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Electronic Companion—“Who Should Be Responsible for Software Security? A Comparative Analysis of Liability Policies in Network Environments” by Terrence August and Tunay I. Tunca, *Management Science*, DOI 10.1287/mnsc.1100.1304.

Online Appendix. Proofs of Propositions

Proof of Lemma 1. If a consumer with valuation v decides to buy and patch, then her expected payoff is $v - p - \tilde{c}_p - \tilde{\pi}_z \alpha b v$. If she decides to buy but not patch, then her expected payoff is $v - p - \tilde{\pi}_a \alpha u v - \tilde{\pi}_z \alpha b v$. Therefore, a consumer buys the product and patches in the third period in case a security vulnerability is revealed if and only if

$$v \geq \max\left(\frac{\tilde{c}_p}{\tilde{\pi}_a \alpha u}, \frac{p + \tilde{c}_p}{1 - \tilde{\pi}_z \alpha b}\right). \tag{A1}$$

Consequently, in equilibrium, if a consumer with valuation v_0 buys and patches the software, then every consumer with valuation $v > v_0$ will also buy and patch, and hence there exists a $v_p \in (0, 1]$ such that a consumer with valuation $v \in \mathcal{V}$ will buy and patch if and only if $v \geq v_p$, in which case $\sigma^*(v) = (B, P)$. We next consider the buying decision in the second period. If a consumer with valuation v decides to buy the product, she will incur a cost p. She will buy the software if and only if

$$v \geq \min\left(\frac{p}{1 - \tilde{\pi}_a \alpha u - \tilde{\pi}_z \alpha b}, \frac{p + \tilde{c}_p}{1 - \tilde{\pi}_z \alpha b}\right). \tag{A2}$$

Let $0 < v_1 \leq 1$ and $\sigma^*(v_1) \in \{(B, P), (B, NP)\}$, then by (A2), for all $v > v_1$, $\sigma^*(v) \in \{(B, P), (B, NP)\}$, and hence there exists a $v_b \in (0, 1]$, such that a consumer with valuation $v \in \mathcal{V}$ will purchase if and only if $v \geq v_b$. By definition, $v_p \geq v_b$. Suppose $0 < v_p = v_b < 1$ and $\tilde{c}_p > 0$ and consider the consumer with valuation v satisfying $v_b - \epsilon < v < v_b$. Because $u = 0$ and, by (A1), $v_b \geq (p + \tilde{c}_p)/(1 - \tilde{\pi}_z \alpha b)$, there exist $\epsilon > 0$ sufficiently small such that $v \geq p + \tilde{\pi}_z \alpha b v$ is satisfied. By (A2), $\sigma^*(v) \in \{(B, P), (B, NP)\}$, which is a contradiction. Therefore, we conclude that when $0 < p \leq 1$ there exist $0 < v_b < v_p \leq 1$ satisfying σ^* given in (4):

The following three inequalities are algebraically equivalent:

$$\frac{p}{1 - \tilde{\pi}_a \alpha u - \tilde{\pi}_z \alpha b} \geq \frac{p + \tilde{c}_p}{1 - \tilde{\pi}_z \alpha b} \quad \Leftrightarrow \quad \frac{p}{1 - \tilde{\pi}_a \alpha u - \tilde{\pi}_z \alpha b} \geq \frac{\tilde{c}_p}{\tilde{\pi}_a \alpha u} \quad \Leftrightarrow \quad \frac{p + \tilde{c}_p}{1 - \tilde{\pi}_z \alpha b} \geq \frac{\tilde{c}_p}{\tilde{\pi}_a \alpha u}. \tag{A3}$$

To see this,

$$\frac{p + \tilde{c}_p}{1 - \tilde{\pi}_z \alpha b} \geq \frac{\tilde{c}_p}{\tilde{\pi}_a \alpha u} \tag{A4}$$

$$\Leftrightarrow \quad (p + \tilde{c}_p)\tilde{\pi}_a \alpha u \geq \tilde{c}_p(1 - \tilde{\pi}_z \alpha b)$$

$$\Leftrightarrow \quad (p + \tilde{c}_p)\tilde{\pi}_a \alpha u \geq (p + \tilde{c}_p)(1 - \tilde{\pi}_z \alpha b) - p(1 - \tilde{\pi}_z \alpha b)$$

$$\Leftrightarrow \quad p(1 - \tilde{\pi}_z \alpha b) \geq (p + \tilde{c}_p)(1 - \tilde{\pi}_z \alpha b) - (p + \tilde{c}_p)\tilde{\pi}_a \alpha u$$

$$\Leftrightarrow \quad p(1 - \tilde{\pi}_z \alpha b) \geq (p + \tilde{c}_p)(1 - \tilde{\pi}_a \alpha u - \tilde{\pi}_z \alpha b)$$

$$\Leftrightarrow \quad \frac{p}{1 - \tilde{\pi}_a \alpha u - \tilde{\pi}_z \alpha b} \geq \frac{p + \tilde{c}_p}{1 - \tilde{\pi}_z \alpha b} \tag{A5}$$

$$\Leftrightarrow \quad p(1 - \tilde{\pi}_z \alpha b) \geq p(1 - \tilde{\pi}_a \alpha u - \tilde{\pi}_z \alpha b) + \tilde{c}_p(1 - \tilde{\pi}_a \alpha u - \tilde{\pi}_z \alpha b)$$

$$\Leftrightarrow \quad p(1 - \tilde{\pi}_z \alpha b) \geq p(1 - \tilde{\pi}_z \alpha b) - p\tilde{\pi}_a \alpha u + \tilde{c}_p(1 - \tilde{\pi}_a \alpha u - \tilde{\pi}_z \alpha b)$$

$$\Leftrightarrow \quad 0 \geq -p\tilde{\pi}_a \alpha u + \tilde{c}_p(1 - \tilde{\pi}_a \alpha u - \tilde{\pi}_z \alpha b)$$

$$\Leftrightarrow \quad p\tilde{\pi}_a \alpha u \geq \tilde{c}_p(1 - \tilde{\pi}_a \alpha u - \tilde{\pi}_z \alpha b)$$

$$\Leftrightarrow \quad \frac{p}{1 - \tilde{\pi}_a \alpha u - \tilde{\pi}_z \alpha b} \geq \frac{\tilde{c}_p}{\tilde{\pi}_a \alpha u} \tag{A6}$$

In words, viewing the expressions in (A3) as functions of the model variables, the three inequalities are valid on the same region of the parameter space. When $v_p < 1$, by (A1) and (A2), in equilibrium

$$v_p = \max\left(\frac{\tilde{c}_p}{\tilde{\pi}_a \alpha u(\sigma^*)}, \frac{p+\tilde{c}_p}{1-\tilde{\pi}_z \alpha b(\sigma^*)}\right) \quad \text{and} \tag{A7}$$

$$v_b = \min\left(\frac{p}{1-\tilde{\pi}_a \alpha u(\sigma^*) - \tilde{\pi}_z \alpha b(\sigma^*)}, \frac{p+\tilde{c}_p}{1-\tilde{\pi}_z \alpha b(\sigma^*)}\right). \tag{A8}$$

Thus, by (A3), (A7), (A8), and because $v_b < v_p$,

$$v_b = \frac{p}{1-\tilde{\pi}_a \alpha u(\sigma^*) - \tilde{\pi}_z \alpha b(\sigma^*)} \quad \text{and} \tag{A9}$$

$$v_p = \frac{\tilde{c}_p}{\tilde{\pi}_a \alpha u(\sigma^*)}. \tag{A10}$$

By (4), it follows that

$$\tilde{\pi}_a \alpha (v_p - v_b) v_p = \tilde{c}_p \quad \text{and} \tag{A11}$$

$$v_b = p + \tilde{\pi}_a \alpha (v_p - v_b) v_b + \tilde{\pi}_z \alpha (1 - v_b) v_b. \tag{A12}$$

Substituting (A11) into (A12) yields

$$v_p = \frac{\tilde{c}_p v_b}{v_b - p - \tilde{\pi}_z \alpha (1 - v_b) v_b}. \tag{A13}$$

For convenience, we define the function $z(p, v_b) \triangleq v_b - p - \tilde{\pi}_z \alpha (1 - v_b) v_b$, and, by substituting (A13) into (A12), we obtain

$$z^2 = \tilde{\pi}_a \alpha v_b^2 (\tilde{c}_p - z). \tag{A14}$$

For $v_p < 1$ to be satisfied, by (A13), $v_b > v_b' \triangleq (-1 + \tilde{\pi}_z \alpha + \tilde{c}_p + \sqrt{(1 - \tilde{\pi}_z \alpha - \tilde{c}_p)^2 + 4\tilde{\pi}_z \alpha p})/(2\tilde{\pi}_z \alpha)$ is a necessary condition, noting that the other root of the underlying quadratic is negative but $v_b \geq 0$ must be satisfied. Substituting this expression into (A14) and because $0 \leq v_b \leq 1$, we obtain that for $v_p < 1$, we must have $p < \bar{p}$. By (A14), we define $g(v_b) \triangleq z^2 - \tilde{\pi}_a \alpha v_b^2 (\tilde{c}_p - z)$, and, further, we let $v_b^{zl} \triangleq \min\{v_b: z(p, v_b) = 0\}$ and $v_b^{zr} \triangleq \max\{v_b: z(p, v_b) = 0\}$. Using the properties $g(v_b^{zl}) = g(v_b^{zr}) = -\tilde{\pi}_a \alpha v_b^2 \tilde{c}_p < 0$, $g(0) = p^2 > 0$, $g(1) = (1-p)^2 + \tilde{\pi}_a \alpha (1 - p - \tilde{c}_p) > 0$, in addition to $v_b' > v_b^{zr}$ being satisfied, it follows that when $p < \bar{p}$, only the largest root of $g(\cdot)$ can satisfy $v_b > v_b'$, and, by (A12), this root must satisfy $v_b > p$. However, when $p \geq \bar{p}$, substituting $v_p = 1$ into (A12), we obtain $(\tilde{\pi}_a \alpha + \tilde{\pi}_z \alpha) v_b^2 + (1 - \tilde{\pi}_a \alpha - \tilde{\pi}_z \alpha) v_b - p = 0$, which has a unique positive root satisfying $v_b \leq 1$ and given by

$$v_b = -\frac{1 - \tilde{\pi}_a \alpha - \tilde{\pi}_z \alpha}{2(\tilde{\pi}_a \alpha + \tilde{\pi}_z \alpha)} + \frac{1}{2(\tilde{\pi}_a \alpha + \tilde{\pi}_z \alpha)} \sqrt{(1 - \tilde{\pi}_a \alpha - \tilde{\pi}_z \alpha)^2 + 4(\tilde{\pi}_a \alpha + \tilde{\pi}_z \alpha) p}. \tag{A15}$$

This completes the proof. □

Proof of Proposition 1. Technically, we will show that there exist bounds $\underline{\omega}$, $\bar{\omega}$, $\underline{\eta}$, and $\bar{\eta}$ such that $dW/d\lambda_z < 0$ if (i) $\alpha > \underline{\omega}$ and $\pi_z < \bar{\eta}$ or (ii) $\alpha > \underline{\omega}$ and $\pi_z > \underline{\eta}$.

