Appendix for

Network Software Security and User Incentives

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Proofs

Proof of Lemma 1: In order to characterize the equilibrium, we first start with the second period decisions for the consumers who purchased the product in the first period. If, in the second period, no vulnerabilities arise then there is no decision to make for a consumer. Suppose a vulnerability arises. If a consumer with valuation v decides to patch the software, her expected total payoff is $v - p - c_p$. Notice that the consumer only incurs a patching cost when vulnerabilities actually occur. Suppose she decides not to patch and the total mass of unpatched population is u. In this case, her expected payoff is $v - p - \pi u \alpha v$. Therefore, a consumer who buys the product patches in the second period in case a security vulnerability is revealed if and only if

$$v \geq \frac{c_p}{\pi u \alpha}\,. \tag{A.1}$$

Consequently, in equilibrium, if a buyer with valuation v_0 patches the software, then every buyer with valuation $v > v_0$ will patch and hence there exists a $v_p \in [0,1]$, such that when a vulnerability arises, a consumer with valuation $v \in \mathcal{V}$ will patch if and only if $v \geq v_p$.

Next, we examine the buying decision in the first period. If a consumer with valuation v decides to buy the product, she will incur a cost p. Her expected security losses are $C(v, \sigma^*)$. Then she will buy the software if and only if

$$v - C(v, \sigma^*) \geq p\,. \tag{A.2}$$

Now first, suppose $v_p < 1$. Then $v_p \geq p + c_p$, and hence, in equilibrium, since (1) implies $C(v, \sigma^*) = \min\{\pi u \alpha v, c_p\}$ and by (A.2), for all $v > v_p$, we have $\sigma^*(v) = (B, P)$. Now let $0 \leq v_1 \leq 1$ and $\sigma^*(v_1) = (B, NP)$. Then, by (A.2),

$$v_1 \geq \frac{p}{1 - \pi u \alpha}\,, \tag{A.3}$$

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and therefore for all $v > v_1$, $\sigma^*(v) \in \{(B,P),(B,NP)\}$, and hence there exists a $v_b \in [0,1]$, such that a consumer with valuation $v \in \mathcal{V}$ will purchase if and only if $v \geq v_b$. By definition $v_p \geq v_b$. Suppose $0 < v_p = v_b < 1$ and $c_p > 0$. But then, there exists $0 < v < v_p$ such that $v \geq p + C(v, \sigma^*) = p$, which is a contradiction. Therefore, we conclude that, when $c_p > 0$ and $0 < p < 1$ there exist $0 \leq v_b < v_p \leq 1$ satisfying (2), from which, it follows that

$$\pi\alpha(v_p - v_b)v_p = c_p\,, \tag{A.4}$$

and

$$v_b = p + \pi\alpha(v_p - v_b)v_b\,. \tag{A.5}$$

Substituting (A.4) into (A.5) yields

$$v_p = \frac{c_p v_b}{v_b - p}\,, \tag{A.6}$$

which, in turn, by substituting into (A.5) gives

$$\pi\alpha v_b^3 + (1 - \pi\alpha(c_p + p))v_b^2 - 2pv_b + p^2 = 0\,. \tag{A.7}$$

Now, for $v_p < 1$ to hold, by (A.6), we must have $v_b > \frac{p}{1-c_p}$. Plugging this in equation (A.7) and since $0 \leq v_b \leq 1$, we obtain that for $v_p < 1$, we must have $p < \overline{p}$. When $p < \overline{p}$, it can be shown that (A.7) has a single root v_b that satisfies $1 > v_b > p$, which is satisfied by (A.2). Further, when $p > 0$, again from (A.7), $v_b < p + c_p$ follows, which by plugging in (A.6) confirms $p + c_p < v_p$.

When $p = 0$ and $\alpha \geq \frac{c_p}{\pi}$, since $\overline{p} > 0$ (A.7) is valid and substituting, $p = 0$ into (A.7) yields $v_b^2\left(v_b - \left(c_p - \frac{1}{\pi\alpha}\right)\right) = 0$, which has two roots, namely $v_b = 0$ and $v_b = c_p - 1/\pi\alpha$. If $c_p < 1/\pi\alpha$, then the only possible solution in $[0,1]$ is $v_b = 0$, and when $v_b = 0$, by (A.4), it follows that

$$v_p = \sqrt{\frac{c_p}{\pi\alpha}}\,. \tag{A.8}$$

If $\alpha > \frac{1}{c_p\pi}$, however, under (A.8), (A.2) cannot be satisfied. Therefore, the only valid root for this region is $v_b = c_p - 1/\pi\alpha$ and by (A.6), the statement follows.

Finally when, $p \geq \overline{p}$, on the other hand, substituting $v_p = 1$ in (A.5), we obtain $\pi\alpha v_b^2 + (1 - \pi\alpha)v_b - p = 0$, which has a unique positive root that satisfies $v_b \leq 1$ and is given by

$$v_b = -\frac{1 - \pi\alpha}{2\pi\alpha} + \frac{1}{2\pi\alpha}\sqrt{(1 - \pi\alpha)^2 + 4\pi\alpha p}. \tag{A.9}$$

This completes the proof. ■

Before we move on to the next proposition, we first state and prove the following lemmas that will be useful for the remaining proofs:

Lemma A.1 *The purchasing threshold v_b is strictly increasing in price. Further, in Region I, $\frac{dv_b}{dp} > 1$.*

Proof: The statement for Region II is immediate from (A.9). For Region I, from (A.7) and by the implicit function theorem, we obtain

$$\frac{dv_b}{dp} = \frac{\pi\alpha v_b^2 + 2(v_b - p)}{3\pi\alpha v_b^2 + 2(1 - \pi\alpha c_p - \pi\alpha p)v_b - 2p} = \frac{1}{1 + \frac{2\pi\alpha v_b(v_b - c_p - p)}{\pi\alpha v_b^2 + 2(v_b - p)}} \,. \tag{A.10}$$

Re-arranging equation (A.7), we have

$$\pi\alpha v_b^2(v_b - c_p - p) = -(v_b - p)^2 \,. \tag{A.11}$$

From (A.10) and (A.11), it then follows that

$$\frac{dv_b}{dp} = \frac{\pi\alpha v_b^2 + 2(v_b - p)}{\pi\alpha v_b^2 + 2\frac{p}{v_b}(v_b - p)} > 1 \,. \; \square \tag{A.12}$$

Lemma A.2

(i) There exists a solution, $p_s^* \in [0, 1]$, to the profit maximization problem of the vendor. The profit function for the vendor is piece-wise strictly concave in price, i.e., it is concave when restricted to price regions $[0, \overline{p})$ and $[\overline{p}, 1]$, where $\overline{p}$ is as given in Lemma 1.

