\pi_r)^2} + \sum_{k=1}^{\infty} a_k \delta^k. \tag{B.159}$$

The derivative of this with respect to π_r is given by:

$$\frac{d}{d\pi_r}\left[T_V(\sigma^*)\right] = \frac{R(1-R\pi_r)}{4(1+R\pi_r)^3} + \sum_{k=1}^{\infty} a_k \delta^k. \tag{B.160}$$

That this is increasing in π_r for sufficiently small δ is equivalent to $R\pi_r < 1$. Since $R \leq \hat{R}_1$ and $\hat{R}_1 < 1$ under the c_p conditions of the focal region (as shown in the proof of the previous proposition), it follows that $R\pi_r < 1$ so that the expected total ransom paid is increasing in π_r in this range of R.

For $\hat{R}_1 < R \leq \hat{R}_2$, the equilibrium outcome is $0 < v_r < 1$ for $\pi_r < \pi_1$ and $0 < v_r < v_p < 1$ for $\pi_r \geq \pi_1$. Note that $R\pi_r < 1$ still holds for $R < \hat{R}_2 = \frac{\alpha}{2}$ and $\pi_r < \pi_1$ under the conditions of the focal region. Hence, the expected total ransom paid is still increasing in π_r in this region when $0 < v_r < 1$ is induced in equilibrium. Moreover, the comparative statics on the vendor's profit, market size, and ransom-paying population remain the same as it was in the lower R region in the preceding paragraphs.

For $\pi_r \geq \pi_1$, the vendor's profit is given in (B.59). For reference, the expression for the profit of this case is provided here: $\Pi^*_{VI} = \frac{1}{4}(1-c_p)^2 + \frac{(1-c_p)c_p^2\alpha\delta}{2R^2\pi_r} + \sum_{k=2}^{\infty} a_k\delta^k$. Taking the derivative with respect to π_r, we have:

$$\frac{d}{d\pi_r}\left[\Pi^*_{VI}\right] = -\frac{(1-c_p)c_p^2\alpha}{2R^2\pi_r^2}\delta + \sum_{k=2}^{\infty} a_k\delta^k. \tag{B.161}$$

This shows that for sufficiently low δ, the vendor's profit is decreasing in π_r, and the sign of this direction does not change with the magnitude of R.

The market size can be found by plugging in the vendor's equilibrium price, given in (B.57), into the expression for v_r, given in (B.55). This gives the expression for v_r in equilibrium under optimal pricing, and to find the market size, one would just subtract this from 1. The asymptotic expression for the market size in this case is given below.

$$M_{VI}(\sigma^*) = \frac{1-c_p}{2} + \frac{c_p^2\alpha}{2R^2\pi_r}\delta + \sum_{k=2}^{\infty} a_k\delta^k. \tag{B.162}$$

Taking the derivative with respect to π_r, we have:

$$\frac{d}{d\pi_r}\left[M_{VI}(\sigma^*)\right] = -\frac{c_p^2\alpha}{2R^2\pi_r^2}\delta + \sum_{k=2}^{\infty} a_k\delta^k. \tag{B.163}$$

This shows that for sufficiently low δ, the equilibrium market size is decreasing in π_r, and the sign of this direction does not change with the magnitude of R.

Similarly, the equilibrium ransom-paying population size is $v_p - v_r$ at the vendor's optimal price. The asymptotic expression for this is given below.

$$r_{VI}(\sigma^*) = \frac{2c_p}{R(1+c_p)} - \frac{c_p\alpha}{R^3}\left(R + \frac{2c_p}{(1+c_p)^2\pi_r}\right)\delta + \sum_{k=2}^{\infty} a_k\delta^k. \tag{B.164}$$

Taking the derivative with respect to π_r, we have:

$$\frac{d}{d\pi_r}\left[r_{VI}(\sigma^*)\right] = \frac{2c_p^2\alpha}{(1+c_p)^2R^3\pi_r^2}\delta + \sum_{k=2}^{\infty} a_k\delta^k. \tag{B.165}$$

This shows that for sufficiently low δ, the equilibrium ransom-paying population size is increasing in π_r, and the sign of this direction does not change with the magnitude of R.

The expected total ransom paid in this case is given by $\pi_r u(\sigma) r(\sigma) R = \pi_r (v_p - v_r)^2 R$. The asymptotic expression for this is given below.

$$T_{VI}(\sigma^*) = \frac{4c_p^2 \pi_r}{(1+c_p)^2 R} + \sum_{k=1}^{\infty} a_k \delta^k. \tag{B.166}$$

Taking the derivative with respect to π_r, we have:

$$\frac{d}{d\pi_r}\left[T_{VI}(\sigma^*)\right] = \frac{4c_p^2}{(1+c_p)^2 R} + \sum_{k=1}^{\infty} a_k \delta^k. \tag{B.167}$$

This shows that for sufficiently low δ, the equilibrium expected total ransom paid is increasing in π_r, and the sign of this direction does not change with the magnitude of R.

Next, consider $\hat{R}_2 < R \leq R_2$. By Lemma B.4, for $\pi_r < \hat{\pi}$, the equilibrium market outcome is $0 < v_{nr} < v_r < 1$. For $\pi_r \in (\hat{\pi}, \pi_1)$, the equilibrium market outcome is $0 < v_r < 1$. For $\pi_r > \pi_1$, the equilibrium outcome is $0 < v_r < v_p < 1$.

For $0 < v_{nr} < v_r < 1$, the vendor's equilibrium profit is given in (B.42). For reference, the expression is given as: $\Pi^*_{III} = \frac{\left(3+3\alpha\pi_r - \sqrt{5+\pi_r\alpha(-2+5\pi_r\alpha)+4\sqrt{1+\pi_r\alpha+(\pi_r\alpha)^3+(\pi_r\alpha)^4}}\right)}{54(\pi_r\alpha)^2} \times \left(-1+\pi_r\alpha(4-\pi_r\alpha)+\sqrt{1+\pi_r\alpha+(\pi_r\alpha)^3+(\pi_r\alpha)^4}\right)$. Taking the derivative of this with respect to π_r, we have:

$$\begin{aligned}\frac{d}{d\pi_r}[\Pi^*_{III}] = ((-4\alpha\pi_r + \frac{\alpha\pi_r(4\alpha\pi_r-1)+1}{\sqrt{\alpha\pi_r(\alpha\pi_r-1)+1}} + 8)(-(4\sqrt{\alpha^4\pi_r^4+\alpha^3\pi_r^3+\alpha\pi_r+1}+ \\ \alpha\pi_r(5\alpha\pi_r-2)+5)^{\frac{1}{2}} + 3\alpha\pi_r+3))(108\alpha\pi_r^2)^{-1}+ \\ ((\sqrt{\alpha^4\pi_r^4+\alpha^3\pi_r^3+\alpha\pi_r+1}+\alpha\pi_r(4-\alpha\pi_r)-1)(\sqrt{4\sqrt{\alpha^4\pi_r^4+\alpha^3\pi_r^3+\alpha\pi_r+1}+\alpha\pi_r(5\alpha\pi_r-2)+5}- \\ 3\alpha\pi_r-3))(27\alpha^2\pi_r^3)^{-1}+ \\ \frac{(\sqrt{\alpha^4\pi_r^4+\alpha^3\pi_r^3+\alpha\pi_r+1}+\alpha\pi_r(4-\alpha\pi_r)-1)(3\alpha - \frac{\alpha(10\alpha\pi_r+\frac{2\alpha\pi_r(4\alpha\pi_r-1)+2}{\sqrt{\alpha\pi_r(\alpha\pi_r-1)+1}}-2)}{2\sqrt{4\sqrt{\alpha^4\pi_r^4+\alpha^3\pi_r^3+\alpha\pi_r+1}+\alpha\pi_r(5\alpha\pi_r-2)+5}})}{54\alpha^2\pi_r^2}.\end{aligned} \tag{B.168}$$