For (i), let $\pi_z = k/\alpha^2$. By definition, $\bar{p} \to 1 - c_p$ as α grows large, hence Region I of Lemma 1 applies. By (A13) and (A14), we obtain

$$v_b = p + c_p + \left(k(p + c_p)(1 - p - c_p)(1 - \lambda_z) - \frac{c_p^2}{(p + c_p)^2 \pi_a}\right)\frac{1}{\alpha} + O\left(\frac{1}{\alpha^2}\right). \tag{A16}$$

By (7), using the implicit function theorem on (A14) to compute dv_b/dp, and substituting (A16) in for v_b, we obtain

$$\frac{d\Pi}{dp} = 1 - c_p - 2p + \frac{A_1}{2(p + c_p)^3 \pi_a \alpha} + O\left(\frac{1}{\alpha^2}\right), \tag{A17}$$

where A_1 is a constant, giving

$$p^* = \frac{1 - c_p}{2} + \left(\frac{k(1 - c_p)(7\lambda_z - 2 + c_p(\lambda_z + 2))}{16} + \frac{2c_p^2(3c_p - 1)}{(1 + c_p)^3 \pi_a}\right)\frac{1}{\alpha} + O\left(\frac{1}{\alpha^2}\right), \tag{A18}$$

and, hence, $p^* < \bar{p}$ is satisfied. Substituting (A16) and (A18) into (8), we obtain

$$W(\lambda_z) = \frac{3(1 - c_p)^2}{8} - \left(\frac{k(1 - c_p)^2(8(1 + c_p) - 3\lambda_z(1 - c_p))}{32} + \frac{c_p^2(3 - c_p(4 - c_p))}{(1 + c_p)^3 \pi_a}\right)\frac{1}{\alpha} + O\left(\frac{1}{\alpha^2}\right), \tag{A19}$$

which upon differentiation yields

$$\frac{dW}{d\lambda_z} = -\frac{3k(1-c_p)^3}{32\alpha} + O\left(\frac{1}{\alpha^2}\right), \tag{A20}$$

thus $dW/d\lambda_z < 0$. For (ii), we refer the reader to the proof of Proposition 3, where $\beta = 0$ is a special case. This completes the proof. □

Proof of Proposition 2. Technically, we will prove the following:

(i) There exist $\underline{\omega} > 0$ and $\underline{\pi}_z \in (0,1)$ such that if $\alpha > \underline{\omega}$ and $\pi_z > \underline{\pi}_z$, then

(a) if $c_p \geq \pi_a/(\pi_a+\pi_z)$, then $\lambda_p^* = 0$; i.e., in the welfare-maximizing patch liability policy, the vendor has no share of patching costs.

(b) if $c_p < \pi_a/(\pi_a+\pi_z)$, then welfare is maximized at a strictly positive vendor patch liability level $\lambda_p^* \in (0,1)$. Furthermore, $\lim_{\alpha\to\infty} \lambda_p^* = 1/2$.

(ii) There exist $\underline{\omega} > 0$ and $\bar{\pi}_z \in (0,1)$ such that if $\alpha > \underline{\omega}$ and $\pi_z < \bar{\pi}_z$, then a patch liability policy strictly increases social welfare if and only if $c_p > 6 - \sqrt{33}$. Furthermore, $\lim_{\alpha\to\infty} \lambda_p^* = (12c_p - 3 - c_p^2)/(16c_p)$.

For part (i), by definition, $\bar{p} \to 1 - (1-\lambda_p)c_p(\pi_a+\pi_z)/\pi_a$. Suppose Region II of Lemma 1 applies. By (A15), we obtain

$$v_b = 1 - \frac{1-p}{(\pi_a+\pi_z)\alpha} + \frac{p(1-p)}{(\pi_a+\pi_z)^2\alpha^2} + O\left(\frac{1}{\alpha^3}\right). \tag{A21}$$

By (5) and (7), we obtain

$$p_{ii}^* = \frac{1}{2} - \frac{1}{8(1-\beta)(\pi_a+\pi_z)\alpha} + O\left(\frac{1}{\alpha^2}\right). \tag{A22}$$

Substituting (A21) and (A22) into (5) and (8) gives

$$\Pi(p_{ii}^*, \lambda_p) = \frac{1}{4(\pi_a+\pi_z)\alpha} - \frac{1}{8(\pi_a+\pi_z)^2\alpha^2} + O\left(\frac{1}{\alpha^3}\right) \tag{A23}$$

and

$$W(\lambda_p) = \frac{1}{4(\pi_a+\pi_z)\alpha} + O\left(\frac{1}{\alpha^2}\right). \tag{A24}$$

Suppose Region I of Lemma 1 applies. Then, by (A14), we obtain

$$v_b = 1 - \frac{1-p-c_p(1-\lambda_p)}{\pi_z\alpha} + \frac{A_1}{\pi_a\pi_z^2\alpha^2} + O\left(\frac{1}{\alpha^3}\right), \tag{A25}$$

where $A_1 = p(1-p)\pi_a + c_p\pi_a(1-2p)(1-\lambda_p) - c_p^2(1-\lambda_p)^2 \cdot (\pi_a+\pi_z)$. By (5) and (7), we obtain

$$p_i^* = \frac{1-c_p}{2} + \lambda_p c_p + \left(\frac{c_p^2(1+\lambda_p(2\lambda_p-3))}{2\pi_a\alpha} - \frac{(1-c_p)^2}{8\pi_z\alpha}\right) + O\left(\frac{1}{\alpha^2}\right), \tag{A26}$$

and, hence, substituting (A25) and (A26) into (5) and (8) yields

$$\Pi(p_i^*, \lambda_p) = \left(\frac{c_p^2(1-\lambda_p)\lambda_p}{\pi_a} + \frac{(1-c_p)^2}{4\pi_z}\right)\frac{1}{\alpha} + O\left(\frac{1}{\alpha^2}\right) \tag{A27}$$

and

$$W(\lambda_p) = \left(\frac{c_p^2\lambda_p(1-\lambda_p)}{\pi_a} + \frac{(1-c_p)^2}{4\pi_z}\right)\frac{1}{\alpha} + O\left(\frac{1}{\alpha^2}\right), \tag{A28}$$

respectively. Differentiating (A28) gives

$$\frac{dW}{d\lambda_p} = \frac{c_p^2(1-2\lambda_p)}{\pi_a\alpha} + O\left(\frac{1}{\alpha^2}\right). \tag{A29}$$

By (A29), it follows that

$$\lambda_p^* = \frac{1}{2} + O\left(\frac{1}{\alpha}\right), \tag{A30}$$

which upon substitution into (A28) yields

$$W(\lambda_p^*) = \frac{(1-c_p)^2\pi_a + c_p^2\pi_z}{4\pi_a\pi_z\alpha} + O\left(\frac{1}{\alpha^2}\right). \tag{A31}$$

Finally, suppose $p = \bar{p}$, which upon substitution into (A15) and subsequently into (5) and (8) gives

$$\Pi(\bar{p}, \lambda_p) = \frac{c_p(1-\lambda_p)(\pi_a(1-c_p(1-\lambda_p)) - c_p\pi_z(1-\lambda_p))}{\pi_a^2\alpha} + O\left(\frac{1}{\alpha^2}\right), \tag{A32}$$

and

$$W(\lambda_p) = \frac{c_p(1-\lambda_p)(\pi_a(1-c_p(1-\lambda_p)) - c_p\pi_z(1-\lambda_p))}{\pi_a^2\alpha} + O\left(\frac{1}{\alpha^2}\right), \tag{A33}$$

respectively. Hence,

$$\frac{dW}{d\lambda_p} = -\frac{c_p(\pi_a(1-2c_p(1-\lambda_p)) - 2c_p\pi_z(1-\lambda_p))}{\pi_a^2\alpha} + O\left(\frac{1}{\alpha^2}\right). \tag{A34}$$

By (A34), we obtain

$$\lambda_p^* = \frac{2c_p(\pi_a+\pi_z) - \pi_a}{2c_p(\pi_a+\pi_z)} + O\left(\frac{1}{\alpha}\right), \tag{A35}$$

which upon substitution into (A33) yields

$$W(\lambda_p^*) = \frac{1}{4(\pi_a+\pi_z)\alpha} + O\left(\frac{1}{\alpha^2}\right). \tag{A36}$$

By the definition of $\bar{p}$, (A22), and (A26), for sufficiently large α, $p_i^* < \bar{p}$ is satisfied if and only if $\lambda_p > 1 - \pi_a(1-c_p)/(2c_p\pi_z)$, and $p_{ii}^* > \bar{p}$ is satisfied if and only if $\lambda_p < 1 - \pi_a/(2c_p(\pi_a+\pi_z))$. Suppose that $c_p < \pi_a/(\pi_a+\pi_z)$. In this case, $1-\pi_a(1-c_p)/(2c_p\pi_z) < 1-\pi_a/(2c_p(\pi_a+\pi_z))$, thus the equilibrium outcome falls in Region I whenever $\lambda_p > 1-\pi_a/(2c_p(\pi_a+\pi_z))$ and in Region II whenever $\lambda_p < 1-\pi_a(1-c_p)/(2c_p\pi_z)$ is satisfied. If $1-\pi_a(1-c_p)/(2c_p\pi_z) < \lambda_p < 1-\pi_a/(2c_p(\pi_a+\pi_z))$, then by (A23) and (A27), for sufficiently large α, $p^* = p_i^*$ is satisfied whenever

$$h(\lambda_p) \triangleq \left(\frac{c_p^2(1-\lambda_p)\lambda_p}{\pi_a} + \frac{(1-c_p)^2}{4\pi_z}\right) - \frac{1}{4(\pi_a+\pi_z)} \geq 0. \tag{A37}$$

Denoting

$$\bar{\lambda}_p \triangleq \frac{1}{2} - \frac{(c_p\pi_z - \pi_a(1-c_p))^2}{2\sqrt{c_p^2\pi_z(\pi_a+\pi_z)(c_p\pi_z-\pi_a(1-c_p))^2}}, \tag{A38}$$

then, by (A37), it follows that $p^* = p_i^*$ is satisfied if either $c_p < 1-\sqrt{\pi_z/(\pi_a+\pi_z)}$ or both $1-\sqrt{\pi_z/(\pi_a+\pi_z)} < c_p < \pi_a/(\pi_a+\pi_z)$ and $\bar{\lambda}_p < \lambda_p < 1-\pi_a/(2c_p(\pi_a+\pi_z))$ are satisfied. On the other hand, $p^* = p_{ii}^*$ is satisfied whenever $1-\pi_a(1-c_p)/(2c_p\pi_z) < \lambda_p < \bar{\lambda}_p$. By (A24) and (A31), because $(\pi_a(1-c_p)^2 + c_p^2\pi_z)/(4\pi_a\pi_z) > 1/(4(\pi_a+\pi_z))$ and the welfare-maximizing liability share is feasible, λ_p^* is given by (A30).