(ii) Let $c_p \in (0, 1)$ be given. There exist $c_p < \underline{\theta} < \overline{\theta}$ such that

(a) When $\pi\alpha > \overline{\theta}$, the software vendor's profit is maximized by pricing in Region I;

(b) When $0 < \pi\alpha < \underline{\theta}$, the software vendor's profit is maximized by pricing in Region II.

Proof: By Lemma 1, $\Pi_s(\cdot)$ is continuous on compact $[0, 1]$. Therefore, the vendor's problem has an optimal solution on this price range. For strict concavity, we first consider Region II. By (A.9), we have

$$\Pi_s^{ii}(p) = \frac{p}{2\pi\alpha}\left(1 + \pi\alpha - \sqrt{(1 - \pi\alpha)^2 + 4\pi\alpha p}\right) \,. \tag{A.13}$$

In order to circumvent having the first derivative ill-defined, we break the analysis into two cases in which the product $\pi\alpha = 1$ and $\pi\alpha \neq 1$. When $\pi\alpha = 1$, we have $\Pi_s^{ii}(p) = p(1 - \sqrt{p})$.

Thus we have $d\Pi_s^{ii}(p)/dp = 1 - \frac{3}{2}\sqrt{p}$ and $d^2\Pi_s^{ii}(p)/dp^2 = -\frac{3}{4\sqrt{p}}$. When $\pi\alpha \neq 1$, we have $d\Pi_s^{ii}(p)/dp = \frac{1}{2\pi\alpha}\left(1 + \pi\alpha - \sqrt{(1-\pi\alpha)^2 + 4\pi\alpha p}\right) - \frac{p}{\sqrt{(1-\pi\alpha)^2+4\pi\alpha p}}$, and $d^2\Pi_s^{ii}(p)/dp^2 = \frac{-2(1-\pi\alpha)^2 - 6\pi\alpha p}{((1-\pi\alpha)^2+4\pi\alpha p)^{3/2}} < 0$. Hence, we conclude that $\Pi_s^{ii}(\cdot)$ is indeed strictly concave.

Now consider Region I. Notice that $d\Pi_s^i(p)/dp = 1 - v_b - p\frac{dv_b}{dp}$ and $d^2\Pi_s^i(p)/dp^2 = -\big(2\frac{dv_b}{dp} + p\frac{d^2v_b}{dp^2}\big)$. By differentiating equation (A.12) and rearranging we obtain

$$\frac{d^2v_b}{dp^2} = \frac{\frac{dv_b}{dp}\left(2\alpha v_b + \frac{4p}{\pi v_b}\right) - \left(\frac{dv_b}{dp}\right)^2\left(\frac{2p^2}{\pi v_b^2} + 2\alpha v_b\right) - \frac{2}{\pi}}{\alpha v_b^2 + \frac{2p}{\pi v_b}(v_b - p)}\,. \tag{A.14}$$

Substituting back into the second derivative of the profit function, we have

$$\frac{d^2\Pi_s^i(p)}{dp^2} = -\frac{2\left(\alpha v_b^2 + \frac{2(v_b-p)}{\pi} + p\frac{dv_b}{dp}\left(\alpha v_b + \frac{2}{\pi}\frac{p}{v_b}\right) - p\left(\frac{dv_b}{dp}\right)^2\left(\frac{1}{\pi}\frac{p^2}{v_b^2} + \alpha v_b\right) - \frac{p}{\pi}\right)}{\alpha v_b^2 + \frac{2p}{\pi v_b}(v_b - p)}\,. \tag{A.15}$$

Now, by (A.12) and Lemma A.1, we have

$$\frac{dv_b}{dp}\alpha v_b(v_b+p) - p\alpha v_b\left(\frac{dv_b}{dp}\right)^2 = \alpha v_b\frac{dv_b}{dp}\left(v_b + p\left(1 - \frac{dv_b}{dp}\right)\right) = \alpha v_b\frac{dv_b}{dp}\left(\frac{\alpha v_b^3 + \frac{2}{\pi}\frac{p^2}{v_b}(v_b-p)}{\alpha v_b^2 + \frac{2}{\pi}\frac{p}{v_b}(v_b-p)}\right) > 0\,. \tag{A.16}$$

Further, again by Lemma A.1 and rearranging

$$\begin{aligned}\frac{2p}{\pi}\cdot\frac{dv_b}{dp} - \frac{p}{\pi}\left(\left(\frac{p}{v_b}\cdot\frac{dv_b}{dp}\right)^2 + 1\right) &= \frac{p}{\pi}\left(2\frac{dv_b}{dp} - \left(\frac{p}{v_b}\cdot\frac{dv_b}{dp}\right)^2 - 1\right)\\ &= \frac{p}{\pi}\left(\frac{\alpha v_b^2(\frac{v_b}{p} - 1)}{\frac{\alpha v_b^3}{p} + \frac{2}{\pi}(v_b - p)}\left(1 + \frac{p}{v_b}\cdot\frac{dv_b}{dp}\right) + 2\left(\frac{dv_b}{dp} - 1\right)\right) > 0\,.\end{aligned} \tag{A.17}$$

Combining (A.16), (A.17) and the fact that $v_b > p$, we find that the right hand side of (A.15) is strictly negative and therefore, Π_s^i is strictly concave. This completes the proof of part (i).