This is strictly negative under the conditions of the focal region (namely, $0 < c_p < 2-\sqrt{3}$ and $\frac{(c_p-2)^2 c_p}{(c_p-1)^2} < \alpha < 2(c_p-2)^2$). The market size is $M_{III} = 1 - v_{nr}$ in this case, and the equilibrium v_{nr} can be found by plugging the vendor's optimal price (B.41) into the expression defining v_{nr} in this case (B.40). The expression is given as:

$$M_{III}(\sigma^*) = \frac{-\sqrt{4\sqrt{\alpha^4\pi_r^4+\alpha^3\pi_r^3+\alpha\pi_r+1}+\alpha\pi_r(5\alpha\pi_r-2)+5}+3\alpha\pi_r+3}{6\alpha\pi_r}. \tag{B.169}$$

The derivative of this with respect to π_r is given by:

$$\frac{d}{d\pi_r}\left[M_{III}(\sigma^*)\right] = -\frac{10\alpha\pi_r + \frac{2\alpha\pi_r(4\alpha\pi_r-1)+2}{\sqrt{\alpha\pi_r(\alpha\pi_r-1)+1}} - 2}{12\pi_r\sqrt{4\sqrt{\alpha^4\pi_r^4+\alpha^3\pi_r^3+\alpha\pi_r+1}+\alpha\pi_r(5\alpha\pi_r-2)+5}} + \frac{\sqrt{4\sqrt{\alpha^4\pi_r^4+\alpha^3\pi_r^3+\alpha\pi_r+1}+\alpha\pi_r(5\alpha\pi_r-2)+5} - 3\alpha\pi_r - 3}{6\alpha\pi_r^2} + \frac{1}{2\pi_r}. \quad \text{(B.170)}$$

The c_p and α conditions of the focal region imply that this is negative for $\pi_r > 0$, so the market size decreases in π_r in this case.

The size of the consumer segment willing to pay ransom in equilibrium is given as $r(\sigma^*) \triangleq 1 - v_r$. In this market structure, $v_r = \frac{R}{\alpha(1-\delta)}$. This is constant in π_r, and so the ransom-paying population size $r_{III}(\sigma^*) = 1 - v_r$ is constant in π_r.

For the expected total ransom paid, that is given by $T(\sigma^*) \triangleq \pi_r u(\sigma^*) r(\sigma^*) R$, where $u(\sigma^*)$ is the size of the consumer segment willing to remain unpatched. The expected total ransom paid is $T_{III}(\sigma^*) = \pi_r u(\sigma^*) r(\sigma^*) R$. For $0 < v_{nr} < v_r < 1$, $u(\sigma^*) = 1 - v_{nr}$ in equilibrium, where the equilibrium v_{nr} can be found by substituting the vendor's optimal price (B.41) into the expression defining v_{nr} in (B.40).

$$T_{III}(\sigma^*) = R\left(-\sqrt{4\sqrt{\alpha^4\pi_r^4+\alpha^3\pi_r^3+\alpha\pi_r+1}+\alpha\pi_r(5\alpha\pi_r-2)+5} + 3\alpha\pi_r + 3\right) \times \frac{(\alpha(\delta-1)+R)}{6\alpha^2(\delta-1)}. \quad \text{(B.171)}$$

The derivative with respect to π_r of this is given by:

$$\frac{d}{d\pi_r}\left[T_{III}(\sigma^*)\right] = R(R-\alpha(1-\delta))\frac{\left(3\alpha - \frac{\alpha\left(10\alpha\pi_r + \frac{2\alpha\pi_r(4\alpha\pi_r-1)+2}{\sqrt{\alpha\pi_r(\alpha\pi_r-1)+1}} - 2\right)}{2\sqrt{4\sqrt{\alpha^4\pi_r^4+\alpha^3\pi_r^3+\alpha\pi_r+1}+\alpha\pi_r(5\alpha\pi_r-2)+5}}\right)}{6\alpha^2(\delta-1)}. \quad \text{(B.172)}$$

For sufficiently small δ, $R(R-\alpha(1-\delta)) < 0$ within this range since $R \in (\hat{R}_2, R_2)$, where $\hat{R}_2 = \frac{\alpha}{2} + \tilde{\kappa}_2(\delta)$ and $R_2 = \frac{\alpha}{2-c_p} + \tilde{\kappa}_3(\delta)$ are asymptotic expressions of the R bounds from Lemma B.4 that govern this region of R. The remaining factor of $\frac{d}{d\pi_r}\left[T_{III}(\sigma^*)\right]$ is negative for sufficiently small δ under the focal region assumptions on c_p and α, as well as $\pi_r > 0$. Consequently, $\frac{d}{d\pi_r}\left[T_{III}(\sigma^*)\right] > 0$ for sufficiently small δ, so that the expected total ransom paid is increasing in π_r.

Next, for $0 < v_r < 1$, again the comparative statics hold for the vendor's profit, market size, and equilibrium ransom-paying population size since those comparative statics results did not depend on the magnitude of R for sufficiently small δ (as was shown in (B.157), (B.158), and (B.160)). Note that $R\pi_r < 1$ still holds for $\pi_r < \pi_1$ (where $\pi_1 = \frac{(2-c_p)c_p}{(1-c_p)^2 R} + \sum_{k=1}^{\infty} a_k\delta^k$ from (B.72) and (B.73)) since $\frac{(2-c_p)c_p}{(1-c_p)^2} < 1$ using $0 < c_p < 2-\sqrt{3}$.

For $0 < v_r < v_p < 1$, the comparative statics still hold in the same way that they had for $\hat{R}_1 < R < \hat{R}_2$, since those did not depend on the value of R or any of the other parameters.

Lastly, for $R_2 < R < \hat{\omega}$, the equilibrium outcome is $0 < v_{nr} < v_r < 1$ for $\pi_r < \hat{\pi}$, $0 < v_r < 1$ for $\pi_r \in (\hat{\pi}, \pi_2)$, and $0 < v_{nr} < v_r < v_p < 1$ for $\pi_r \in (\pi_2, 1)$ by Lemma B.4.

For $0 < v_{nr} < v_r < 1$, the comparative statics remains the same as for $R < R_2$, since those results did not depend on the magnitude of R, π_r, or any other parameters.

For $0 < v_r < 1$, again the comparative statics results do not change for R close to R_2. Focusing on the region of R close to $R = R_2$, we have that the expected total ransom paid increases in R as long as $c_p < 1 - \frac{1}{\sqrt{2}}$, which holds under the focal region since $1 - \frac{1}{\sqrt{2}} < 2 - \sqrt{3}$.