Suppose $c_p > \pi_a/(\pi_a+\pi_z)$. Similarly, $p^* = p_i^*$ is satisfied if $\lambda_p > 1-\pi_a(1-c_p)/(2c_p\pi_z)$, and $p^* = p_{ii}^*$ is satisfied if $\lambda_p < 1-\pi_a/(2c_p(\pi_a+\pi_z))$. However, for $1-\pi_a/(2c_p(\pi_a+\pi_z)) < \lambda_p < 1-\pi_a(1-c_p)/(2c_p\pi_z)$, it follows that $p^* = \bar{p}$. By (A30) and (A35), the interior, welfare-maximizing liability shares are not feasible, and because

$$\lim_{\lambda_p \to (1-\pi_a(1-c_p)/(2c_p\pi_z))^+} W(\lambda_p) < W(0), \tag{A39}$$

we obtain $\lambda_p^* = 0$. The proof of part (ii) is similar to that of the proof of Proposition 3 in August and Tunca (2006), hence we skip it here for conciseness. □

Proof of Proposition 3. For part (i), we will prove that there exist $\underline{\omega}$ and $\bar{\eta}$ such that if $\alpha > \underline{\omega}$ and $\pi_z < \bar{\eta}$, then, denoting the unique solution to the Equation (15) in $(0,1)$ by z^*,

(a) if $C'(z^*)/C''(z^*) \geq 3(1-z^*)$, then both welfare and vendor investment in security increase with λ_z, i.e., $\lambda_z^* = 1$;

(b) if $C'(z^*)/C''(z^*) < 3(1-z^*)$, then $\lambda_z^* = 0$. Furthermore, if $c_p < 1/4$ or $C'(1-1/4c_p) < 3c_p/8$, then vendor investment in security increases, but welfare decreases with λ_z. Otherwise both welfare and vendor investment in security decrease with λ_z.

Let $\pi_z = k/\alpha^2$. By definition, $\bar{p} \to 1-c_p(1-\beta)$ as $\alpha \to \infty$, hence Region I of Lemma 1 applies. By (A13) and (A14) under parameters $(1-\beta)\pi_a$, $(1-\lambda_z)(1-\beta)\pi_z$, and $(1-\beta)c_p$, we obtain

$$v_b = p + c_p(1-\beta) - \frac{(1-\beta)(c_p^2 - k(-1+p+c_p(1-\beta))(p+c_p(1-\beta))^3(-1+\lambda_z)\pi_a)}{\pi_a(p+c_p(1-\beta))^2\alpha} + O\left(\frac{1}{\alpha^2}\right) \tag{A40}$$

and

$$v_p = v_b + \frac{c_p}{(p+c_p(1-\beta))\pi_a\alpha} + O\left(\frac{1}{\alpha^2}\right). \tag{A41}$$

By (13), differentiating $\Pi(\cdot)$, and substituting for (A40) and (A41), it follows that

$$\frac{d\Pi}{dp} = (1-2p-c_p(1-\beta)) - \frac{(1-\beta)A_1}{2\alpha} + O\left(\frac{1}{\alpha^2}\right), \tag{A42}$$

where $A_1 = k[-\lambda_z + p(3p-2)(3\lambda_z-2) + c_p^2(1-\beta)^2(5\lambda_z-2) + 2c_p(1-\beta)(1-2\lambda_z+p(7\lambda_z-4))] + 2c_p^2(p-c_p(1-\beta))/(\pi_a(p+c_p(1-\beta))^3)$. Hence, we obtain

$$p^* = \frac{1-c_p(1-\beta)}{2} - \frac{(1-\beta)A_2}{4\alpha} + O\left(\frac{1}{\alpha^2}\right), \tag{A43}$$

where $A_2 = (k/4)(1-c_p(1-\beta))[2-7\lambda_z - c_p(1-\beta)(2+\lambda_z)] + 8c_p^2(1-3c_p(1-\beta))/(\pi_a(1+c_p(1-\beta))^3)$, thus $p^* < \bar{p}$ is satisfied for sufficiently large α. The vendor's optimal investment satisfies

$$\frac{d\Pi}{d\beta} = \frac{\partial\Pi}{\partial\beta} + \frac{\partial\Pi}{\partial v_b}\cdot\frac{dv_b}{d\beta} + \frac{\partial\Pi}{\partial p}\cdot\frac{dp}{d\beta}, \tag{A44}$$

which, by using (A12), the first-order equation for price, the implicit function theorem, and subsequently substituting in (A40) and (A43), yields

$$\frac{d\Pi(p^*(\beta),\beta)}{d\beta} = \frac{c_p(1-c_p(1-\beta))}{2} - C'(\beta) + \frac{A_3}{16\pi_a(1+c_p(1-\beta))^3\alpha} + O\left(\frac{1}{\alpha^2}\right), \tag{A45}$$

where A_3 is a constant. By (11), (A40), and (A43), we obtain

$$W(\lambda_z, \beta^*(\lambda_z)) = \frac{3(1-c_p(1-\beta^*))^2}{8} - C(\beta^*) - \frac{A_4}{32\pi_a(1+c_p(1-\beta^*))^3\alpha} + O\left(\frac{1}{\alpha^2}\right), \tag{A46}$$

where A_4 is a constant. By (13) and (A45), it follows that $\beta^* = z^* + O(1/\alpha)$ for sufficiently large α, where z^* is the solution to (15), and exists and is unique because $C'(0) < c_p(1-c_p)$ and C'' is increasing. Thus, by (A45), we obtain

$$\frac{d\beta^*}{d\lambda_z} = \frac{k(1-c_p(1-z^*))^2(1-4c_p(1-z^*))}{8(2C''(z^*)-c_p^2)\alpha} + O\left(\frac{1}{\alpha^2}\right), \tag{A47}$$

hence for sufficiently large α, β^* is increasing in λ_z if and only if $z^* > 1-1/(4c_p)$, which, by (A45), is satisfied whenever $C'(1-1/(4c_p)) < 3c_p/8$ or $c_p < 1/4$. By (11) and (A47), it follows that

$$\frac{dW}{d\lambda_z} = \frac{k(1-c_p(1-z^*))^2A_5}{32(2C''(z^*)-c_p^2)\alpha} + O\left(\frac{1}{\alpha^2}\right), \tag{A48}$$

where

$$A_5 = -4(1-4c_p(1-z^*))C'(z^*) + 3(1-c_p(1-z^*))\big(c_p(1-3c_p(1-z^*)) - 2(1-z^*)\big)C''(z^*). \tag{A49}$$

Plugging $C'(z^*) = c_p(1-c_p(1-z^*))/2$ into (A49) and, in turn, into (A48) and simplifying, we find that $dW/d\lambda_z \geq 0$ if and only if

$$c_p(1-c_p(1-z^*)) - 6(1-z^*)C''(z^*) \geq 0, \tag{A50}$$

which, in turn, is satisfied when

$$C''(z^*) < \frac{c_p(1-c_p(1-z^*))}{6(1-z^*)}. \tag{A51}$$

Substituting $C'(z^*) = c_p(1-c_p(1-z^*))/2$, we obtain $dW/d\lambda_z \geq 0$ if an only if $C'(z^*)/C''(z^*) \geq 3(1-z^*)$. Now (A50) is satisfied at $C''(z^*) = c_p^2/2$, if and only if $c_p(1-c_p(1-z^*))(1-4c_p(1-z^*)) > 0$, or equivalently, when $C'(1-1/(4c_p)) < 3c_p/8$ as we have shown above. Now, because $C''(z^*) > c_p^2/2$, it follows that $C'(z^*)/C''(z^*) \geq 3(1-z^*)$ implies $C'(1-1/(4c_p)) < 3c_p/8$, and vendor investment in security increases in λ_z as well as the social welfare with increased λ_z. Conversely, if $C'(1-1/(4c_p)) \geq 3c_p/8$, then $C'(z^*)/C''(z^*) < 3(1-z^*)$, and both social welfare and vendor security investment decrease with λ_z. Finally, if $C'(z^*)/C''(z^*) < 3(1-z^*)$ and $C'(1-1/(4c_p)) < 3c_p/8$, vendor security investment increases but the welfare decreases with an increase in λ_z. This completes the proof of part (i).

For part (ii), by definition, $\bar{p} \to 1-c_p-(1-\lambda_z)c_p\pi_z/\pi_a$ as $\alpha \to \infty$. If $c_p < 1-\sqrt{\pi_z/(\pi_a+\pi_z)}$, then, by Lemma 1, Region I can apply. From (A14), we obtain

$$v_b = 1 - \frac{1-p-c_p(1-\beta)}{(1-\beta)(1-\lambda_z)\pi_z\alpha} + \frac{A_1}{(1-\beta)^2(1-\lambda_z)^2\pi_a\pi_z^2\alpha^2} + O\left(\frac{1}{\alpha^3}\right), \tag{A52}$$

where $A_1 = (1-p-c_p(1-\beta))(p+c_p(1-\beta)) - \pi_z c_p^2(1-\beta)^2(1-\lambda_z)$. By (12) and (13), the unconstrained optimizing price for Region I satisfies

$$p_i^*(\beta) = \frac{(1-c_p(1-\beta))(1+\lambda_z)}{2} + \frac{A_2}{16\pi_a\pi_z(1-\beta)\alpha} + O\left(\frac{1}{\alpha^2}\right), \tag{A53}$$

where $A_2 = \pi_a(1-c_p(1-\beta))(c_p(1-\beta)(1-\lambda_z)(2-3\lambda_z)-2-3\lambda_z(1+\lambda_z))+8\pi_z c_p^2(1-\beta)^2(1+\lambda_z)$. Substituting (A52) and (A53) into (13) and differentiating, we obtain

$$\frac{d\Pi(p_i^*(\beta),\beta)}{d\beta} = -C'(\beta)+\frac{1-c_p^2(1-\beta)^2}{4(1-\beta)^2\pi_z\alpha}+\frac{A_3}{16(1-\beta)^3\pi_a\pi_z^2\alpha^2}+O\left(\frac{1}{\alpha^3}\right), \tag{A54}$$

where $A_3 = (1-c_p(1-\beta))[c_p(-1-c_p(1-\beta))(1-\beta)(2-\lambda_z)-2(2+\lambda_z)]\pi_a - 8c_p^3(1-\beta)^3\pi_z$. Thus, if $C'(0)>0$, then $\beta_i^*=0$. However, if $C'(0)=0$, then

$$\beta_i^* = \frac{1-c_p^2}{4\pi_z C''(0)\alpha}+\frac{A_4}{32\pi_a\pi_z^2C''(0)^3\alpha^2}+O\left(\frac{1}{\alpha^3}\right), \tag{A55}$$

where $A_4 = 2C''(0)^2\{\pi_a(1-c_p)[c_p(1+c_p)(\lambda_z-2)-2(\lambda_z+2)]+8c_p^3\pi_z\}+\pi_a(1-c_p^2)[4C''(0)-(1-c_p^2)C'''(0)]$. By (A53), (A55), and since, for sufficiently large α, $p_i^*(\beta_i^*)<\bar{p}$ if and only if $c_p<\pi_a/(\pi_a+2\pi_z)$, which is satisfied because $c_p<1-\sqrt{\pi_z/(\pi_a+\pi_z)}$, by (12), we obtain

$$\Pi(p_i^*(\beta_i^*),\beta_i^*) = -C(0)+\frac{(1-c_p)^2}{4\pi_z\alpha}+O\left(\frac{1}{\alpha^2}\right). \tag{A56}$$