To see part (ii), first by part (i), there exists an optimal price that solves the vendor's profit maximization problem. To see part (a), notice that by (A.9), in Region II, $\lim_{\pi\alpha\to\infty} v_b = 1$. Therefore as $\pi\alpha\to\infty$, profit in Region II for any feasible p approaches zero. By (A.7), $v_b < p + c_p$ is always satisfied. Therefore for any given $p \in [0, \overline{p}\,)$, $\Pi_s^i(p) > p(1 - c_p - p)$, which has a maximum at $p = (1 - c_p)/2$, which is in $[0, \overline{p}\,)$ for sufficiently large $\pi\alpha$ as desired. For part (b), notice that by Lemma 1, the feasible price range for Region I is

$p \in [0, \overline{p})$. At $\pi\alpha = c_p$, this range gets reduced to $\{0\}$ and as $\pi\alpha$ approaches this threshold the vendor's profit vanishes on $[0, \overline{p})$. For any $\pi\alpha \leq c_p$, there is no feasible price for Region I. On the other hand, Region II becomes feasible for all values of $\pi\alpha$ in this range and by (A.9), for any given $p \geq \overline{p}$, the profit in Region II increases as $\pi\alpha$ decreases. Hence, there exists an $\underline{\theta} \geq c_p > 0$ such that the vendor's profit is maximized in Region II for $\pi\alpha < \underline{\theta}$. This completes the proof. $\square$

Lemma A.3 *For the proprietary software, if* $v_b \leq v_m$ *then* $W_s > W_m$.

Proof: Consider each consumer $v \in [v_m, 1]$. Under self patching decisions, each of these consumers contributes $v - C(v, \sigma^*)$ to the expected social welfare. Note that this contribution incorporates the externalities created by all other users in equilibrium. Under the mandatory patching policy, each of these consumers contributes $v - c_p$. However, $C(v, \sigma^*) \leq c_p$ for all these consumers since c_p is the greatest loss that any purchaser will accept. Each consumer $v \in [v_b, v_m]$ will purchase only if they make a positive contribution to the welfare. Furthermore, by (A.4) and since $v_p > v_b$, $\pi\alpha(v_p - v_b)v_b < c_p$. Thus, the expected social welfare under self patching is strictly greater than the expected social welfare under mandatory patching when $v_b \leq v_m$. $\square$

Proof of Proposition 1: To see part (i), first note that $v_m = p_m^* + c_p = (1 + c_p)/2$ and consider the associated purchasing threshold as a function of c_p, i.e. $v_m(c_p) = (1 + c_p)/2$. Since $\pi\alpha < c_p$ and $v_m(\cdot)$ is increasing in c_p, it follows that $v_m(\pi\alpha) < v_m(c_p)$.

Now from (A.9), we have that $v_b(\pi\alpha) = -\frac{1-\pi\alpha}{2\pi\alpha} + \frac{1}{2\pi\alpha}\sqrt{(1-\pi\alpha)^2 + 4\pi\alpha p_s^*}$. By Lemma A.2, Π_s^{ii} is concave and since $\Pi_s^{ii}(0) = \Pi_s^{ii}(1) = 0$ the optimal price can be found through the first order condition, which yields

$$p_s^* = \frac{1}{9\pi\alpha}\sqrt{-1 + 4\pi\alpha - (\pi\alpha)^2 + (1+\pi\alpha)\sqrt{1 - \pi\alpha + (\pi\alpha)^2}}\,. \tag{A.18}$$

Plugging (A.18) into (A.9), we obtain

$$v_m(\pi\alpha) - v_b(\pi\alpha) = \frac{3 + 3(\pi\alpha)^2 - \sqrt{5 - 2\pi\alpha + 5(\pi\alpha)^2 + 4(1+\pi\alpha)\sqrt{1 - \pi\alpha + (\pi\alpha)^2}}}{6\pi\alpha} \geq 0\,. \tag{A.19}$$

(A.19) can be easily established by rearranging the inequality and taking the square of both sides twice. Therefore $v_b \leq v_m$ and the result follows from Lemma A.3.

To see part (ii), suppose that $v_b > v_m$. Define $p_c > 0$ as the price such that

$$\pi\alpha v_m^3 + (1 - \pi\alpha(c_p + p_c))\, v_m^2 - 2p_c v_m + p_c^2 = 0. \tag{A.20}$$

Plugging $v_m = (1+c_p)/2$ in (A.20) and solving for p_c, we find that

$$p_c = \frac{1}{8}\left(4 + 4c_p + \pi\alpha(1+c_p)^2 - \sqrt{\pi\alpha(1+c_p)^2(16c_p + \pi\alpha(1+c_p)^2)}\right). \tag{A.21}$$

By Lemma A.2, at the optimal price for Region I, p_s^*, we have $\left.\frac{d\Pi_s^i(p)}{dp}\right|_{p=p_s^*} \geq 0$. Then, by Lemma A.1 and again by Lemma A.2, $\left.\frac{d\Pi_s^i(p)}{dp}\right|_{p=p_c} > 0$ also holds. Now,

$$\frac{d\Pi_s^i(p)}{dp} = 1 - v_b - p\left(\frac{v_b^2 + \frac{2}{\pi\alpha}(v_b - p)}{v_b^2 + \frac{2}{\pi\alpha}\frac{p}{v_b}(v_b - p)}\right) = \frac{\pi\alpha v_b^3(1-p) + 2p(v_b - p)(1-2v_b) - \pi\alpha v_b^4}{\pi\alpha v_b^3 + 2p(v_b - p)}. \tag{A.22}$$

Plugging (A.20) in (A.22), we find that $\left.\frac{d\Pi_s^i(p)}{dp}\right|_{p=p_c} > 0$ if and only if

$$\begin{aligned}&\pi^2\alpha^2(1+c_p)^3(-1+3c_p) + 32c_p^2\pi\alpha(1+c_p)\\ &\qquad - (8c_p - \pi\alpha(1+c_p)(1-3c_p))\sqrt{\pi\alpha(1+c_p)^2(16c_p + \pi\alpha(1+c_p)^2)} > 0.\end{aligned} \tag{A.23}$$