Lastly, for $0 < v_{nr} < v_r < v_p < 1$, the vendor's profit was given in (B.48). This decreases in π_r under the assumptions of the focal region. For reference, it is $\Pi_{IV}^* = \frac{R - c_p\alpha}{4R} + \frac{c_p\alpha^2(2c_p + R\pi_r)\delta}{8R^3\pi_r} + \sum_{k=2}^{\infty} a_k\delta^k$.

Then the derivative of the profit with respect to π_r is given by:

$$\frac{d}{d\pi_r}\left[\Pi_{IV}\right] = -\frac{c_p^2\alpha^2}{4R^2\pi_r^2}\delta + \sum_{k=2}^{\infty} a_k\delta^k. \tag{B.173}$$

So the profit shrinks in π_r in this case.

The market size of this case is $M_{IV} = 1 - v_{nr}$. The equilibrium v_{nr} can be found by substituting the vendor's optimal price (B.46) into the expression defining v_{nr} for this case in (B.44). The asymptotic expression for the market size of this case is given below.

$$M_{IV}(\sigma^*) = \frac{1}{2} - \frac{\alpha^2 c_p}{8R^2 - 8\alpha c_p R}\delta + \frac{\alpha^3 c_p\left(-4\alpha c_p^2 + c_p R(6 - \alpha\pi_r) + 2\pi_r R^2\right)}{16\pi_r R^3(R - \alpha c_p)^2}\delta^2 + \sum_{k=3}^{\infty} a_k\delta^k. \tag{B.174}$$

Taking the derivative with respect to π_r, we have:

$$\frac{d}{d\pi_r}\left[M_{IV}(\sigma^*)\right] = \frac{\alpha^3 c_p^2(2\alpha c_p - 3R)}{8\pi_r^2 R^3(R - \alpha c_p)^2}\delta^2 + \sum_{k=3}^{\infty} a_k\delta^k. \tag{B.175}$$

Since $R \in (R_2, \hat{\omega})$, R is arbitrarily close to $\frac{\alpha}{2-c_p}$. Substituting this expression for R into the above, the derivative of the market size within this region for this market structure is given as:

$$\frac{d}{d\pi_r}\left[M_{IV}(\sigma^*)\right] = -\frac{(c_p - 2)^4 c_p^2(2(c_p - 2)c_p + 3)}{8\alpha(c_p - 1)^4\pi_r^2}\delta^2 + \sum_{k=3}^{\infty} a_k\delta^k. \tag{B.176}$$

That $-\frac{(c_p-2)^4 c_p^2(2(c_p-2)c_p+3)}{8\alpha(c_p-1)^4\pi_r^2} < 0$ follows from the focal conditions on c_p and α, as well as $\pi_r > 0$. Hence, the market size shrinks under the conditions of the focal region for $R \in (R_2, \hat{\omega})$ for sufficiently small δ.

The size of the consumer segment willing to pay ransom in equilibrium is given as $r(\sigma^*) \triangleq v_p - v_r$. In this market structure, v_r and v_p were given by Case (IV) in Lemma B.2. In particular, $v_r = \frac{R}{\pi_r\alpha(1-\delta)}$ and $v_p = v_{nr} + \frac{v_{nr}-p}{\pi_r\alpha v_{nr}}$. The asymptotic expression for v_{nr} is given in (B.44). Substituting in the vendor's price, given in (B.46) and simplifying $r(\sigma^*)$, we have the equilibrium size of the consumer segment willing to pay ransom in this case is

$$r_{IV}(\sigma^*) = \frac{1}{2} - \frac{R}{\alpha} + \frac{c_p}{R\pi_r} + \sum_{k=1}^{\infty} a_k\delta^k. \tag{B.177}$$

The derivative of this with respect to π_r is given by

$$\frac{d}{d\pi_r}\left[r_{IV}(\sigma^*)\right] = -\frac{c_p}{R\pi_r^2} + \sum_{k=1}^{\infty} a_k\delta^k. \tag{B.178}$$

Hence, $r_{IV}(\sigma^*)$ is decreasing in π_r for sufficiently small δ.

For the total expected ransom paid, this is given as

$$T_{IV}(\sigma^*) = c_p\left(\frac{1}{2} - \frac{R}{\alpha} + \frac{c_p}{R\pi_r}\right) + \sum_{k=1}^{\infty} a_k\delta^k. \tag{B.179}$$

The derivative of this with respect to π_r is given by:

$$\frac{d}{d\pi_r}\left[T_{IV}(\sigma^*)\right] = -\frac{c_p^2}{R\pi_r^2} + \sum_{k=1}^{\infty} a_k\delta^k. \tag{B.180}$$

Therefore, $\frac{d}{d\pi_r}\left[T_{IV}(\sigma^*)\right] < 0$ for sufficiently small δ in this case.

To complete the proof of Proposition 3, we compare $(B.177)$ to $(B.156)$ at their π_r boundary $\pi_r = \pi_2$ when $R \in (R_2, \hat{\omega})$ to show that, for sufficiently small δ, $r_{IV}(\sigma^*) < r_V(\sigma^*)$ at the $\pi_r = \pi_2$ boundary. Specifically, recall $\pi_2 = \frac{c_p\alpha}{R^2 - c_pR\alpha} + \sum_{k=1}^{\infty} a_k\delta^k$ from (B.79) and $R_2 = \frac{\alpha}{2-c_p} + \sum_{k=1}^{\infty} a_k\delta^k$ from (B.67) and $\hat{\omega}$ is arbitrarily close to R_2. Comparing the ransom-paying population size at this π_r boundary within this R range, for sufficiently small δ, we have the following.

Substituting in $\pi_r = \pi_2$ for $(B.177)$ gives:

$$r_{IV}(\sigma^*) = \frac{1}{2} - c_p + \sum_{k=1}^{\infty} a_k\delta^k. \tag{B.181}$$

Substituting in $\pi_r = \pi_2$ for $(B.156)$ gives:

$$r_V(\sigma^*) = \frac{1}{2}(1-c_p)^2 + \sum_{k=1}^{\infty} a_k \delta^k. \tag{B.182}$$

That $\frac{1}{2}(1-c_p)^2 > \frac{1}{2} - c_p$ follows from the focal region condition on c_p, so there is a drop in $r(\sigma^*)$ at this boundary.

Comparing the expected total ransom paid at this boundary, for sufficiently small δ:

Substituting in $\pi_r = \pi_2$ for $(B.179)$ gives:

$$T_{IV}(\sigma^*) = \frac{c_p^2(8 - 8c_p + 2c_p^2 - \pi_r\alpha)}{2(2-c_p)\pi_r\alpha} + \sum_{k=1}^{\infty} a_k \delta^k. \tag{B.183}$$

Substituting in $\pi_r = \pi_2$ for $(B.159)$ gives:

$$T_V(\sigma^*) = \frac{(2-c_p)\pi_r\alpha}{4(2-c_p+\pi_r\alpha)^2} + \sum_{k=1}^{\infty} a_k \delta^k. \tag{B.184}$$

That $\frac{c_p^2(8-8c_p+2c_p^2-\pi_r\alpha)}{2(2-c_p)\pi_r\alpha} > \frac{(2-c_p)\pi_r\alpha}{4(2-c_p+\pi_r\alpha)^2}$ follows from $0 < c_p < 2-\sqrt{3}$. Therefore, $T_{IV}(\sigma^*) > T_V(\sigma^*)$ at that π_r boundary for sufficiently small δ. ■

Proof of Proposition A.1: From Lemma B.3, under the conditions of this proposition, the consumer market equilibrium outcome is $0 < v_{nr} < v_r < v_p < 1$.