Differentiating (A55), we obtain

$$\frac{d\beta_i^*}{d\lambda_z} = -\frac{c_p^3-3c_p+2}{16\pi_z^2C''(0)\alpha^2}+O\left(\frac{1}{\alpha^3}\right), \tag{A57}$$

thus β_i^* is decreasing in λ_z for sufficiently large α. By (11), (A52), (A53), and (A55), it follows that

$$\frac{dW_i}{d\lambda_z} = -\frac{3(1-c_p)^2(\pi_a(1-c_p)(1+3c_p(1-\lambda_z)+3\lambda_z)-8c_p^2\pi_z)}{128\pi_a\pi_z^3\alpha^3}+O\left(\frac{1}{\alpha^4}\right), \tag{A58}$$

hence $dW_i/d\lambda_z<0$ for sufficiently large α because $c_p<1-\sqrt{\pi_z/(\pi_a+\pi_z)}$. By Lemma 1, Region II can also apply. From (A15), we obtain

$$v_b = 1-\frac{1-p}{(1-\beta)(\pi_a+\pi_z(1-\lambda_z))\alpha}+\frac{p(1-p)}{(1-\beta)^2(\pi_a+\pi_z(1-\lambda_z))^2\alpha^2}+O\left(\frac{1}{\alpha^3}\right). \tag{A59}$$

By (12) and (13), the unconstrained optimizing price for Region II satisfies

$$p_{ii}^*(\beta) = \frac{\pi_a+\pi_z(1+\lambda_z)}{2(\pi_a+\pi_z)}-\frac{2\pi_a^2+\pi_z\pi_z(4+3\lambda_z)+\pi_z^2(2+3\lambda_z(1+\lambda_z))}{16(1-\beta)(\pi_a+\pi_z)^3\alpha}+O\left(\frac{1}{\alpha^2}\right). \tag{A60}$$

Substituting (A59) and (A60) into (12) and differentiating, we obtain

$$\frac{d\Pi(p_{ii}^*(\beta),\beta)}{d\beta} = -C'(\beta)+\frac{1}{4(1-\beta)^2(\pi_a+\pi_z)\alpha}-\frac{2\pi_a+\pi_z(2+\lambda_z)}{8(1-\beta)^3(\pi_a+\pi_z)^3\alpha^2}+O\left(\frac{1}{\alpha^3}\right). \tag{A61}$$

Similar to the analysis from Region I, $C'(0)>0$ implies $\beta_i^*=0$ whereas $C'(0)=0$ implies

$$\beta_{ii}^* = \frac{1}{4(\pi_a+\pi_z)C''(0)\alpha}+\frac{A_5}{32(\pi_a+\pi_z)^3C''(0)^3\alpha^2}+O\left(\frac{1}{\alpha^3}\right), \tag{A62}$$

where $A_5 = 4C''(0)(\pi_a+\pi_z-C''(0)[2\pi_a+\pi_z(2+\lambda_z)]-C'''(0)^3(\pi_a+\pi_z))$. By (A60), $p_{ii}^*>\bar{p}$ is satisfied if and only if $c_p>\pi_a/(2(\pi_a+\pi_z))$ for sufficiently large α. In this case, by (12), we obtain

$$\Pi(p_{ii}^*(\beta_{ii}^*),\beta_{ii}^*) = -C(0)+\frac{1}{4(\pi_a+\pi_z)\alpha}+O\left(\frac{1}{\alpha^2}\right). \tag{A63}$$

By (A56) and (A63), $\Pi(p_i^*(\beta_i^*),\beta_i^*)>\Pi(p_{ii}^*(\beta_{ii}^*),\beta_{ii}^*)$ when $\pi_a/(2(\pi_a+\pi_z))<c_p<1-\sqrt{\pi_z/(\pi_a+\pi_z)}$, hence (A57) and (A58) apply for this range of c_p and for $c_p<\pi_a/(2(\pi_a+\pi_z))$ because $\Pi(\cdot)$ is continuous in price. Suppose $c_p>1-\sqrt{\pi_z/(\pi_a+\pi_z)}$. Then, by similar logic, Region II maximizes profits. Differentiating (A62), we obtain

$$\frac{d\beta_{ii}^*}{d\lambda_z} = -\frac{\pi_z}{8(\pi_a+\pi_z)^3C''(0)\alpha^2}+O\left(\frac{1}{\alpha^3}\right), \tag{A64}$$

thus β_i^* is decreasing in λ_z for sufficiently large α. By (11), (A59), (A60), and (A62), it follows that

$$\frac{dW_{ii}}{d\lambda_z} = -\frac{3\pi_z(\pi_a+\pi_z(1+3\lambda_z))}{128(\pi_a+\pi_z)^5\alpha^3}+O\left(\frac{1}{\alpha^4}\right), \tag{A65}$$

hence $dW_{ii}/d\lambda_z<0$, which completes the proof. □

Proof of Proposition 4. Technically, we will prove that, when $C''(0) > c_p^2/2$, there exist $\underline{\omega}, \bar{\eta}, \bar{\kappa} > 0$, such that if $\alpha > \underline{\omega}$, $\pi_z < \bar{\eta}$, and $c_p(1-c_p)/2 - \bar{\kappa} < C'(0)$, then

(i) for $C''(0) \le w_1$, if $c_p \le 1/3$, then $\lambda_p^* = 1$. Furthermore, β^* increases under the optimal liability policy, i.e., $\lambda_p^* = 1$, compared to investment under no liability. If $c_p > 1/3$, $\lambda_p^* = 0$;

(ii) for $w_1 < C''(0) \le w_2$, $\lambda_p^* = 0$;

(iii) for $C''(0) > w_2$, if $c_p \le 6 - \sqrt{33}$, then $\lambda_p^* = 0$; if $c_p > 6 - \sqrt{33}$, then $\lambda_p^* = \lambda_0 \in (0, 1/2)$. As $\alpha \to \infty$, $\pi_z \to 0$, and $C'(0) \to c_p(1-c_p)/2$,

$$\lambda_0 \to \frac{c_p(1+c_p)G(c_p) - 2H(c_p)C''(0)}{8c_p(4C''(0) - c_p(1+c_p))}. \tag{A66}$$

Furthermore, β^* decreases under the optimal liability policy, i.e., $\lambda_p^* = \lambda_0$, compared to investment under no liability.

Let $\pi_z = k/\alpha^2$. By definition, $\bar{p} \to 1-(1-\lambda_p)(1-\beta)c_p$ for sufficiently large α, hence Region I of Lemma 1 applies. By (A13) and (A14) under parameters $(1-\beta)\pi_a$, $(1-\beta)\pi_z$ and $(1-\lambda_p)(1-\beta)c_p$, we obtain

$$v_b = p + c_p(1-\beta)(1-\lambda_p) - \frac{A_1}{A_2\alpha} + O\left(\frac{1}{\alpha^2}\right), \tag{A67}$$

where $A_1 = (1-\beta)[c_p^2(1-\lambda_p)^2 + k\pi_a(p + c_p(1-\beta)(1-\lambda_p))^3(p + c_p(1-\beta)(1-\lambda_p) - 1)]$ and $A_2 = (p + c_p(1-\beta)(1-\lambda_p))^2\pi_a$. By (13), differentiating (12), and substituting in (A67), we obtain

$$\frac{d\Pi}{dp} = 1 - 2p - c_p(1-\beta)(2\lambda_p(\beta-1)) + O\left(\frac{1}{\alpha}\right), \tag{A68}$$

from which we derive

$$p^* = \frac{1-c_p(1-\beta+2\lambda_p(\beta-1))}{2} - \frac{(1-\beta)A_3}{8\alpha} + O\left(\frac{1}{\alpha^2}\right), \tag{A69}$$

where $A_3 = k(1-c_p(1-\beta))^2 + 16c_p^2(1-\lambda_p)[1+c_p(1-\beta)(4\lambda_p-3)]/(\pi_a(1+c_p(1-\beta))^3)$, hence $p^* < \bar{p}$ is satisfied for sufficiently large α. Using (A14), the first-order condition on price, the implicit function theorem, and subsequently substituting in (A67) and (A69), yields

$$\frac{d\Pi(p^*(\beta), \beta)}{d\beta} = \frac{c_p(1-c_p(1-\beta))}{2} - C'(\beta) + \frac{A_4}{8\pi_a(1+c_p(1-\beta))^3\alpha} + O\left(\frac{1}{\alpha^2}\right), \tag{A70}$$

where $A_4 = k\pi_a + c_p k\pi_a(1-\beta) + c_p^4 k\pi_a(1-\beta)^4 + 9c_p^5 k\pi_a(1-\beta)^5 + 4c_p^6 k\pi_a(1-\beta)^6 - 2c_p^2 M_1 + 2c_p^3(1-\beta)M_2$, $M_1 = 8(1-\lambda_p) + 3k\pi_a(1-\beta)^2$, and $M_2 = 24 - 8(7-4\lambda_p)\lambda_p - 5k\pi_a(1-\beta)^2$. By (11), (A67), and (A69), we obtain

$$W(\lambda_p, \beta^*(\lambda_p)) = \frac{3(1-c_p(1-\beta^*))^2}{8} - C(\beta^*) + \frac{A_5}{4\pi_a(1+c_p(1-\beta^*))^3\alpha} + O\left(\frac{1}{\alpha^2}\right), \tag{A71}$$

where A_5 is a constant. By (13), (A70), and because $C''(\cdot) > c_p^2/2$, it follows that

$$\beta^* = z^* + \frac{A_6}{A_7(c_p^2 - 2C''(z^*))\alpha} + O\left(\frac{1}{\alpha^2}\right), \tag{A72}$$

where $A_6 = (k/4)[1-c_p(1-z^*)][-1+c_p(-1-4c_p(1-z^*))(-1+z^*)]A_7 + 2c_p^2(1-\lambda_p)[-1-c_p(1-z^*)(4\lambda_p-3)]$, $A_7 = (-1-c_p(1-z^*))^3\pi_a$, and z^* satisfies (15). By (A70), we obtain

$$\frac{d\beta^*}{d\lambda_p} = \frac{4(c_p^2(1+c_p(1-z^*)(8\lambda_p-7)))}{(1+c_p(1-z^*))^3(2C''(z^*)-c_p^2)\pi_a\alpha} + O\left(\frac{1}{\alpha^2}\right). \tag{A73}$$