Suppose that $c_p \geq 1/3$. Moving the radical in (A.23) and squaring yields the equivalent condition,

$$\pi\alpha c_p^2(1+c_p)^2(16c_p + \pi\alpha(3-c_p)(3c_p-1)) < 0, \tag{A.24}$$

and hence (A.23) is not satisfied. Now suppose $c_p < 1/3$ and define $s(\pi\alpha) \triangleq \pi\alpha(1+c_p)^3(-1+3c_p)+32c_p^2(1+c_p)$ and $t(\pi\alpha) \triangleq (8c_p+\pi\alpha(1+c_p)(-1+3c_p))$. Notice that $s(\pi\alpha) > 0$ if and only if $\pi\alpha < a_s \triangleq \frac{32c_p^2}{(1+c_p)^2(1-3c_p)}$ and $t(\pi\alpha) > 0$ if and only if $\pi\alpha < a_t \triangleq \frac{8c_p}{(1+c_p)(1-3c_p)}$. Further (A.24) is violated if and only if $\pi\alpha \geq a_\tau \triangleq \frac{16c_p}{(3-c_p)(1-3c_p)}$. Notice that, $c_p < 1/3$ implies $a_s < a_t$. When $a_s \leq \pi\alpha < a_t$, (A.23) does not hold. It then follows that when $\pi\alpha \geq a_t$, (A.23) is violated if and only if (A.24) is violated, which is true since $a_t \geq a_\tau$. Further, when $\pi\alpha < a_s$ (A.23) is violated if and only if (A.24) is violated which is true since $a_s \leq a_\tau$. Therefore $v_b \leq v_m$ and, again by Lemma A.3, the result follows. ■

Proof of Proposition 2: For any given $p > 0$, $v_b > p$ and $0 < r < c_p$, by (A.11)

$$p + c_p - r - v_b = \frac{1}{\pi\alpha}\left(\frac{v_b - p}{v_b}\right)^2. \tag{A.25}$$

Define $\xi = \frac{1}{\pi\alpha}\left(\frac{v_b - p}{v_b}\right)^2$. By Lemma A.2, for sufficiently large $\pi\alpha$ the vendor will price in Region I. The first order condition for $\Pi_v(p,r)$ is given by

$$\frac{\partial \Pi_v(p,r)}{\partial p} = 1 - v_b - p\frac{dv_b}{dp} + \frac{r(c_p - r)}{(v_b - p)^2}\left(v - p\frac{dv_b}{dp}\right) = 0\,, \tag{A.26}$$

which, by combining with (A.12) and (A.25) yields

$$p_v^* = \frac{1 - c_p}{2} + r + c_p\left(\frac{2}{1 + c_p} - \frac{1}{2(c_p - r)}\right)\xi + O(\xi^2)\,. \tag{A.27}$$

Therefore, combining (A.25) and (A.27), for $\pi\alpha$ sufficiently large, $p_v^* < \overline{p}$ and the unconstrained optimum of Π_v^i will be feasible for Region I.

Now consider the optimal price as a function of the rebate denoted $p(r)$ and define the optimal expected vendor profit as a function of the rebate by $\Pi_v^*(r) = \Pi_v(r, p(r), v_b(p(r), r))$. By Lemma A.2 and the envelope theorem, we obtain the first order condition for the optimal rebate as

$$\begin{aligned}\frac{d\Pi_v^*(r)}{dr} &= \frac{\partial \Pi_v\Big(r, p(r), v_b(p(r), r)\Big)}{\partial r} + \frac{\partial \Pi_v\Big(r, p(r), v_b(p(r), r)\Big)}{\partial v_b}\frac{\partial v_b(p(r), r)}{\partial r} \\ &= -1 + \frac{v_b}{v_b - p}(c_p - 2r) + \left(1 + \frac{r(c_p - r)}{(v_b - p)^2}\right)\frac{p v_b^2}{v_b^2 + \frac{2}{\pi\alpha}(v_b - p)\frac{p}{v_b}} = 0\,.\end{aligned} \tag{A.28}$$

Substituting in for (A.25),

$$\frac{d\Pi_v^*(r)}{dr} = -1 + \frac{p + c_p - r - \xi}{c_p - r - \xi}(c_p - 2r) + \left(1 + \frac{r(c_p - r)}{(c_p - r - \xi)^2}\right)\frac{p(c_p - r - \xi)}{c_p - r - \xi + \frac{2p\xi}{p + c_p - r - \xi}}\,,$$

which, evaluated at (A.27), yields

$$\frac{d\Pi_v^*(r)}{dr} = \frac{c_p(3c_p - 1 - 4r)}{2(c_p - r)^2}\xi + O(\xi^2)\,. \tag{A.29}$$

Hence there exists an $\overline{\omega} > 0$ such that when $\pi\alpha > \overline{\omega}$, $\left.\frac{d\Pi_v^*(r)}{dr}\right|_{r=0} \geq 0$ if and only if $c_p > \frac{1}{3}$. Therefore, a rebate policy will be effective if and only if $c_p > 1/3$. By (A.29), we have $r_v^* \to (3c_p - 1)/4$ and hence, by (A.27), $p_v^* \to (1 + c_p)/4$. Further, there exists a constant k such that $\lim_{\xi\to 0}\frac{r_v^* - (3c_p - 1)/4}{\xi} = \lim_{\xi\to 0}\frac{p_v^* - (1 + c_p)/4}{\xi} = k$. Substituting into (A.29), it follows that $k = (1 - c_p)/(8c_p) > 0$. Therefore, r_v^* and p_v^* are increasing in ξ, and hence decreasing in $\pi\alpha$. This completes the proof of part (i).

To see part (ii) first notice that under the hypothesis $\pi\alpha < c_p$ holds and in this region, for a rebate $r > 0$ to be effective, by Lemma 1, we must have $c_p - \pi\alpha < r < c_p$, since, only in this case the consumers will face a patching cost that will induce at least some of them to patch. For a fixed p, let $v_b(r)$ and $v_p(r)$ denote the purchasing and patching thresholds when a rebate r is offered, respectively. Clearly, when $r > c_p - \pi\alpha$, $v_b(r) < v_b(0)$, since otherwise $\Pi_v(p,r) < \Pi_v(p,0)$ holds. But then, by (A.5),

$$v_b(r) = \frac{p}{1 - \pi\alpha(v_p(r) - v_b(r))} < \frac{p}{1 - \pi\alpha(1 - v_b(0))} = v_b(0)\,, \tag{A.30}$$

which implies $1 - v_p(r) > v_b(0) - v_b(r)$ and therefore, for $\Pi_v(p,r) > \Pi_v(p,0)$, $p > r$ has to hold.