If $0 < v_{nr} < v_r < v_p < 1$ is induced in equilibrium, then from (B.48), the asymptotic expression for the vendor's profit is given by

$$\Pi_{IV}^* = \frac{R - c_p\alpha}{4R} + \frac{c_p\alpha^2(2c_p + R\pi_r)\delta}{8R^3\pi_r} + \sum_{k=2}^{\infty} a_k \delta^k. \tag{B.185}$$

Differentiating with respect to δ, we have that $\frac{d}{d\delta}\left[\Pi_{IV}^*\right] = \frac{c_p\alpha^2(2c_p+R\pi_r)}{8R^3\pi_r} + \sum_{k=1}^{\infty} a_k\delta^k$. It follows that for sufficiently small δ, $\frac{d}{d\delta}\left(\Pi_{IV}^*\right) > 0$.

For the market size to be decreasing in δ, we will show that the equilibrium v_{nr} under optimal pricing is increasing in δ. By substituting in (B.46) into (B.44), the asymptotic expression for this threshold in equilibrium is given by

$$v_{nr}^* = \frac{1}{2} + \frac{c_p\alpha^2}{8R(R-c_p\alpha)}\delta + \sum_{k=2}^{\infty} a_k \delta^k. \tag{B.186}$$

Taking the derivative of this with respect to δ and focusing on the zero-order term, we have that $\frac{d}{d\delta}\left[v_{nr}^*\right] = \frac{c_p\alpha^2}{8R(R-c_p\alpha)} + \sum_{k=2}^{\infty} a_k\delta^k$. This is positive for sufficiently small δ since $R > c_p\alpha$

is a condition of this case (for $p^*_{IV} > 0$). Hence, the market size shrinks in δ for sufficiently small δ under the conditions of this case.

The aggregate unpatched loss measure given as

$$UL \triangleq \int_{\mathcal{V}} \mathbb{1}_{\{\sigma^*(v)=(B,NP,NR)\}} \pi_r \alpha u(\sigma^*) v dv + \int_{\mathcal{V}} \mathbb{1}_{\{\sigma^*(v)=(B,NP,R)\}} \pi_r u(\sigma^*)(R + \delta\alpha v) dv \,.$$

Consequently, aggregate unpatched losses are given as

$$UL^*_{IV} = \int_{v_{nr}}^{v_r} \pi_r \alpha (v_p - v_{nr}) v dv + \int_{v_r}^{v_p} \pi_r (v_p - v_{nr})(R + \delta\alpha v) dv \,. \tag{B.187}$$

Substituting (B.46) into (B.44), we can characterize the equilibrium v_{nr} threshold as

$$v_{nr}(p^*_{IV}) = \frac{1}{2} + \frac{c_p \alpha^2 \delta}{8R(R - c_p\alpha)} + \sum_{k=2}^{\infty} a_k \delta^k. \tag{B.188}$$

Then substituting (B.46) and (B.188) into (B.24), we can characterize the equilibrium v_p threshold as

$$v_p(p^*_{IV}) = \frac{1}{2} + \frac{c_p}{R\pi_r} + \frac{c_p\alpha(8c_p^2\alpha + R^2\pi_r(-4 + \pi_r\alpha) + 4c_pR(-2 + \pi_r\alpha))\delta}{8R^3(R - c_p\alpha)\pi_r^2} + \sum_{k=2}^{\infty} a_k \delta^k. \tag{B.189}$$

Finally, the asymptotic expansion of $v_r = \frac{R}{\alpha(1-\delta)}$ is given by

$$v_r(p^*_{IV}) = \frac{R(1+\delta)}{\alpha} + \sum_{k=2}^{\infty} a_k \delta^k. \tag{B.190}$$

Substituting in (B.188), (B.190), and (B.189) into the above expression, the asymptotic characterization of the aggregate unpatched losses is given as

$$UL^*_{IV} = \frac{c_p(8c_p\alpha - (-2R + \alpha)^2\pi_r)}{8R\pi_r\alpha} + \frac{c_p\left(\frac{R(2R-\alpha)(4R^3 - 4c_pR^2\alpha + R\alpha^2 - 2c_p\alpha^3)}{\alpha(-R + c_p\alpha)} - \frac{24c_p^2\alpha}{\pi_r^2} + \frac{2c_p(4R^2 - 8R\alpha + \alpha^2)}{\pi_r}\right)}{16R^3}\delta + \sum_{k=2}^{\infty} a_k \delta^k. \tag{B.191}$$

Taking the derivative with respect to δ, we have

$$\frac{d}{d\delta}[UL^*_{IV}] \;=\; \frac{c_p\left(\frac{R(2R-\alpha)(4R^3-4c_pR^2\alpha+R\alpha^2-2c_p\alpha^3)}{\alpha(-R+c_p\alpha)} - \frac{24c_p^2\alpha}{\pi_r^2} + \frac{2c_p(4R^2-8R\alpha+\alpha^2)}{\pi_r}\right)}{16R^3} \;+\; \sum_{k=1}^{\infty} b_k\delta^k. \tag{B.192}$$

Under the assumptions of the focal region along with $\pi_r > \bar{\pi}$ and $R > \frac{\alpha}{2-c_p}$, we have that $\frac{c_p\left(\frac{R(2R-\alpha)(4R^3-4c_pR^2\alpha+R\alpha^2-2c_p\alpha^3)}{\alpha(-R+c_p\alpha)} - \frac{24c_p^2\alpha}{\pi_r^2} + \frac{2c_p(4R^2-8R\alpha+\alpha^2)}{\pi_r}\right)}{16R^3} < 0$ for sufficiently small δ.

Similarly, denote consumer surplus as $CS \triangleq \int_{\mathcal{V}} \mathbb{1}_{\{\sigma^*(v)\in\{(B,NP,NR),(B,NP,R),(B,P)\}\}} U(v,\sigma)dv$. For the case of $0 < v_{nr} < v_r < v_p < 1$, this becomes

$$CS^*_{IV} = \int_{v_{nr}}^{v_r} (v - p^*_{IV} - \pi_r\alpha(v_p - v_{nr})v)\, dv + \int_{v_r}^{v_p} (v - p^*_{IV} - \pi_r(v_p - v_{nr})(R + \delta\alpha v))\, dv + \int_{v_p}^{1} (v - p^*_{IV} - c_p)\, dv\,. \tag{B.193}$$

Substituting in (B.46), (B.188), (B.190), and (B.189) into the above expression, the asymptotic characterization of consumer surplus is given as

$$CS^*_{IV} = \frac{1}{8}\left(1 + c_p\left(-8 + \frac{4R}{\alpha} + \frac{3\alpha}{R}\right)\right) + \frac{c_p(4c_p^2\alpha^2 + c_p\alpha(-4R^2 + 4R\alpha - 3\alpha^2)\pi_r + R(4R^3 - 2R^2\alpha + R\alpha^2 - 2\alpha^3)\pi_r^2)}{8R^3\alpha\pi_r^2}\delta + \sum_{k=2}^{\infty} a_k\delta^k. \tag{B.194}$$