By (11) and (A73), it follows that

$$\frac{dW}{d\lambda_p} = \frac{A_8}{(1+c_p(1-z^*))^3(2C''(z^*)-c_p^2)\pi_a\alpha} + O\left(\frac{1}{\alpha^2}\right), \tag{A74}$$

where $A_8 = c_p^2(c_p(3 + A_9(3(8\lambda_p - 7) - A_9(8\lambda_p - 9 - A_9))) - 4(1 + A_9(8\lambda_p - 7))C'(z^*) - 2(1-z^*)(3 - A_9(12 - 16\lambda_p - A_9))C''(z^*))$ and $A_9 = c_p(1-z^*)$. Reorganizing (A74) by collecting the terms with respect to λ_p, we obtain that $dW/d\lambda_p > 0$ if and only if

$$c_p^2(1+\eta)G(\eta) - 2\eta H(\eta)C''(z^*) + 8\eta(c_p^2(1+\eta) - 4\eta C''(z^*))\lambda_p > 0, \tag{A75}$$

where $\eta = c_p(1-z^*)$ and G and H are as defined in the statement of the proposition. Now let $c_p(1-c_p)/2 - \kappa < C'(0) < c_p(1-c_p)/2$. For κ small enough, z^* converges to 0 and η converges to c_p. Therefore, for small enough κ, $dW/d\lambda_p > 0$ if and only if

$$\varrho(\lambda_p) \triangleq c_p(1+c_p)G(c_p) - 2H(c_p)C''(0) + 8(c_p^2(1+c_p) - 4c_pC''(0))\lambda_p > 0. \tag{A76}$$

Now $W|_{\lambda_p=1} \geq W|_{\lambda_p=0}$ if and only if

$$\int_0^1 \varrho(\lambda_p)\, d\lambda_p = c_p(c_p(1-c_p^2) - 2(3-c_p)C''(0)) > 0, \tag{A77}$$

or equivalently

$$C''(0) \leq \frac{c_p(1-c_p^2)}{2(3-c_p)}. \tag{A78}$$

Furthermore, on $[0,1]$, $G(c_p) > 0$ and $H(c_p) > 0$ if and only if $c_p < 3-2\sqrt{2}$ and $c_p < 6-\sqrt{33}$, respectively. Therefore, for $c_p \leq 6-\sqrt{33}$, $\varrho(0) > 0$ if and only if

$$C''(0) \leq \frac{c_p(1+c_p)G(c_p)}{2H(c_p)}, \tag{A79}$$

while for $c_p > 6-\sqrt{33}$, $\varrho(0) > 0$ if and only if (A79) does not hold. Comparing the right-hand sides of (A78) and (A79), and simplifying, we see that

$$\frac{c_p(1+c_p)G(c_p)}{2H(c_p)} > \frac{c_p(1-c_p^2)}{2(3-c_p)} \tag{A80}$$

if and only if $c_p > 6-\sqrt{33}$. Also note that ϱ is increasing in λ_p if and only if

$$C''(0) \leq \frac{c_p(1+c_p)}{4}, \tag{A81}$$

$$\frac{c_p(1+c_p)}{4} \geq \frac{c_p(1+c_p)G(c_p)}{2H(c_p)} \tag{A82}$$

if and only if $c_p \leq 6-\sqrt{33}$, and

$$\frac{c_p(1+c_p)}{4} \geq \frac{c_p(1-c_p^2)}{2(3-c_p)}, \tag{A83}$$

for all $c_p \in [0,1]$. Finally,

$$\frac{c_p(1-c_p^2)}{2(3-c_p)} > \frac{c_p^2}{2}, \tag{A84}$$

if and only if $c_p < 1/3$.

Given these observations, first for $c_p \leq 6-\sqrt{33}$, when $\varrho(0) > 0$, i.e., when (A79) is satisfied, (A81) will be satisfied. Therefore, if W is increasing at $\lambda_p = 0$, it is increasing on $\lambda_p \in [0,1]$, i.e., it cannot have an interior maximizer. Hence, when (A78) is satisfied, $\lambda_p^* = 1$, otherwise, $\lambda_p^* = 0$.

For $6-\sqrt{33} < c_p \leq 1/3$, when $\varrho(0) > 0$, i.e., when (A79) is not satisfied, (A81) is violated, and ϱ is decreasing. Furthermore, by (A76), $\varrho(1) < 0$ if and only if

$$C''(0) < \frac{c_p(1+c_p)^2}{2(3+c_p)}, \tag{A85}$$

and because

$$\frac{c_p(1+c_p)^2}{2(3+c_p)} < \frac{c_p(1+c_p)G(c_p)}{2H(c_p)} \tag{A86}$$

for $c_p > 6-\sqrt{33}$, when (A79) is not satisfied, $\varrho(1) < 0$ implying that expected welfare is maximized at a λ_p^* in the interior of $[0,1]$. Therefore, for $6-\sqrt{33} < c_p \leq 1/3$, $\lambda_p^* = 1$ if

$$\frac{c_p^2}{2} < C''(0) \leq \frac{c_p(1-c_p^2)}{2(3-c_p)}, \tag{A87}$$

$\lambda_p^* = 0$ if

$$\frac{c_p(1-c_p^2)}{2(3-c_p)} < C''(0) \leq \frac{c_p(1+c_p)G(c_p)}{2H(c_p)}, \tag{A88}$$

and λ_p^* is in the interior otherwise.

When $c_p > 1/3$, (A78) cannot be satisfied for $C''(0) > c_p^2/2$. Therefore, for

$$C''(0) \leq \frac{c_p(1+c_p)G(c_p)}{2H(c_p)}, \tag{A89}$$

$\lambda_p^* = 0$. For larger $C''(0)$ values, the same analysis as above is valid and an interior λ_p^* is optimal.

When λ_p^* is in the interior, its value in the limit as $\alpha \to \infty$ and $\kappa \to 0$ can be found by solving $\varrho(\lambda_p) = 0$, which yields (A66). Note that

$$\frac{d}{dC''(0)}\left(\frac{c_p(1+c_p)G(c_p) - 2H(c_p)C''(0)}{8c_p(4C''(0) - c_p(1+c_p))}\right) = -\frac{(1-c_p^2)(2-13c_p+c_p^2)}{4c_p(c_p^2+c_p-4C''(0))^2} > 0 \tag{A90}$$

for all $c_p > (13-\sqrt{161})/2$. Because $(13-\sqrt{161})/2 < 6-\sqrt{33}$, λ_0 is increasing for all $C''(0) > w_2$ when $c_p > 6-\sqrt{33}$. Now as $C''(0) \to \infty$, $\lambda_0 \to 12c_p - c_p^2 - 3/16c_p$, and $d/d(c_p)(12c_p - c_p^2 - 3/(16c_p)) = (3-c_p^2)/(16c_p) > 0$. That is, $\lim_{C''(0)\to\infty} \lambda_0$ is increasing in c_p. Furthermore, at $c_p = 1$, $\lim_{C''(0)\to\infty} \lambda_0 = 1/2$. It follows that $\lambda_0 < 1/2$.

Finally, by (A73), as $\alpha \to \infty$ and $\kappa \to 0$,

$$\frac{d\beta^*}{d\lambda_p} \longrightarrow 1 - 7c_p + 8c_p\lambda_p. \tag{A91}$$

As we have shown above, for $c_p < 1/3$ and $C''(0) < w_1$, $\lambda_p^* = 1$. For a given λ_p^*, the corresponding change in β^* is

$$\beta^*|_{\lambda_p=\lambda_p^*} - \beta^*|_{\lambda_p=0} = \int_0^{\lambda_p^*} (1 - 7c_p + 8c_p\lambda_p)\, d\lambda_p = \lambda_p^*(1 - 7c_p + 4c_p\lambda_p^*). \tag{A92}$$

Plugging in $\lambda_p^* = 1$, $\beta^*|_{\lambda_p=1} - \beta^*|_{\lambda_p=0} = 1 - 3c_p \geq 0$ for the domain where $\lambda_p^* = 1$. That is when $\lambda_p^* = 1$, β^* increases compared to the base case with no patch liability, i.e., $\beta^* = 0$. Now, by (A92), $\beta^*|_{\lambda_p=\lambda_0} - \beta^*|_{\lambda_p=0} \geq 0$ if and only if $\lambda_0 \geq (7c_p - 1)/(4c_p)$. However, $(7c_p - 1)/(4c_p) > 1/2$ if and only if $c_p > 1/5$, and as we have shown above for $c_p > 6 - \sqrt{33}$ and $C''(0) > w_2$, $\lambda_p^* = \lambda_0 < 1/2$. Because $6 - \sqrt{33} > 1/5$, it follows that in this region $\lambda_0 < 1/2 < (7c_p - 1)/(4c_p)$, and hence $\beta^*|_{\lambda_p=\lambda_0}$ is less than $\beta^*|_{\lambda_p=0}$; i.e., vendor investment decreases at the optimal patch liability policy. This completes the proof. □

Proof of Proposition 5. Technically, we will prove that there exist $\underline{\omega} > 0$, $\underline{\pi}_z \in (0,1)$, and $\bar{\lambda}_p \in (0,1)$ such that if $\alpha > \underline{\omega}$ and $\pi_z > \underline{\pi}_z$, then part (i) of Proposition 2 holds in the long run as well. Furthermore, there exists a $\lambda_p^o \in (0,1)$ such that β^* is weakly decreasing in λ_p for $\lambda_p \leq \lambda_p^o$, and strictly increasing in λ_p otherwise.