When the vendor offers such a rebate, r, his expected profit function can be written as $\Pi_v(p,r) = p(1-v_b) - r(1-v_p)$ where $p \in [0, (1-(c_p - r))(1 - \frac{c_p - r}{\pi\alpha})]$. Also note that the purchasing threshold is now governed by the equation $\pi\alpha v_b^3 + \big(1 - \pi\alpha(c_p - r + p)\big)v_b^2 - 2pv_b + p^2 = 0$. then, by the implicit function theorem, we obtain:

$$\frac{dv_b}{dr} = -\frac{v_b^2}{v_b^2 + \frac{2}{\pi\alpha}(v_b - p)\frac{p}{v_b}}\,, \tag{A.31}$$

and hence $-1 \leq \frac{dv_b}{dr} \leq 0$. Differentiating the expected profit function, we obtain

$$\frac{d\Pi_r}{dr} = -p\frac{dv_b}{dr} - 1 + v_p + r\frac{dv_p}{dr} = -p\frac{dv_b}{dr} - 1 + v_p + r\left(-\frac{p(c_p - r)\frac{dv_b}{dr}}{(v_b - p)^2} - \frac{v_b}{v_b - p}\right). \tag{A.32}$$

Notice that the first three terms are bounded and that r approaches c_p as $\pi\alpha$ approaches zero. Substituting $c_p - r$ in place of c_p in (A.11) and re-arranging we obtain

$$c_p - r = (v_b - p)\Big(\frac{v_b - p}{\pi\alpha v_b^2} + 1\Big)\,. \tag{A.33}$$

Therefore

$$\frac{p(c_p - r)\frac{dv_b}{dp}}{(v_b - p)^2} - \frac{v_b}{v_b - p} = \frac{-\pi\alpha v_b^2 - p}{\pi\alpha v_b^2 + 2(v_b - p)\frac{p}{v_b}} = \frac{dv_b}{dr}r\left(1 + \frac{p}{\pi\alpha v_b^2}\right). \tag{A.34}$$

Now since $\pi\alpha < \frac{c_p^2}{1+c_p}$ and $p \leq 1$, $\pi\alpha(p - (c_p - \pi\alpha)) < (c_p - \pi\alpha)^2$, and since $p > r > c_p - \pi\alpha$, we have

$$\pi\alpha < \frac{(c_p - \pi\alpha)^2}{p - (c_p - \pi\alpha)} < \frac{pr}{p - r}\,. \tag{A.35}$$

From (A.35), and since $v_b \leq 1$, it follows that

$$p - r - \frac{pr}{\pi\alpha v_b^2} < 0\,. \tag{A.36}$$

Combining (A.32), (A.34) and (A.36), we obtain $d\Pi_r/dr < 0$ and therefore, it is suboptimal for the vendor to offer a rebate. This completes the proof. ■

Proof of Proposition 3: By (4), (A.6) and (A.7),

$$W_g^i(p,r) = \frac{1}{2}\left(1 - v_b^2 - \frac{\pi\alpha v_b^3(v_b - p - c_p + r)^2(v_b + c_p - p - r)}{(v_b - p)^3} + \frac{2c_p\,(p - v_b(1 + r - c_p))}{v_b - p}\right). \tag{A.37}$$

Taking the total derivative with respect to r, substituting (A.12) and (A.31), utilizing the implicit function theorem on (A.26), and defining ξ as in the proof of Proposition 2, by (A.25) and (A.27) we then obtain

$$\lim_{\xi\to 0}\frac{1}{\xi}\frac{dW_g^*(r)}{dr} = \frac{c_p\,(c_p(12 - c_p) - 3 - 16r)}{4(1 + c_p)(c_p - r)^2}\,. \tag{A.38}$$

Notice that $c_p(12 - c_p) - 3$ is a concave quadratic expression in c_p with roots $6 - \sqrt{33}$ and $6 + \sqrt{33}$. Since $6 - \sqrt{33} < 1 < 6 + \sqrt{33}$, we conclude that there exists an $\varpi > 0$ such that when $\pi\alpha > \varpi$, $\left.\frac{dW_g^*(r)}{dr}\right|_{r=0} \geq 0$ if and only if $c_p > 6 - \sqrt{33}$. Hence, in this region, a rebate policy is effective at increasing social welfare if and only if c_p is large enough as stated in the proposition. By (A.38), as $\pi\alpha$ becomes large, we have $r_g^* \to (c_p(12 - c_p) - 3)/16$ and, by substituting into (A.27), $p_g^* \to (5 - c_p)(1 + c_p)/16$. Clearly, both r_g^* and p_g^* are strictly increasing in c_p.

Further, substituting r_g^* back into (A.38) we obtain

$$\lim_{\xi\to 0}\frac{r_g^* - (c_p(12 - c_p) - 3)/16}{\xi} = f(c_p)\,, \tag{A.39}$$

where $f(c_p)$ is a fifth order polynomial with three real roots only one of which (denoted by θ') in $(6 - \sqrt{33}, 1)$ and for all $c_p \in (\theta', 1)$, $f(c_p) < 0$. Thus, for $\pi\alpha$ sufficiently large, r_g^* is decreasing in $\pi\alpha$ if $c_p \in (6 - \sqrt{33}, \theta')$ and increasing in $\pi\alpha$ if $c_p \in (\theta', 1)$. Substituting r_g^* into (A.27) and carrying out the analysis in a similar way shows that there exists a θ in $(6 - \sqrt{33}, 1)$ such that p_v^* is decreasing in $\pi\alpha$ if $c_p \in (6 - \sqrt{33}, \theta)$ and increasing in $\pi\alpha$ if $c_p \in (\theta, 1)$. This completes the proof of part (i).