Taking the derivative with respect to δ, we have

$$\frac{d}{d\delta}[CS^*_{IV}] = \frac{c_p(4c_p^2\alpha^2 + c_p\alpha(-4R^2 + 4R\alpha - 3\alpha^2)\pi_r + R(4R^3 - 2R^2\alpha + R\alpha^2 - 2\alpha^3)\pi_r^2)}{8R^3\alpha\pi_r^2} + \sum_{k=2}^{\infty} b_k\delta^k. \tag{B.195}$$

We will show that $\frac{c_p(4c_p^2\alpha^2+c_p\alpha(-4R^2+4R\alpha-3\alpha^2)\pi_r+R(4R^3-2R^2\alpha+R\alpha^2-2\alpha^3)\pi_r^2)}{8R^3\alpha\pi_r^2} < 0$ under the conditions of the proposition. This is equivalent to $4c_p^2\alpha^2 + c_p\alpha(-4R^2 + 4R\alpha - 3\alpha^2)\pi_r + R(4R^3 -$

$2R^2\alpha + R\alpha^2 - 2\alpha^3)\pi_r^2 < 0$. This is a quadratic in π_r, with negative second-order term for R sufficiently close to the $R = \frac{\alpha}{2-c_p}$ boundary (R_2). For this to hold, either there are two real roots and π_r is larger than the larger root of that quadratic or smaller than the smaller root of that quadratic, or there are not two real roots (in which case the inequality is always satisfied). Under the conditions the proposition, the larger of the two roots is given by

$$\pi_{r,2} = \frac{8c_p\alpha}{4R^2 - 4R\alpha + 3\alpha^2 + \sqrt{-48R^4 + 24R^2\alpha^2 + 8R\alpha^3 + 9\alpha^4}}. \tag{B.196}$$

Under the conditions of the proposition, the smaller root is negative, so we want to show that $\pi_r > \pi_{r,2}$ in this case. Note that $\frac{c_p\alpha}{R^2 - c_pR\alpha} > \pi_{r,2}$ from the conditions of the proposition and the focal region assumptions, and since $\pi_r > \frac{c_p\alpha}{R^2 - c_pR\alpha}$ for this proposition, it follows that $\pi_r > \pi_{r,2}$. Therefore, $\frac{d}{d\delta}[CS_{IV}^*] < 0$ for sufficiently small δ under the conditions of the proposition. ■

Proof of Proposition A.2: From the consumer utility function (B.1), a consumer of valuation v prefers (B, NP, R) over (B, NP, NR) if and only if $v - p - \pi_r u(\sigma)(R + \delta\alpha v) \geq v - p - \pi_r u(\sigma)\alpha v$. This is equivalent to $v \geq \frac{R}{\alpha(1-\delta)}$. Consequently, if $\frac{R}{\alpha(1-\delta)} > 1$ (or $\delta > 1 - \frac{R}{\alpha}$), then no consumer would prefer (B, NP, R) over (B, NP, NR).

As (B, NP, R) is a strictly dominated option under this condition, consumers are left with (NB), (B, NP, NR), and (B, P) as incentive-compatible choices. Consequently, when $\delta > 1 - \frac{R}{\alpha}$, the equilibrium characterization of the consumption subgame no longer depends on R or δ, as in Lemma B.11.

In particular, if Case (I) of Lemma B.11 holds, then $0 < v_{nr} < 1$ would be the equilibrium outcome. From the first-order condition, the interior optimal price of this case is given by $p_I^* = \frac{-1+\pi_r\alpha(4-\pi_r\alpha)+\sqrt{1+\pi_r\alpha+(\pi_r\alpha)^3+(\pi_r\alpha)^4}}{9\pi_r\alpha}$. If $\pi_r\alpha > \frac{(2-3c_p)c_p}{1-2c_p}$, then p_I^* will not satisfy the conditions in (I). Note that $\frac{(2-3c_p)c_p}{1-2c_p} > 0$ holds under the conditions of the focal region. Hence, if $\pi_r\alpha > \frac{c_p(-2+3c_p)}{-1+2c_p}$ and $\delta > 1 - \frac{R}{\alpha}$, then $0 < v_{nr} < v_p < 1$ is the equilibrium outcome, and no measures of interest change in R or δ.

On the other hand, the condition of Case (II) of Lemma B.11 is $c_p + (-1 + c_p + p)\pi_r\alpha < c_p^2$. Therefore, if $\alpha < \frac{c_p}{\pi_r}$, then this condition cannot be met by any price $p > 0$. Consequently, if $\alpha < \frac{c_p}{\pi_r}$ and $\delta > 1 - \frac{R}{\alpha}$, then the equilibrium outcome will be $0 < v_{nr} < 1$, and no measures of interest change in R or δ. ■

Proof of Proposition 4: Lemma B.9 provides the characterization of the consumer market equilibrium under optimal pricing as R changes. To complete the proof, we compute the vendor's price, profit, and market size for each market outcome to do comparative statics with respect to R.

When $0 < v_r < 1$ is induced in equilibrium, the vendor's price is given in (B.127) and given again for reference here: $p_V^* = \Big(-1 - 2R\pi_r + 4\delta\pi_r\alpha - R^2\pi_r^2 - 2R\delta\alpha\pi_r^2 - (\delta\pi_r\alpha)^2 +$

$(1+R\pi_r+\delta\pi_r\alpha)\sqrt{1+\pi_r(2R-\delta\alpha+(R+\delta\alpha)^2\pi_r)}\Big)\Big(9\delta\pi_r\alpha\Big)^{-1}$. The asymptotic expression for this is given as:

$$p_V^* = \frac{1}{2} - \frac{\pi_r\alpha}{8(1+R\pi_r)^2}\delta + \sum_{k=2}^{\infty} a_k\delta^k. \tag{B.197}$$

Taking the derivative of this with respect to R gives:

$$\frac{d}{dR}\left[p_V^*\right] = \frac{\alpha\pi_r^2}{4(1+R\pi_r)^3}\delta + \sum_{k=2}^{\infty} a_k\delta^k. \tag{B.198}$$

For sufficiently small δ, we have $\frac{d}{dR}\left[p_V^*\right] > 0$ so the price increases in R.

The asymptotic expansion in δ for the vendor's profit at this price is given in (B.128). For reference, it is provided again here as: $\Pi_V^* = \frac{1}{4(1+R\pi_r)} + \sum_{k=1}^{\infty} a_k\delta^k$. The derivative of this with respect to R is:

$$\frac{d}{dR}\left[\Pi_V^*\right] = -\frac{\pi_r}{4(1+R\pi_r)^2} + \sum_{k=1}^{\infty} a_k\delta^k. \tag{B.199}$$

For sufficiently small δ, we have $\frac{d}{dR}\left[\Pi_V^*\right] < 0$ so the profit decreases in R.

The market size is $M_V = 1 - v_r$, and the equilibrium v_r can be found by substituting (B.127) into (B.126). This has an asymptotic expansion given by

$$M_V^* = \frac{1}{2(1+R\pi_r)} + \sum_{k=1}^{\infty} a_k\delta^k. \tag{B.200}$$

Taking the derivative with respect to R of this, we have:

$$\frac{d}{dR}\left[M_V^*\right] = -\frac{\pi_r}{2(1+R\pi_r)^2} + \sum_{k=1}^{\infty} a_k\delta^k. \tag{B.201}$$

Altogether, this shows that, for sufficiently small δ, the vendor's equilibrium price increases in R while the profit and market size shrinks in R.