First, note that for $C'(0) > 0$, $\beta^* = 0$ and the analysis collapses to that of part (i) of Proposition 2. Now for $C'(0) = 0$, by definition,

$$\bar{p} = [1 - (1-\beta)(1-\lambda_p)c_p(1 + \pi_z/\pi_a)][1 - (1-\lambda_p)c_p/(\pi_a\alpha)], \tag{A93}$$

and, in equilibrium, one of three cases can occur: Region I with interior price, Region II with interior price, and a boundary outcome priced at $\bar{p}$. First, suppose Region I applies. Then, by (12) and (13), we obtain

$$p_i^* = \frac{1 - c_p(1-\beta-2\lambda_p(1-\beta))}{2} + \left(\frac{c_p^2(1-\beta)(1+\lambda_p(2\lambda_p-3))}{2\pi_a\alpha} - \frac{(1-c_p(1-\beta))^2}{8(1-\beta)\pi_z\alpha}\right) + O\left(\frac{1}{\alpha^2}\right) \tag{A94}$$

and

$$\beta^* = \frac{\pi_a(1-c_p^2) - 4c_p^2(1-\lambda_p)\lambda_p\pi_z}{4\pi_a\pi_z C''(0)\alpha} + O\left(\frac{1}{\alpha^2}\right). \tag{A95}$$

Substituting (A94) and (A95) into (12) yields

$$\Pi(p_i^*(\beta^*), \beta^*) = \left(\frac{c_p^2(1-\lambda_p)\lambda_p}{\pi_a} + \frac{(1-c_p)^2}{4\pi_z}\right)\frac{1}{\alpha} + O\left(\frac{1}{\alpha^2}\right), \tag{A96}$$

and similarly substituting into (11) gives

$$W(\lambda_p, \beta^*(\lambda_p)) = \left(\frac{c_p^2\lambda_p(1-\lambda_p)}{\pi_a} + \frac{(1-c_p)^2}{4\pi_z}\right)\frac{1}{\alpha} + O\left(\frac{1}{\alpha^2}\right). \tag{A97}$$

Differentiating (A95) and (A97) with respect to λ_p, we obtain

$$\frac{d\beta^*}{d\lambda_p} = \frac{c_p^2(2\lambda_p - 1)}{\pi_a C''(0)\alpha} + O\left(\frac{1}{\alpha^2}\right) \quad \text{and} \tag{A98}$$

$$\frac{dW}{d\lambda_p} = \frac{c_p^2(1-2\lambda_p)}{\pi_a\alpha} + O\left(\frac{1}{\alpha^2}\right), \tag{A99}$$

respectively. By (A99), it follows that

$$\lambda_p^* = \frac{1}{2} + O\left(\frac{1}{\alpha}\right), \tag{A100}$$

which upon substitution into (A97) yields

$$W(\lambda_p^*, \beta^*(\lambda_p^*)) = \frac{(1-c_p)^2\pi_a + c_p^2\pi_z}{4\pi_a\pi_z\alpha} + O\left(\frac{1}{\alpha^2}\right). \tag{A101}$$

If Region II applies, by (12), (13), and (A15), we obtain

$$p_{ii}^* = \frac{1}{2} - \frac{1}{8(1-\beta)(\pi_a+\pi_z)\alpha} + O\left(\frac{1}{\alpha^2}\right) \tag{A102}$$

and

$$\beta^* = \frac{1}{4(\pi_a+\pi_z)C''(0)\alpha} + O\left(\frac{1}{\alpha^2}\right). \tag{A103}$$

Substituting (A102) and (A103) into (12) yields

$$\Pi(p_{ii}^*(\beta^*), \beta^*) = \frac{1}{4(\pi_a+\pi_z)\alpha} + \frac{1-4C''(0)}{32(\pi_a+\pi_z)^2C''(0)\alpha^2} + O\left(\frac{1}{\alpha^3}\right), \tag{A104}$$

and similarly substituting into (11) gives

$$W(\lambda_p, \beta^*(\lambda_p)) = \frac{1}{4(\pi_a+\pi_z)\alpha} + \frac{1-2C''(0)}{32(\pi_a+\pi_z)^2C''(0)\alpha^2} + O\left(\frac{1}{\alpha^3}\right). \tag{A105}$$

Finally, suppose the vendor optimally prices at the boundary between Region I and II, i.e., $p = \bar{p}$ as given in (A93). By (12) and (13), we obtain

$$\beta^* = \frac{c_p^2(1-\lambda_p)^2(\pi_a+\pi_z)}{\pi_a^2C''(0)\alpha} + O\left(\frac{1}{\alpha^2}\right), \tag{A106}$$

and, hence,

$$\frac{d\beta^*}{d\lambda_p} = -\frac{2c_p^2(1-\lambda_p)(\pi_a+\pi_z)}{\pi_a^2C''(0)\alpha} + O\left(\frac{1}{\alpha^2}\right). \tag{A107}$$

Substituting (A106) into (A93) and both into (12) yields

$$\Pi(\bar{p}(\beta^*), \beta^*) = \frac{c_p(1-\lambda_p)[\pi_a(1-c_p(1-\lambda_p)) - c_p\pi_z(1-\lambda_p)]}{\pi_a^2\alpha} + O\left(\frac{1}{\alpha^2}\right). \tag{A108}$$

Making similar substitutions into (11) gives

$$W(\lambda_p, \beta^*(\lambda_p)) = \frac{c_p(1-\lambda_p)[\pi_a(1-c_p(1-\lambda_p)) - c_p\pi_z(1-\lambda_p)]}{\pi_a^2\alpha} + O\left(\frac{1}{\alpha^2}\right), \tag{A109}$$

and, hence,

$$\frac{dW}{d\lambda_p} = -\frac{c_p[\pi_a(1-2c_p(1-\lambda_p)) - 2c_p\pi_z(1-\lambda_p)]}{\pi_a^2\alpha} + O\left(\frac{1}{\alpha^2}\right). \tag{A110}$$

By (A110), we obtain

$$\lambda_p^* = \frac{2c_p(\pi_a+\pi_z) - \pi_a}{2c_p(\pi_a+\pi_z)} + O\left(\frac{1}{\alpha}\right), \tag{A111}$$

which upon substitution into (A109) yields

$$W(\lambda_p^*, \beta^*(\lambda_p^*)) = \frac{1}{4(\pi_a+\pi_z)\alpha} + O\left(\frac{1}{\alpha^2}\right). \tag{A112}$$

Given (A93)–(A112), the determination of λ_p^* proceeds with similar steps as given in the proof of Proposition 2. To see the behavior of investment β^* with the vendor's liability share λ_p, note from the proof of Proposition 2 that if $c_p < \pi_a/(\pi_a+\pi_z)$, then the optimal price is p_i^* and β is given by (A95), and by (A98) β^* is decreasing in λ_p if and only if $\lambda_p < 1/2$. For $c_p > \pi_a/(\pi_a+\pi_z)$, on the other hand, again similar to the proof of Proposition 2, $p^* = p_{ii}^*$ for $\lambda_p < 1 - \pi_a/(2c_p(\pi_a+\pi_z))$ and users are not patching for these λ_p values; $p^* = \bar{p}$, for $1 - \pi_a/(2c_p(\pi_a+\pi_z)) < \lambda_p < 1 - \pi_a(1-c_p)/(2c_p\pi_z)$; and $p^* = p_i^*$ for $\lambda_p > 1 - \pi_a(1-c_p)/(2c_p\pi_z)$. Hence, first, for $\lambda_p < 1 - \pi_a/(2c_p(\pi_a+\pi_z))$, β^* is independent of λ_p. Second, for $1 - \pi_a/(2c_p(\pi_a+\pi_z)) < \lambda_p < 1 - \pi_a(1-c_p)/(2c_p\pi_z)$, by (A107) β^* is decreasing in λ_p. Finally, when $\lambda_p > 1 - \pi_a(1-c_p)/(2c_p\pi_z)$, because $c_p > \pi_a/(\pi_a+\pi_z)$, we have $\lambda_p > 1/2$, and hence, by (A98), it follows that β^* is increasing in λ_p. Therefore, we conclude that β^* is weakly decreasing in λ_p up to a certain λ_p value and increasing beyond that threshold. This completes the proof. □

Proof of Proposition 6. Technically, we will prove that there exist $\underline{\omega}$, $\underline{\eta}$, and $\bar{\eta}$ such that when $\alpha > \underline{\omega}$,

(i) if $\pi_z < \bar{\eta}$, then $\beta_s^* \to \xi^*$ as $\alpha \to \infty$. Furthermore, there exist $0 < \underline{c} < \bar{c} < 1$ such that (a) if $c_p < \underline{c}$, then β_s^* is increasing in c_p; and (b) if $c_p > \bar{c}$, then β_s^* is decreasing in c_p if and only if $\xi^* < 1/2$.

(ii) if $\pi_z > \underline{\eta}$, then $\beta_s^* \to 0$ as $\alpha \to \infty$. Furthermore, (a) if $C'(0) = 0$ and $c_p < 1 - \sqrt{\pi_z/(\pi_a + \pi_z)}$, then β_s^* is decreasing in c_p; (b) otherwise β_s^* is constant in c_p.

For part (i), following closely to the proof of Proposition 3, we obtain

$$v_b = p + c_p(1-\beta_s) - \frac{(1-\beta_s)[c_p^2 + k\pi_a(-1+p+c_p(1-\beta_s))(p+c_p(1-\beta_s))^3]}{(p+c_p(1-\beta_s))^2\pi_a\alpha} + O\left(\frac{1}{\alpha^2}\right) \quad \text{(A113)}$$

and

$$p^* = \frac{1-c_p(1-\beta_s)}{2} - \frac{(1-\beta_s)A_1}{8\alpha} + O\left(\frac{1}{\alpha^2}\right), \quad \text{(A114)}$$

where $A_1 = k(1-c_p(1-\beta_s))^2 + 16c_p^2(1-3c_p(1-\beta_s))/(\pi_a(1+c_p(1-\beta_s))^3)$, hence $p^* < \bar{p}$ is satisfied. By (14) and substituting in (A113) and (A114), we obtain

$$W(\beta_s) = \frac{3(1-c_p(1-\beta_s))^2}{8} - C(\beta_s) + O\left(\frac{1}{\alpha}\right), \quad \text{(A115)}$$

which upon differentiation yields

$$\frac{dW}{d\beta_s} = \frac{3c_p(1-c_p(1-\beta_s))}{4} - C'(\beta_s) - \frac{A_2}{4\pi_a(1+c_p(1-\beta_s))^4\alpha} + O\left(\frac{1}{\alpha^2}\right), \quad \text{(A116)}$$

where A_2 is a constant; hence, we obtain $\beta_s^* = \xi^* + O(1/\alpha)$. By (21) and (A115), for sufficiently large c_p, we obtain

$$\frac{d\beta_s^*}{dc_p} = \frac{3-6\xi^*}{(3-4C''(\xi^*))\alpha} + O\left(\frac{1}{\alpha^2}\right), \quad \text{(A117)}$$

hence $d\beta_s^*/dc_p < 0$ if and only if $\xi^* < 1/2$. Similarly, for sufficiently small c_p, we obtain

$$\frac{d\beta_s^*}{dc_p} = \frac{3}{4C''(\xi^*)\alpha} + O\left(\frac{1}{\alpha^2}\right), \quad \text{(A118)}$$

hence $d\beta_s^*/dc_p > 0$.