For part (ii), when $\pi\alpha < c_p$ and $r = 0$, by Lemmas 1 and A.2, the optimal price, p_s^*, is found in Region II. Plugging (A.18) in (A.22), we find that

$$\lim_{\pi\alpha\to 0}\pi\alpha\left.\frac{d\Pi_s^i}{dp}\right|_{p=p_s^*} = \frac{(c_p - r)r}{\nu(1 + 8\nu)}\,, \tag{A.40}$$

where, from (A.7), $\nu = \lim_{\pi\alpha\to 0}(v_b - p)/\pi\alpha > 0$. Therefore, when a planner imposed rebate is effective, i.e., when a large enough $r < c_p$ induces the vendor to price so that there is a patching population, since the vendor's profit curve is strictly piecewise concave in p, $p_g^* > p_s^*$ follows. Now define

$$n' = \sup\{n : \lim_{\pi\alpha \to 0} \frac{(p_g^*(r) - p_s^*)}{(\pi\alpha)^n} < \infty\}\,. \tag{A.41}$$

Further, define v_b^s as given in (A.9), which is the purchasing threshold for $r = 0$ and

$$n'' = \sup\{n : \lim_{\pi\alpha \to 0} \frac{(v_b(p_g^*(r), r) - v_b^s)}{(\pi\alpha)^n} < \infty\}\,. \tag{A.42}$$

By (A.4)

$$\lim_{\pi\alpha \to 0} \frac{v_p(p_g^*(r), r)}{(\pi\alpha)^{\min\{n',n''\}}} < \infty\,, \tag{A.43}$$

and hence,

$$\lim_{\pi\alpha \to 0} \frac{v_p(p_g^*(r), r) - v_b(p_g^*(r), r)}{(\pi\alpha)^{\min\{n',n''\}}} < \infty\,. \tag{A.44}$$

Since $p_g^*(r) > p_s^*$, it then follows that there exists a $\underline{\theta} > 0$ such that when $0 < \pi\alpha < \underline{\theta}$, for any r such that $v_p(p_g^*(r), r) < 1$,

$$W_g^i(p_g^*(r), r) - W_g^{ii}(p_s^*, 0) < \pi\alpha(v_p(p_g^*(r), r) - v_b(p_g^*(r), r))v_p(p_g^*(r), r) - (1 - v_p(p_g^*(r), r)c_p < 0\,. \tag{A.45}$$

This completes the proof. ■

Proof of Proposition 4: We first have to consider how the equilibrium region changes when a rebate is offered. By Lemma 1, when $\pi\alpha < c_p - r$ equilibrium outcome is in Region II with all consumers are purchasing and the expected social welfare is $W_g^{ii} = \frac{1}{2}(1 - \pi\alpha)$. When $c_p - r \le \pi\alpha \le \frac{1}{c_p - r}$, the equilibrium outcome is in Region I with $p = 0$, all consumers are purchasing, only consumers with valuations $v > \sqrt{\frac{c_p - r}{\pi\alpha}}$ are patching, and the expected social welfare is $W_g^i = \frac{1}{2} - c_p + \frac{c_p + r}{2}\sqrt{\frac{c_p - r}{\pi\alpha}}$. Finally, when $\pi\alpha > \frac{1}{c_p - r}$, the equilibrium outcome is in Region I with only the consumers with valuations $v > c_p - r - \frac{1}{\pi\alpha}$ purchasing and only the consumers with valuations $v > c_p - r$ are patching. The expected social welfare in this region is $W_g^i = \frac{1}{2}(1 - c_p)^2 - \frac{r^2}{2}$.

Which of the above regions are reachable is determined by whether $\pi\alpha < c_p$, $c_p \le \pi\alpha \le \frac{1}{c_p}$, or $\pi\alpha > \frac{1}{c_p}$. When $\pi\alpha > \frac{1}{c_p}$, for any rebate such that $c_p - \frac{1}{\pi\alpha} \le r \le c_p$, the equilibrium outcome will be in Region I, with $v_b = 0$. For $0 \le r < c_p - \frac{1}{\pi\alpha}$, on the other hand, the equilibrium outcome will be in Region I, with $v_b > 0$. When $\pi\alpha < c_p$, for any rebate such that $0 \le r < c_p - \pi\alpha$, the equilibrium outcome will remain in Region II, while for $c_p - \pi\alpha \le r \le c_p$, it will move into Region I with $v_b = 0$. Finally, when $c_p \le \pi\alpha \le \frac{1}{c_p}$, the equilibrium outcome will remain in Region I, with $v_b = 0$ for all r in $0 \leq r \leq c_p$.

With these ranges in mind, we first address the case where $\pi\alpha > \frac{1}{c_p}$. For r such that $0 \leq r < c_p - \frac{1}{\pi\alpha}$, the expected social welfare is $W_g^i = \frac{1}{2}(1-c_p)^2 - \frac{r^2}{2}$ and is decreasing in r. Thus, the highest expected social welfare achievable under this rebate range is $\frac{1}{2}(1-c_p)^2$. For $r \in [c_p - \frac{1}{\pi\alpha}, c_p]$, expected social welfare is given by $W_g^i = \frac{1}{2} - c_p + \frac{c_p+r}{2}\sqrt{\frac{c_p-r}{\pi\alpha}}$. Let $g(r) \triangleq \frac{1}{2} - c_p + \frac{c_p+r}{2}\sqrt{\frac{c_p-r}{\pi\alpha}}$. Then, we have $dg(r)/dr = \frac{c_p-3r}{4\sqrt{\pi\alpha(c_p-r)}}$ and hence g is increasing on $r \in [0, \frac{c_p}{3}]$ and decreasing on $r \in [\frac{c_p}{3}, c_p]$. Since $r_g^* = \frac{c_p}{3}$ maximizes this function, it remains to find when r_g^* is feasible, i.e. $c_p - \frac{1}{\pi\alpha} \leq \frac{c_p}{3}$. This condition is equivalent to $\pi\alpha \leq \frac{3}{2c_p}$ and when it holds along with $\frac{1}{2}(1-c_p)^2 \geq g(\frac{c_p}{3})$, then there does not exist an $r > 0$ such that the expected social welfare can be increased by offering a rebate of r . The latter holds if and only if

$$\frac{1}{2}(1-c_p)^2 - \left(\frac{1}{2} - c_p + \frac{2c_p}{3}\sqrt{\frac{2c_p}{3\pi\alpha}}\right) \geq 0\,, \tag{A.46}$$

which, in turn, is satisfied if and only if $\pi\alpha \geq \frac{32}{27c_p}$. Now if $\pi\alpha > \frac{3}{2c_p}$ then r_g^* is not feasible. However, $g(r_g^*) \geq g(r)$ for any other r. Thus when $\pi\alpha \geq \frac{32}{27c_p}$, there is no $r > 0$ such that the expected social welfare can be increased by offering a rebate r, while when for $\pi\alpha \in [\frac{1}{c_p}, \frac{32}{27c_p})$, offering a rebate of $r_g^* = c_p/3$ maximizes the expected social welfare.