For the cases of $0 < v_{nr} < v_r < 1$ and $0 < v_{nr} < 1$, the equilibrium prices are given in (B.124) and (B.122), respectively. Note that they are equal, and neither depend on R. Hence, as R changes, the equilibrium price remains constant across R in these regimes. It follows that the vendor's equilibrium profit is constant in R as well. Also, since v_{nr} does not directly depend on R in either market outcome (and since it does not indirectly depend on R through the price p), it follows that the market size is also constant in R. ■

Proof of Proposition 5: Lemmas B.9 and B.10 provide the characterization of the consumer market equilibrium under optimal pricing as π_r changes. Lemma B.9 states that when $R \leq \tilde{R}_2$, the equilibrium consumer market structure is $0 < v_r < 1$. Lemma B.10 states that for a higher range of R (specifically, $\tilde{R}_2 < R < \tilde{R}_3$), the market structure is $0 < v_{nr} < v_r < 1$ for $\pi_r < \hat{\pi}$ and is $0 < v_r < 1$ otherwise. To complete the proof, it suffices to do comparative statics on the equilibrium price for the three cases. Then we will compare the prices.

For the case of $0 < v_r < 1$, the price comparative statics are the same as provided in the proof of Proposition 2. In particular, for sufficiently low δ, the price decreases in π_r if $\pi_r < \frac{1}{R}$. This implies that the price decreases for all $\pi_r \in (0, 1)$ if and only if $R < 1$. Since R is bounded above by $\tilde{R}_2$, for sufficiently low δ, $R > 1$ can only happen is $\alpha > 2$. Hence, if $\alpha > 2$ and $R > 1$, then the vendor's price decreases in π_r for $(0, \frac{1}{R})$ and increases in π_r for $\pi_r > \frac{1}{R}$. Otherwise, if either $\alpha \leq 2$ or $R \leq 1$, then the vendor's price will decrease in π_r for all π_r.

For the case of $0 < v_{nr} < v_r < 1$, the equilibrium price is given in (B.122). The price is decreasing in π_r as long as $\pi_r < \frac{1}{\alpha}$ (for the same reason as in Proposition 2). We want to show that $\frac{1}{\alpha} < \pi'$. For sufficiently low δ, this is equivalent to $R < \frac{11\alpha}{16}$. For sufficiently small δ and R sufficiently close to $R = \tilde{R}_2$, there exists an $\tilde{R}_3$ such that for $R \in (\tilde{R}_2, \tilde{R}_3)$, we have that $R < \frac{11\alpha}{16}$. Hence, the price is decreasing in π_r for R close to $R = \tilde{R}_2$. Repeating the same argument as in the proof of Proposition 2 by comparing the prices (which are the same expressions as in Proposition 2), we have that there is a price hike at $\pi_r = \pi'$. ■

Proof of Proposition 6: Lemma B.10 provides the characterization of the consumer market equilibrium under optimal pricing as π_r changes for $R \in (\tilde{R}_2, \tilde{R}_3)$, and Lemma B.9 provides the consumer market outcome characterization for $R \leq \tilde{R}_2$. To complete the proof, we compute the vendor's equilibrium profit, market size, the size of the population willing to pay ransom, and the expected total ransom paid.

For the case of $0 < v_r < 1$, the expected total ransom paid is increasing in π_r for sufficiently low δ if $R\pi_r < 1$. This is the same condition as for the price comparative statics for the same case in the previous proposition and is also given in the proof of Proposition 2 (with the equations for the expected total ransom paid given in (B.159) and its derivative with respect to π_r given in (B.160)). In particular, for sufficiently low δ, expected total ransom paid increases in π_r for all π_r if $\alpha \leq 2$ or if $\alpha > 2$ and $R \leq 1$. Otherwise, the expected total ransom paid will be non-monotonic in π_r.

For $0 < v_{nr} < v_r < 1$, the the size of the ransom-paying population is $r(\sigma) = 1 - v_r$. Since $v_r = \frac{R}{\alpha(1-\delta)}$ in this case is constant in π_r, it follows that the size of the ransom-paying population is constant in π_r in this case. The vendor's profit is given in (B.125), and the derivative of the profit is given in (B.168). The equilibrium market size is $M = 1 - v_{nr}$, where the equilibrium v_{nr} can be found by substituting (B.124) into (B.123). The expression for the market size is given in (B.169), and its derivative with respect to π_r is given in (B.170). As was the case in Proposition 2 (and identically following the proof of that proposition), we have that both the market size and vendor's profit decreases in π_r for any $\alpha > 0$ and

$\pi_r > 0$. The expected total ransom paid in this case is $T(\sigma) = \pi_r(1 - v_{nr})(1 - v_r)R$. Again from Proposition 2, we have:

$$T_{III}(\sigma^*) = R\left(-\sqrt{4\sqrt{\alpha^4\pi_r^4 + \alpha^3\pi_r^3 + \alpha\pi_r + 1} + \alpha\pi_r(5\alpha\pi_r - 2) + 5} + 3\alpha\pi_r + 3\right) \times \frac{(\alpha(\delta - 1) + R)}{6\alpha^2(\delta - 1)}. \tag{B.202}$$

and

$$\frac{d}{d\pi_r}\left[T_{III}(\sigma^*)\right] = R(R - \alpha(1 - \delta))\frac{\left(3\alpha - \frac{\alpha\left(10\alpha\pi_r + \frac{2\alpha\pi_r(4\alpha\pi_r - 1) + 2}{\sqrt{\alpha\pi_r(\alpha\pi_r - 1) + 1}} - 2\right)}{2\sqrt{4\sqrt{\alpha^4\pi_r^4 + \alpha^3\pi_r^3 + \alpha\pi_r + 1} + \alpha\pi_r(5\alpha\pi_r - 2) + 5}}\right)}{6\alpha^2(\delta - 1)}. \tag{B.203}$$

$R < \alpha$ (which follows from $R < \tilde{R}_3$) and $R > \frac{1}{2}\alpha$. Furthermore, the remaining factor of $\frac{d}{d\pi_r}\left[T_{III}(\sigma^*)\right]$ is negative for sufficiently small δ under the focal region (as in Proposition 2), so again we have that $\frac{d}{d\pi_r}[T(\sigma^*)] > 0$ for sufficiently small δ.

For $0 < v_r < 1$ when $R > \tilde{R}_2$, the vendor's equilibrium profit is given in (B.128). The asymptotic expression was derived in (B.53) and is provided again here for reference:

$$\Pi_V^* = \frac{1}{4(1 + R\pi_r)} + \sum_{k=1}^{\infty} a_k\delta^k. \tag{B.204}$$

The derivative with respect to π_r is given as:

$$\frac{d}{d\pi_r}\left[\Pi_V^*\right] = -\frac{R}{4(1 + R\pi_r)^2} + \sum_{k=1}^{\infty} a_k\delta^k. \tag{B.205}$$

The equilibrium market size is $M = 1 - v_r$, where the equilibrium v_r can be found by substituting the price (B.127) into the expression of v_r in this case (B.126). The asymptotic expression in δ is given as:

$$M_V(\sigma^*) = \frac{1}{2(1 + R\pi_r)} + \sum_{k=1}^{\infty} a_k\delta^k. \tag{B.206}$$

The derivative with respect to π_r is:

$$\frac{d}{d\pi_r}\left[M_V(\sigma^*)\right] = -\frac{R}{2(1 + R\pi_r)^2} + \sum_{k=1}^{\infty} a_k\delta^k. \tag{B.207}$$

From the derivatives of the market size and profit with respect to π_r being negative for sufficiently small δ, both the profit and market size decrease in π_r in this case for sufficiently small δ.