For part (ii), if $C'(0) > 0$ it is easy to show that $\beta^* = 0$. Now suppose $C'(0) = 0$. When Region I applies, we obtain

$$v_b = 1 - \frac{1-p-c_p(1-\beta_s)}{(1-\beta_s)\pi_z\alpha} + O\left(\frac{1}{\alpha^2}\right), \quad \text{(A119)}$$

and, by (19), it follows that

$$p_i^* = \frac{1-c_p(1-\beta_s)}{2} + \left(\frac{c_p^2(1-\beta_s)}{2\pi_a\alpha} + \frac{(1-c_p(1-\beta_s))^2}{8(1-\beta_s)\pi_z\alpha}\right) + O\left(\frac{1}{\alpha^2}\right), \quad \text{(A120)}$$

and, which upon substitution into the profit function yields

$$\Pi(p_i^*(\beta_s), \beta_s) = -C(\beta_s) + \frac{(1-c_p(1-\beta_s))^2}{4\pi_z(1-\beta_s)\alpha} + O\left(\frac{1}{\alpha^2}\right). \quad \text{(A121)}$$

Furthermore, $p_i^* < \bar{p}$ if and only if $c_p(1-\beta_s) < \pi_a/(\pi_a + 2\pi_z)$ for sufficiently large α.

When Region II applies, we obtain

$$v_b = 1 - \frac{1-p}{(1-\beta)(\pi_a+\pi_z)\alpha} + O\left(\frac{1}{\alpha^2}\right) \quad \text{(A122)}$$

and

$$p_{ii}^* = \frac{1}{2} - \frac{1}{8(1-\beta)(\pi_a+\pi_z)\alpha} + O\left(\frac{1}{\alpha^2}\right), \quad \text{(A123)}$$

where $p_{ii}^* \geq \bar{p}$ if and only if $c_p(1-\beta_s) > \pi_a/(2(\pi_a+\pi_z))$ for sufficiently large α. Plugging (A123) into the profit function yields

$$\Pi(p_{ii}^*(\beta_s), \beta_s) = -C(\beta_s) + \frac{1}{4(1-\beta_s)(\pi_a+\pi_z)\alpha} + O\left(\frac{1}{\alpha^2}\right). \quad \text{(A124)}$$

Thus, if $c_p(1-\beta_s) \leq \pi_a/(2(\pi_a+\pi_z))$, the vendor prices in Region I; if $c_p(1-\beta_s) \geq \pi_a/(\pi_a+2\pi_z)$, the vendor prices in Region II; and, in between, the outcome is determined by comparing (A121) and (A124), in which case, $\Pi(p_i^*(\beta_s), \beta_s) > \Pi(p_{ii}^*(\beta_s), \beta_s)$ if and only if $c_p(1-\beta_s) < 1 - \sqrt{\pi_z/(\pi_a+\pi_z)}$. Since $\pi_a/(2(\pi_a+\pi_z)) < 1 - \sqrt{\pi_z/(\pi_a+\pi_z)} < \pi_a/(\pi_a+2\pi_z)$, it follows that (A121) applies if $c_p(1-\beta_s) < 1 - \sqrt{\pi_z/(\pi_a+\pi_z)}$, and (A124) applies if $c_p(1-\beta_s) \geq 1 - \sqrt{\pi_z/(\pi_a+\pi_z)}$.

For Region I, substituting (A119) and (A120) into $W(\beta_s)$ and optimizing gives

$$\beta_s^* = \frac{1-c_p^2}{4\pi_z C''(0)\alpha} + O\left(\frac{1}{\alpha^2}\right), \tag{A125}$$

and, hence,

$$W(\beta_s^*) = \frac{(1-c_p)^2}{4\pi_z \alpha} + O\left(\frac{1}{\alpha^2}\right). \tag{A126}$$

Similarly, for Region II, we obtain

$$\beta_s^* = \frac{1}{4(\pi_a+\pi_z)C''(0)\alpha} + O\left(\frac{1}{\alpha^2}\right) \tag{A127}$$

and

$$W(\beta_s^*) = \frac{1}{4(\pi_a+\pi_z)\alpha} + O\left(\frac{1}{\alpha^2}\right). \tag{A128}$$

By (A126) and (A128) and because, by (A125) and (A127), $\beta_s^* = O(1/\alpha)$, it follows that the welfare-maximizing outcome lies in Region I if $c_p < 1-\sqrt{\pi_z/(\pi_a+\pi_z)}$ and in Region II otherwise. By (A125), β_s^* is decreasing in c_p in Region I and clearly constant in c_p otherwise. This completes the proof. □

Proof of Proposition 7. Technically, we will prove that there exist $\underline{\omega} > 0$, $\bar{\eta} > 0$ such that if $\alpha > \underline{\omega}$ and $\pi_z < \bar{\eta}$, then

(i) $\beta_s^* > \max\{\beta^*(\lambda_z^*), \beta^*(\lambda_p^*)\}$; and
(ii) $W(\beta_s^*) > \max(W(\lambda_z^*), W(\lambda_p^*))$.

For part (i), let $\pi_z = k/\alpha^2$. As $\alpha \to \infty$, by Proposition 3, $\beta^*(\lambda_z^*)$ converges to z^*, which satisfies (15). Similarly, by Proposition 4, under a patching liability policy, $\beta^*(\lambda_p^*)$ also converges to z^*. Under a security standards policy, by part (i) of Proposition 6, the optimal investment satisfies $\beta_s^* = \xi^* + 1/\alpha$, where ξ^* satisfies (21). We define the function

$$\psi(x, \psi_0) \triangleq \psi_0 c_p(1-c_p(1-x)) - 2C'(x), \tag{A129}$$

where ψ_0 is a constant satisfying $\psi_0 > 0$. By (A129), there exists $K > 0$ such that if $C'(0) < K$, then, by the assumptions on $C(\cdot)$ and its derivatives, $\psi(x, \psi_0) = 0$ has a unique solution. Defining

$$\tilde{x}(\psi_0) \triangleq \{x\colon \psi(x, \psi_0) = 0\}, \tag{A130}$$

it follows that $d\tilde{x}/d\psi_0 > 0$, and, by comparing (15) and (21), it follows that $\xi^* > z^*$. Therefore, we conclude $\beta_s^* > \max\{\beta^*(\lambda_z^*), \beta^*(\lambda_p^*)\}$ for sufficiently large α.

Referring to the proofs of Propositions 3 and 4, by (A46) and (A71), under both loss and patching liability policies, we obtain

$$W(\lambda_\tau^*) = \frac{3(1-c_p(1-\beta^*))^2}{8} - C(\beta^*) + O\left(\frac{1}{\alpha}\right), \tag{A131}$$

where $\tau \in \{p, z\}$. Comparing (A115) and (A131), noting that $\beta_s^* = \xi^* + O(1/\alpha)$ is the unique maximizer of (A115), and because $\xi^* > z^*$, we conclude $W(\beta_s^*) > \max(W(\lambda_z^*), W(\lambda_p^*))$. □

Proof of Proposition 8. For part (i), by Proposition 3, $\lambda_z^* = 0$. Thus, by (14) and (20), it follows that $W(\lambda_z^*) \leq \min(W(\lambda_p^*), W(\beta_s^*))$ because $\lambda_p = 0$ and $\beta_s = \beta(\lambda_z^*)$ can replicate the outcome of the loss liability policy. For part (ii), suppose $\pi_z \leq \pi_a(1-c_p)/c_p$. Then, by Proposition 5, the optimal liability share is given by (A100), and the outcome is in Region I. Substituting (A100) into (A95), we obtain

$$\beta^*(\lambda_p^*) = \frac{\pi_a(1-c_p^2) - c_p^2\pi_z}{4\pi_a\pi_z C''(0)\alpha} + O\left(\frac{1}{\alpha^2}\right). \tag{A132}$$

Further, suppose $\pi_z < \pi_a(1-c_p)^2/(c_p(2-c_p))$. By the proof of Proposition 6, under this condition, the welfare-maximizing investment induces Region I in equilibrium and is given by (A125). Comparing (A125) and (A132), it follows that $\beta_s^* > \beta^*(\lambda_p^*)$ is always satisfied in this range of parameters. Suppose instead that $\pi_a(1-c_p)^2/(c_p(2-c_p)) < \pi_z < \pi_a(1-c_p)/c_p$ is satisfied. By the proof of Proposition 6, β_s^* satisfies (A127). However $\pi_z < \pi_a(1-c_p)/c_p$ implies that $1/(4(\pi_a+\pi_z)C''(0)) < (\pi_a(1-c_p^2) - c_p^2\pi_z)/(4\pi_a\pi_z C''(0))$, hence $\beta_s^* < \beta^*(\lambda_p^*)$, which proves part (a) of (ii).

For part (b), if $\pi_z < \pi_a(1-c_p)^2/(c_p(2-c_p))$, then, by Proposition 6, $W(\beta_s^*)$ satisfies (A126). By Proposition 5, $W(\lambda_p^*)$ satisfies (A101). Comparing (A101) and (A126), $W(\lambda_p^*) > W(\beta_s^*)$ is always satisfied. However, if $\pi_a(1-c_p)^2/(c_p(2-c_p)) < \pi_z < \pi_a(1-c_p)/c_p$, then, by the proof of Proposition 6, $W(\beta_s^*)$ satisfies (A128). In this case, comparing (A101) and (A128), it again follows that $W(\lambda_p^*) > W(\beta_s^*)$. Finally, if $\pi_z > \pi_a(1-c_p)/c_p$, then $\lambda_p^* = 0$, which is established in the last part of the proof of Proposition 5. Therefore, $W(\lambda_p^*) \leq W(\beta_s^*)$ by similar reasoning as in part (i). This completes the proof. □

Proof of Proposition 9. For part (i) of the proposition, technically, we will prove that there exist $\underline{\omega}$, $\bar{\eta} > 0$ such that if $\alpha > \underline{\omega}$ and $\pi_z < \bar{\eta}$, then

(a) if $c_p > 6 - \sqrt{33}$ and $C'(1-(6-\sqrt{33})/c_p) > 3c_p(\sqrt{33}-5)/4$, then there exists $M > 0$ such that $W(\mathbf{\Lambda}^*) \le W(\beta_s^*) + M/\alpha$; and

(b) otherwise, $W(\mathbf{\Lambda}^*) = W(\beta_s^*)$.