Second, when $\pi\alpha \in [c_p, \frac{1}{c_p}]$ as we showed above, for all r, the equilibrium outcome will be in Region I, with $v_b = 0$, and the expected social welfare will be $g(r)$ as described above. Clearly, in this range, it is optimal to offer a rebate precisely equal to $r_g^* = c_p/3$.

Finally, when $\pi\alpha < c_p$ as we have shown above, for all rebates such that $0 \leq r < c_p - \pi\alpha$ we are still operating in Region II. Thus, the expected social welfare is unchanged as no consumer elects to patch even with the rebate. We focus our attention on r such that $c_p - \pi\alpha \leq r \leq c_p$ in which case the equilibrium outcome will be in Region I, with $v_b = 0$. In order for r_g^* to be feasible, we require that $c_p - \pi\alpha \leq r_g^* = \frac{c_p}{3}$ which can be equivalently written as $\pi\alpha \geq \frac{2c_p}{3}$.

For $\pi\alpha \leq \frac{2c_p}{3}$, we compare the expected social welfare $W_g^{ii} = \frac{1-\pi\alpha}{2}$ against $g(c_p - \pi\alpha)$ as $g(\cdot)$ is decreasing in this range of rebates. However, it can be easily seen that $g(c_p - \pi\alpha) = W_g^{ii}$ and hence, for $\pi\alpha \leq \frac{2c_p}{3}$ it is clearly suboptimal to offer a rebate.

For $\pi\alpha > \frac{2c_p}{3}$, we must compare $g(r_g^*) = g(\frac{c_p}{3})$ against W_g^{ii}. Let $h(\pi\alpha) \triangleq g(\frac{2c_p}{3}) - \frac{1-\pi\alpha}{2} = \frac{2c_p}{3}\sqrt{\frac{2c_p}{3\pi\alpha}} - c_p + \frac{\pi\alpha}{2}$. We first establish that h is increasing in $\pi\alpha$. Taking the first derivative, we obtain $dg(\pi\alpha)/d(\pi\alpha) = \frac{1}{2} - \frac{\sqrt{6}}{9}(\frac{c_p}{\pi\alpha})^{3/2}$. Taking the second derivative, we obtain $d^2h(\pi\alpha)/(d\pi\alpha)^2 = \frac{(\frac{c_p}{\pi\alpha})^{3/2}}{\pi\alpha\sqrt{6}} \geq 0$. Hence, h is convex and a lower bound on $dh(\pi\alpha)/d(\pi\alpha)$ is $dh(\pi\alpha)/d(\pi\alpha)|_{\pi\alpha=2c_p/3}$, which is positive. Therefore, h is increasing as well. Again since $\pi\alpha \geq 2c_p/3$, we obtain that $h(\pi\alpha) \geq 0$ for all $\pi\alpha$ in this range. Therefore when $\pi\alpha \in (\frac{2c_p}{3}, c_p]$, offering a rebate of $r_g^* = \frac{c_p}{3}$ increases (and maximizes) the expected social welfare. ■

Proof of Proposition 5: For part (i), first suppose $\pi\alpha > \frac{1}{c_p}$. Then

$$W_s^i(p) = \frac{1}{2}\left(1 - v_b^2 + \frac{\pi\alpha(p + c_p - v_b)^2 v_b^3 (c_p - p + v_b)}{(p - v_b)^3} - 2c_p\left(1 + \frac{c_p v_b}{p - v_b}\right)\right). \tag{A.47}$$

Taking the derivative with respect to p, we obtain

$$\begin{aligned} \frac{dW_s^i(p)}{dp} = \quad & \pi\alpha(c_p + p - v_b)^2 \left(3c_p v_b^2 \frac{dv_b}{dp} - v_b^3 - 3pv_b^2 \frac{dv_b}{dp} + 4v_b^3 \frac{dv_b}{dp}\right) \\ - \quad & v_b \frac{dv_b}{dp} + \frac{1}{2(p - v_b)^3}\left(2\pi\alpha(c_p + p - v_b)\left(1 - \frac{dv_b}{dp}\right) v_b^3 (c_p - p + v_b)\right. \\ - \quad & \frac{3\pi\alpha(c_p + p - v_b)^2 v_b^3 (c_p - p + v_b)\left(1 - \frac{dv_b}{dp}\right)}{2(p - v_b)^4} - \frac{c_p^2 \frac{dv_b}{dp}}{p - v_b} + \frac{c_p^2 v_b \left(1 - \frac{dv_b}{dp}\right)}{(p - v_b)^2}. \end{aligned} \tag{A.48}$$

Furthermore, since $\pi\alpha > \frac{1}{c_p}$ and $p = 0$, by Lemma 1, we have $v_b = c_p - \frac{1}{\pi\alpha}$. Evaluating at $v_b = c_p - \frac{1}{\pi\alpha}$, we obtain $\frac{dv_b}{dp} = 1 + \frac{2}{\pi\alpha v_b}$. Simplifying, we obtain

$$\left.\frac{dW_s^i(p)}{dp}\right|_{p=0} = \frac{1 - 2\pi\alpha c_p}{2\pi\alpha(1 - \pi\alpha c_p)} > 0. \tag{A.49}$$

Next suppose $c_p \le \pi\alpha \le \frac{1}{c_p}$. From (A.11), we see that v_b approaches $\frac{p}{1 - \sqrt{\pi\alpha c_p}}$ as p approaches zero. Plugging (A.12) into (A.48) and taking the limit as $p \rightarrow 0$, we have

$$\lim_{p \rightarrow 0} \frac{dW_s^i(p)}{dp} = \frac{c_p}{4\left(1 - \sqrt{\pi\alpha c_p}\right)} > 0. \tag{A.50}$$