Also, since $r(\sigma) = 1 - v_r$ is the same as the market size in this case, it follows that the population willing to pay ransom decreases in π_r. The expected total ransom paid is $T(\sigma) = \pi_r n(\sigma) r(\sigma) R = \pi_r(1 - v_r)(1 - v_r)R$. The asymptotic expression in δ for this is given below.

$$T_V(\sigma^*) = \frac{R\pi_r}{4(1 + R\pi_r)^2} + \sum_{k=1}^{\infty} a_k \delta^k. \tag{B.208}$$

The asymptotic expression for the derivative is given in (B.160) and provided again below:

$$\frac{d}{d\pi_r}\left[T_V(\sigma^*)\right] = \frac{R(1 - R\pi_r)}{4(1 + R\pi_r)^3} + \sum_{k=1}^{\infty} a_k \delta^k. \tag{B.209}$$

That this derivative is positive in π_r for sufficiently small δ holds for $\pi_r < \frac{1}{R}$. If $R \leq 1$, then the expected total ransom paid is increasing in π_r for all π_r. Otherwise, if $R > 1$, then the expected total ransom paid increases in π_r for $\pi_r < \frac{1}{R}$ and decreases in π_r for $\pi_r > \frac{1}{R}$.

Finally, since there is a price hike at $\pi_r = \pi'$ from Proposition 5, the market size shrinks at that discontinuity so the usage risk shrinks due to the price hike. Consequently, the expected total ransom paid decreases due to the price hike. This completes the proof. ■

Proof of Proposition 7: To compare the benchmark and ransomware scenarios, we will use a common π parameter to denote the risk factor. In the ransomware case, $\pi = \pi_r$, and in the benchmark case, $\pi = \pi_n$.

We will show that for sufficiently low π, the equilibrium price of the benchmark case and the equilibrium price of the ransomware case match. Then we will show that for an intermediate range of π, the equilibrium price of the ransomware case is greater than the price of the benchmark scenario. Lastly, we will show that for a high range of π, the equilibrium price of the benchmark case is greater than that of the ransomware case.

First, by Proposition A.2, for sufficiently low π in the benchmark case, we have $0 < v_n < 1$ as the equilibrium outcome (in which nobody patches). By Lemma B.10, for sufficiently low π, we have that $0 < v_{nr} < v_r < 1$ is the equilibrium outcome under the conditions of the proposition. Comparing the prices (B.37) and (B.41), we note that they are equal, so for sufficiently low π, the equilibrium price of the benchmark case and the equilibrium price of the ransomware case match.

Next, we will show that under the conditions of the proposition, whether $0 < v_n < 1$ or $0 < v_n < v_p < 1$ arises under the benchmark case, we will have $p^*_{RW} > p^*_{BM}$ when $0 < v_r < 1$ is the induced equilibrium outcome. If $0 < v_n < 1$ is the equilibrium outcome under the benchmark case, then the equilibrium price would be the same as that of $0 < v_{nr} < v_r < 1$ (again, comparing the prices (B.37) and (B.41). We showed in Proposition 2 that there is a price jump at the point of discontinuity. Hence, if $0 < v_n < 1$ is still the equilibrium outcome under the benchmark case, then $p^*_{RW} > p^*_{BM}$.

On the other hand, suppose that $0 < v_n < v_p < 1$ is the equilibrium outcome under the benchmark case. At the value of π at which point the vendor switches price to induce $0 < v_n < v_p < 1$ instead of $0 < v_n < 1$ in the benchmark case, it cannot be the case that the vendor does so by hiking the price. This is because a price hike would restrict unpatched usage, making patching even less incentive-compatible than with lower π. Therefore, a regime switch must be accompanied by a price drop, and the vendor's price in $0 < v_n < v_p < 1$ is lower than the price in the case of $0 < v_n < 1$ (which was shown to be lower than the price in the ransomware case). To sum up, whether $0 < v_n < 1$ or $0 < v_n < v_p < 1$ arises under the benchmark case, we will have $p^*_{RW} > p^*_{BM}$ when $0 < v_r < 1$ is the induced equilibrium outcome.

Lastly, consider when $\pi > \pi_2$ of Lemma B.4. By Lemma B.4, the equilibrium outcome under the ransomware case is $0 < v_{nr} < v_r < v_p < 1$. By Proposition A.2, we have that if $\pi > \frac{c_p(2-3c_p)}{\alpha(1-2c_p)}$, then the equilibrium outcome under the benchmark case if $0 < v_n < v_p < 1$. Under the conditions of the focal region, when R is close to the boundary $R = R_2$, then $\pi_2 > \frac{c_p(2-3c_p)}{\alpha(1-2c_p)}$. Hence, when R is close to $R = R_2$ and $\pi > \pi_2$, then the equilibrium outcome under the ransomware case is $0 < v_{nr} < v_r < v_p < 1$ while the equilibrium outcome under the benchmark case is $0 < v_n < v_p < 1$.

We compare the price of $0 < v_{nr} < v_r < v_p < 1$ given in (B.46) to the benchmark price given in (B.97). Since $v_{nr} > \frac{1}{2}$ by Lemma B.5 and $p^*_{II}(v_{nr})$ is increasing in v_{nr} (shown in the proof of that lemma), it follows that a lower bound on p^*_{II} is $p^*_{II}\left(\frac{1}{2}\right) = \frac{1}{8}\left(4 + \pi_r\alpha - \sqrt{\pi_r\alpha(16c_p + \pi_r\alpha)}\right)$. The expression $\frac{1}{8}\left(4 + \pi_r\alpha - \sqrt{\pi_r\alpha(16c_p + \pi_r\alpha)}\right) > \frac{R-c_p\alpha}{2R}$ (where the right side of the inequality comes from the constant term of the asymptotic expansion for the price in $0 < v_{nr} < v_r < v_p < 1$) is equivalent to $2c_p\alpha + R(-2R + \alpha)\pi_r > 0$. This holds for all π_r for R sufficiently close to $R = R_2$ under the conditions of the focal region. Hence, the benchmark price is greater than the ransomware price for $\pi > \pi_2$ of Lemma B.4. ■

Proof of Proposition 8: Similar to Proposition 7, we will use the π notation to denote a risk factor parameter across both the ransomware scenario as well as the benchmark scenario, and the proof of this proposition follows a similar structure.

Under the ransomware scenario, the consumer market equilibrium across π was given in Lemma B.4. When π is low in the ransomware scenario, then the equilibrium outcome is $0 < v_{nr} < v_r < 1$ under the conditions of the proposition. By Proposition A.2, the consumer market outcome under the benchmark scenario is $0 < v_n < 1$.

When the equilibrium market outcome is $0 < v_{nr} < v_r < 1$, for sufficiently small π_r, the equilibrium welfare is given as

$$SW_{III} = \frac{3}{8} + \frac{(R^2 - 2R\alpha)\pi_r}{4\alpha} + \sum_{k=1}^{\infty} a_k \pi_r^k. \tag{B.210}$$

For $0 < v_n < 1$, the asymptotic expression for the welfare in π_r for sufficiently small π_r

is given by

$$SW_I = \frac{3}{8} - \frac{\pi_r \alpha}{4} + \sum_{k=1}^{\infty} a_k \pi_r^k. \tag{B.211}$$

Comparing against (B.211), we have that ransomware dominates the benchmark for sufficiently low π_r. Therefore, there exists a bound an $\pi_L > 0$ such that if $0 < \pi < \pi_L$, then $SW_{RW} \geq SW_{BM}$ for sufficiently low δ.

Next, consider the high π case. Under ransomware, the equilibrium outcome for $\pi > \pi_2$ (from Lemma B.4) is $0 < v_{nr} < v_r < v_p < 1$. As shown in the proof of Proposition 7, under this high π range, the equilibrium outcome under the benchmark would be $0 < v_n < v_p < 1$. Under the ransomware scenario, the asymptotic expression in δ of the equilibrium welfare of this case is given as

$$SW_{IV} = \frac{1}{8}\left(3 + c_p\left(-8 + \frac{4R}{\alpha} + \frac{\alpha}{R}\right)\right) + \sum_{k=1}^{\infty} a_k \delta^k. \tag{B.212}$$

On the other hand, in the benchmark case, the equilibrium price satisfies (B.97).

We also have that $v_p = \frac{c_p v_{nr}}{v_{nr} - p}$ from (B.14). The welfare of this case is given by

$$SW_{II} = \int_{v_{nr}}^{v_p} (v - \pi_r \alpha (v_p - v_{nr}) v)\, dv + \int_{v_p}^{1} (v - c_p)\, dv\,. \tag{B.213}$$

Substituting in (B.14) for v_p and substituting in (B.97) for the price as a function of v_{nr}, we have that the welfare expression as a function of v_{nr} is given as

$$SW_{II}(v_{nr}) = \frac{1}{4}\Bigg(v_{nr}^2\left(\sqrt{\alpha}\sqrt{\pi_r}\sqrt{4c_p + \alpha\pi_r v_{nr}^2} + \alpha\pi_r(-v_{nr}) - 2\right) + c_p\left(\frac{\sqrt{4c_p + \alpha\pi_r v_{nr}^2}}{\sqrt{\alpha}\sqrt{\pi_r}} + v_{nr} - 4\right) + 2\Bigg). \tag{B.214}$$

Assuming $\pi_r > \frac{1}{c_p \alpha}$ and using $v_{nr} > \frac{1+c_p^2}{2}$ from Lemma B.5, that (B.214) is strictly decreasing in v_{nr} for $v_{nr} > \frac{1+c_p^2}{2}$ follows from the conditions on c_p and α of the focal region. Therefore, a lower bound on social welfare of this benchmark case is (B.214) evaluated at $v_{nr} = \frac{1+c_p}{2}$, and an upper bound is (B.214) evaluated at $v_{nr} = \frac{1+c_p^2}{2}$.

Focusing on high π_r ($\pi_r \approx 1$) and using $v_{nr} \geq \frac{1+c_p^2}{2}$ to get an upper bound, we have that an upper bound of the benchmark welfare in this case is:

$$SW_{II} \leq -\frac{1}{32}\left(c_p^2+1\right)^2\left(\alpha+\alpha c_p^2-\sqrt{\alpha\left(\alpha\left(c_p^2+1\right)^2+16c_p\right)}+4\right)+ \\ \frac{1}{8}c_p\left(\sqrt{\left(c_p^2+1\right)^2+\frac{16c_p}{\alpha}}+c_p^2-7\right)+\frac{1}{2}. \tag{B.215}$$

On the other hand, the first-order term of (B.212) (the welfare in the ransomware case for high π_r) when $R = \frac{\alpha}{2-c_p}$ gives

$$SW_{IV} = \frac{(1-c_p)^2(6+c_p)}{8(2-c_p)} + + \sum_{k=1}^{\infty} a_k\delta^k. \tag{B.216}$$

Comparing the two expressions (the upper bound on SW_{II} and the first-order term of the ransomware welfare expression SW_{IV}), we have that the welfare of the ransomware case is greater than the upper bound on social welfare if $\alpha > \max(2c_p+1, \frac{1}{c_p})$ (where the bound $\alpha > \frac{1}{c_p}$ is needed to guarantee $\frac{1}{c_p\alpha} < 1$ for the π_r bound to be feasible) along with the conditions in the focal region. Note that $\max(2c_p+1, \frac{1}{c_p}) = \frac{1}{c_p}$ using the focal region assumptions on c_p: $0 < c_p < 2-\sqrt{3}$. Also note that this is a non-empty region of the parameter space since $\frac{1}{c_p} < 2(2-c_p)^2$ for all $0.15 < c_p < 2-\sqrt{3}$. Hence, if $c_p > \tilde{c}_p$ for some $c_p \in (0, 2-\sqrt{3})$ and $\alpha > \frac{1}{c_p}$, then the welfare of the ransomware case is higher than the welfare of the benchmark case for π close to 1.

Lastly, consider an intermediate range of π, for π in a range $(\pi_2-\epsilon, \pi_2)$ (where π_2 comes from Lemma B.4). By Lemma B.4, the equilibrium market outcome under the ransomware case is $0 < v_r < 1$. Then the equilibrium v_r has an asymptotic expression given by

$$v_r^* = \left(1-\frac{1}{2+2R\pi_r}\right) + \sum_{k=1}^{\infty} a_k\delta^k. \tag{B.217}$$

The welfare function of this case is given by

$$SW \triangleq \int_{\mathcal{V}} \mathbb{1}_{\{\sigma^*(v)=(B,NP,R)\}}\left(v-\pi_r u(\sigma^*)(R+\delta\alpha v)\right) dv\,.$$

Consequently, welfare is given as:

$$SW_V^* = \int_{v_r}^{1} \left(v-\pi_r(1-v_{nr})(R+\delta\alpha v)\right) dv\,. \tag{B.218}$$

Substituting in (B.217) into the above expression, the asymptotic characterization of the welfare is given as

$$SW_V^* = \frac{3 + 2R\pi_r}{8(1 + R\pi_r)^2} + \sum_{k=1}^{\infty} a_k \delta^k. \tag{B.219}$$

In particular, when $\pi_r = \pi_2$, then

$$SW_V^* = \frac{(R - c_p\alpha)(3R - c_p\alpha)}{8R^2} + \sum_{k=1}^{\infty} a_k \delta^k. \tag{B.220}$$

On the other hand, in the benchmark case, the welfare of $0 < v_{nr} < v_p < 1$ will be bounded below by (B.214) evaluated at $v_{nr} = \frac{1+c_p}{2}$. Comparing (B.220) with $SW_{II}|_{v_{nr}=\frac{1+c_p}{2}}$ at R sufficiently close to $R = R_2$, there exists some $\epsilon > 0$ such that the welfare under the ransomware case is less than that of the benchmark case for $\pi_r \in (\pi_2 - \epsilon, \pi_2)$. Together with the other two cases, this proves the proposition statement. ■