Finally let $\pi\alpha < c_p$, i.e., the market can only be in Region II as described in Lemma 1. Consequently

$$W_s^{ii}(p) = \frac{1}{2}(1 - v_b^2)(1 - \pi\alpha(1 - v_b)) = \frac{(\pi\alpha + p)(1 - \pi\alpha) + (\pi\alpha - p)\sqrt{(1 - \pi\alpha)^2 + 4\pi\alpha p}}{4\pi\alpha}. \tag{A.51}$$

Taking the derivative, we obtain

$$\left.\frac{dW_s^{ii}(p)}{dp}\right|_{p=0} = \frac{1 - \pi\alpha}{4\pi\alpha} - \frac{\sqrt{(1 - \pi\alpha)^2 + 4\pi\alpha p}}{4\pi\alpha} + \left.\frac{\pi\alpha - p}{2\sqrt{(1 - \pi\alpha)^2 + 4\pi\alpha p}}\right|_{p=0} = \frac{\pi\alpha}{2(1 - \pi\alpha)} > 0. \tag{A.52}$$

Therefore for all $\pi\alpha > 0$, there exists a $\tau > 0$ such that the expected social welfare can be increased by imposing a tax τ.

For part (ii), first consider $\pi\alpha < c_p$. By Region II of Lemma 1, $v_p = 1$ and v_b is given by (A.9). Substituting into (4), the first order condition yields

$$\tau_t^* = \frac{-1 + 2\pi\alpha(1+\pi\alpha) + \sqrt{(1-\pi\alpha)^2\left(1 - 2\pi\alpha + 4\left(\pi\alpha\right)^2\right)}}{9\pi\alpha}, \tag{A.53}$$

which is clearly increasing in $\pi\alpha$ in this range. By Lemma 1 and continuity of the welfare function, there exists a $\underline{\theta} > c_p$ such that for all $c_p < \pi\alpha < \underline{\theta}$, the optimal tax is given by (A.53). Defining ξ as in the proof of Proposition 2 and by (A.48), we obtain $\tau_t^* = \xi - \frac{3}{2c_p}\xi^2 + O\left(\xi^3\right)$. Therefore, for large enough $\pi\alpha$, τ_t^* is decreasing in $\pi\alpha$ and increasing in c_p. ■

Proof of Proposition 6: By part (ii) of Proposition 4, the social welfare under the optimal rebate is given by $W_g^* \triangleq W_g(\frac{c_p}{3}) = \frac{1}{2} - c_p + \frac{1}{\sqrt{\pi\alpha}}\left(\frac{2c_p}{3}\right)^{3/2}$. When a tax is imposed the resulting equilibrium is either in Region I or Region II as given in Lemma 1. Suppose that the equilibrium falls in Region II. By Lemma 1, $v_p = 1$ and v_b is given by (A.9). Substituting into (4), the social welfare is given by

$$W_t^{ii}(\tau) = \frac{(\pi\alpha+\tau)(1-\pi\alpha) + (\pi\alpha-\tau)\sqrt{(1-\pi\alpha)^2 + 4\pi\alpha\tau}}{4\pi\alpha}. \tag{A.54}$$

$W_t^{ii}(\cdot)$ is concave and the optimal tax given by

$$\tau_t^* = \frac{-1 + 2\pi\alpha(1+\pi\alpha) + \sqrt{(1-\pi\alpha)^2(1 - 2\pi\alpha + 4\pi\alpha^2)}}{9\pi\alpha}. \tag{A.55}$$

Define $W_t^* \triangleq W_t^{ii}(\tau_t^*)$ and let $\pi\alpha = kc_p$. We then have

$$W_g^* = \frac{1}{2} - \left(1 - \frac{2}{3}\sqrt{\frac{2}{3k}}\right)c_p + O(c_p^2), \tag{A.56}$$

and $W_t^* = \frac{1}{2} - \frac{kc_p}{2} + O(c_p^2)$. Comparing the two expressions, it follows that for sufficiently small c_p, $W_g^* > W_t^*$ if and only if $k > 2/3$. Now suppose that the optimal tax induces Region I equilibrium behavior. In this case, the social welfare is given by

$$W_t^i(\tau) = \frac{1}{2}\left(1 - v_b^2 - \frac{\pi\alpha v_b^3(v_b-\tau-c_p)^2(v_b-\tau+c_p)}{(v_b-\tau)^3} - 2c_p\left(1 - \frac{c_p v_b}{v_b-\tau}\right)\right), \tag{A.57}$$

where v_b solves (A.7) with $p = \tau$. By (A.7), as $c_p \to 0$, $z_1 \triangleq \lim_{c_p\to 0}(v_b-\tau)/c_p^2$ is constant. Further, taking the derivative with respect to τ, substituting $\pi\alpha = kc_p$, writing the first order condition and by (A.12), it follows that for the optimal tax τ_t^*, $z_2 \triangleq \lim_{c_p\to 0}\tau_t^*/c_p$

is constant. Substituting in (A.7) and taking the limit of both sides as $c_p \to 0$, we obtain $z_2 = z_1/\sqrt{k}$. Further, substituting these two limits back into the first order condition and by taking the limit as $c_p \to 0$, we find that for the optimal tax

$$\lim_{c_p \to 0} \frac{\tau_t^*}{c_p} = \frac{27 z_2^3}{16} + \frac{81 k^4 z_2^9}{256 z_1^8} + \frac{81 k^2 z_2^6}{64 z_1^4} + \frac{z_1^2}{k} + \frac{3 z_1^4}{4k^2}\,. \tag{A.58}$$

Substituting in $z_2 = z_1/\sqrt{k}$ in (A.58) and solving for z_1, we obtain $z_1 = \sqrt{k}/4$. It follows that $z_2 = 1/4$. Substituting back into (A.57) yields $W_t^* \triangleq W_t^i(\tau_t^*) = \frac{1}{2} - \left(1 - \frac{1}{2\sqrt{k}}\right) c_p + O(c_p^2)$. Comparing with (A.56), we see that $W_g^* > W_t^*$, which completes the proof. ■