Let $\pi_z = k/\alpha^2$. Under the coupled policy $\mathbf{\Lambda}$, by definition, $\bar{p} \to 1-(1-\lambda_p^c)(1-\beta_s^c)c_p$ for sufficiently large α, hence Region I of Lemma 1 applies. By (A13) and (A14), under parameters $(1-\beta_s^c)\pi_a$, $(1-\lambda_z^c)(1-\beta_s^c)\pi_z$ and $(1-\lambda_p^c)(1-\beta_s^c)c_p$, we obtain

$$v_b = p + c_p(1-\beta_s^c)(1-\lambda_p^c) - \frac{A_1}{A_2\alpha} + O\left(\frac{1}{\alpha^2}\right), \tag{A133}$$

where

$$A_1 = (1-\beta_s^c)(c_p^2(1-\lambda_p^c)^2 + k\pi_a(1-\lambda_z^c)[p+c_p(1-\beta_s^c)(1-\lambda_p^c)]^3[p+c_p(1-\beta_s^c)(1-\lambda_p^c)-1]) \quad \text{and}$$
$$A_2 = (p+c_p(1-\beta_s^c)(1-\lambda_p^c))^2\pi_a.$$

By (23), differentiating (22), and substituting in (A133), we obtain

$$p^* = \frac{1-c_p(1-\beta_s^c+2\lambda_p^c(\beta_s^c-1))}{2} - \frac{(1-\beta_s^c)A_3}{16\alpha} + O\left(\frac{1}{\alpha^2}\right), \tag{A134}$$

where $A_3 = k(1-c_p(1-\beta_s^c))[2-7\lambda_z^c - c_p(1-\beta_s^c)(2+\lambda_z^c)] + 32c_p^2(1-\lambda_p^c)[1+c_p(1-\beta_s^c)(4\lambda_p^c-3)]/(\pi_a(1+c_p(1-\beta_s^c))^3)$, hence $p^* < \bar{p}$ is satisfied for sufficiently large α. Substituting (A133) and (A134) into (11), we obtain

$$W(\mathbf{\Lambda}) = \frac{3(1-c_p(1-\beta_s^c))^2}{8} - C(\beta_s^c) + \frac{A_4}{32\pi_a(1+c_p(1-\beta_s^c))^3\alpha} + O\left(\frac{1}{\alpha^2}\right), \tag{A135}$$

where A_4 is a constant, and which upon differentiation yields

$$\frac{\partial W(\mathbf{\Lambda})}{\partial \lambda_z^c} = -\frac{3k[1-c_p(1-\beta_s^c)]^3(1-\beta_s^c)}{32\alpha} + O\left(\frac{1}{\alpha^2}\right), \tag{A136}$$

hence $\lambda_z^c = 0$ for sufficiently large α. Similarly, differentiating $W(\mathbf{\Lambda})$ with respect to λ_p^c, then

$$\lambda_p^c = \frac{c_p(1-\beta_s^c)(12-c_p(1-\beta_s^c))-3}{16c_p(1-\beta_s^c)} + O\left(\frac{1}{\alpha}\right), \tag{A137}$$

satisfies the first-order condition. Now suppose that $c_p > 6-\sqrt{33}$ is satisfied such that, by (A137), $\lambda_p^c > 0$ can be satisfied for some $\beta_s^c \in [0,1]$. Further, suppose that $C'(1-(6-\sqrt{33})/c_p) > 3c_p(\sqrt{33}-5)/4$ is satisfied as well. Differentiating $W(\mathbf{\Lambda})$ with respect to β_s^c and substituting in (A137), we obtain

$$\frac{\partial W(\mathbf{\Lambda})}{\partial \beta_s^c} = -\frac{3c_p(1-c_p(1-\beta_s^c))}{4} - C'(\beta_s^c) + \frac{A_5}{32(1+c_p(1-\beta_s^c))^2\pi_a\alpha} + O\left(\frac{1}{\alpha^2}\right), \tag{A138}$$

where $A_5 = (1-c_p(1-\beta_s^c))\{c_p^2(3+c_p(1-\beta_s^c)) - 8k[-1-c_p(1+4c_p(1-\beta_s^c))(-1+\beta_s^c)](1+c_p(1-\beta_s^c))^2\pi_a\}$. Thus, by (21) and (A138), β_s^c satisfies

$$\beta_s^c = \xi^* + \frac{(-1+c_p(1-\xi^*))A_6}{8\pi_a(1+c_p(1-\xi^*))^2(3c_p^2-4C''(\xi^*))\alpha} + O\left(\frac{1}{\alpha^2}\right), \tag{A139}$$

where $A_6 = 3c_p^2 + c_p^3(1-\xi^*) - 8k[-1+c_p(-1+4c_p(-1+\xi^*))(-1+\xi^*)](1+c_p(1-\xi^*))^2\pi_a$. By (21), (A139), and because $C'(1-(6-\sqrt{33})/c_p) > 3c_p(\sqrt{33}-5)/4$, it follows that $\beta_s^c < 1-(6-\sqrt{33})/c_p$, which, by (A137), implies that $\lambda_p^c > 0$. Thus, substituting (A137) and (A139) into $W(\mathbf{\Lambda})$, we obtain

$$W(\mathbf{\Lambda}^*) = \frac{3(1-c_p(1-\xi^*))^2}{8} - C(\xi^*) + O\left(\frac{1}{\alpha}\right). \tag{A140}$$

Under the conditions of this proposition, part (ii) of Proposition 7 applies. Thus, under a security standards policy only, by part (i) of Proposition 6, (A116), and substituting β_s^* into (A115), we obtain

$$W(\beta_s^*) = \frac{3(1-c_p(1-\xi^*))^2}{8} - C(\xi^*) + O\left(\frac{1}{\alpha}\right). \tag{A141}$$

Comparing (A140) and (A141), the result in part (a) of this proposition follows. On the other hand, if either $c_p \le 6-\sqrt{33}$ or $C'(1-(6-\sqrt{33})/c_p) \le 3c_p(\sqrt{33}-5)/4$ is satisfied, then, by Lemma 1, (21), (A137), and (A139), it follows that $\lambda_p^c = 0$ and part (i) of Proposition 6 again applies. This proves part (b) of the proposition.

For part (ii) of the proposition, technically, we will prove that there exist $\underline{\omega}$, $\underline{\eta} > 0$ such that if $\alpha > \underline{\omega}$, then

(a) if $\underline{\eta} < \pi_z < \pi_a(1-c_p)/c_p$, then there exists $M > 0$ such that $W(\mathbf{\Lambda}^*) \le \max(W(\lambda_p^*), W(\beta_s^*)) + M/\alpha^2$; and

(b) if $\overline{\pi}_z \ge \pi_a(1-c_p)/c_p$, then we have $W(\mathbf{\Lambda}^*) = \max(W(\lambda_p^*), W(\beta_s^*))$.

Under the coupled policy $\mathbf{\Lambda}$, by definition, $\bar{p} \to 1-(1-\lambda_p^c)(1-\beta_s^c)c_p(\pi_a+\pi_z(1-\lambda_z^c))/\pi_a$ for sufficiently large α. Suppose Region I of Lemma 1 applies, which implies that $\beta > w_1 \triangleq 1-\pi_a/(c_p(1-\lambda_p^c)(\pi_a+(1-\lambda_z^c)\pi_z))$ must be satisfied for sufficiently large α. By (A13) and (A14), under parameters $(1-\beta_s^c)\pi_a$, $(1-\lambda_z^c)(1-\beta_s^c)\pi_z$, and $(1-\lambda_p^c)(1-\beta_s^c)c_p$, we obtain

$$v_b = 1 - \frac{1-p-c_p(1-\beta_s^c)(1-\lambda_p^c)}{(1-\beta_s^c)(1-\lambda_z^c)\pi_z\alpha} + O\left(\frac{1}{\alpha^2}\right). \tag{A142}$$

By (23), differentiating (22), and substituting in (A142), we obtain

$$p_i^* = \frac{1-c_p(1-\beta_s^c)(1-2\lambda_p^c+\lambda_z^c)+\lambda_z^c}{2} - \frac{A_1}{16(1-\beta_s^c)\pi_a\pi_z\alpha} + O\left(\frac{1}{\alpha^2}\right), \tag{A143}$$

where $A_1 = (1-c_p(1-\beta_s^c))(2+3\lambda_z^c(1+\lambda_z^c)+c_p(1-\beta_s^c)(1-\lambda_z^c)(3\lambda_z^c-2))\pi_a + 8c_p^2(1-\beta_s^c)^2(-1+\lambda_p^c)(1-2\lambda_p^c+\lambda_z^c)\pi_z$. Substituting (A142) and (A143) into (22) yields

$$\Pi(p_i^*(\mathbf{\Lambda}), \mathbf{\Lambda}) = -C(\beta_s^c) + \frac{1}{\alpha}\left(\frac{c_p^2(1-\beta_s^c)(1-\lambda_p^c)\lambda_p^c}{\pi_a} + \frac{(1-c_p(1-\beta_s^c))^2}{4\pi_z(1-\beta_s^c)}\right) + O\left(\frac{1}{\alpha^2}\right), \tag{A144}$$

noting that, for sufficiently large α, $p_i^* < \bar{p}$ if and only if $\beta_s^c > w_2 \triangleq 1-\pi_a/(c_p\pi_a + 2c_p\pi_z(1-\lambda_p^c))$.

Suppose Region II of Lemma 1 applies. By (A15), we obtain

$$v_b = 1 - \frac{1-p}{(1-\beta_s^c)(\pi_a+\pi_z(1-\lambda_z^c))\alpha} + O\left(\frac{1}{\alpha^2}\right). \tag{A145}$$

By (23), differentiating (22), and substituting in (A145), we obtain

$$p_{ii}^* = \frac{\pi_a+\pi_z(1+\lambda_z^c)}{2(\pi_a+\pi_z)} - \frac{2\pi_a^2+\pi_a\pi_z(4+3\lambda_z^c)+\pi_z^2(2+3\lambda_z^c(1+\lambda_z^c))}{16(1-\beta_s^c)(\pi_a+\pi_z)^3\alpha} + O\left(\frac{1}{\alpha^2}\right). \tag{A146}$$

Substituting (A145) and (A146) into (22) yields

$$\Pi(p_{ii}^*(\mathbf{\Lambda}), \mathbf{\Lambda}) = -C(\beta_s) + \frac{1}{4(\pi_a+\pi_z)(1-\beta_s^c)\alpha} + O\left(\frac{1}{\alpha^2}\right), \tag{A147}$$

noting that, for sufficiently large α, $p_{ii}^* \ge \bar{p}$ if and only if $\beta_s^c \le w_3 \triangleq 1-\pi_a/(2c_p(1-\lambda_p^c)(\pi_a+\pi_z))$.

Suppose $\lambda_p^c \le 1/2$. Then, $w_1 \le w_2 \le w_3$; hence, as long as $\beta_s^c \le w_2$ is satisfied, we obtain $p^* = p_{ii}^*$ with the optimal profit given by (A147). Similarly, if $\beta_s^c > w_3$, then $p^* = p_i^*$ because only p_i^* is interior, and the optimal profit is given by (A144). When $w_2 < \beta_s^c \le w_3$, both region unconstrained prices are feasible, and by (A144) and (A147), $p^* = p_i^*$ if and only if

$$g(\beta_s^c) \triangleq \pi_a(\pi_a+\pi_z)(1-c_p(1-\beta_s^c))^2 + 4\pi_z(\pi_a+\pi_z)c_p^2\lambda_p^c(1-\lambda_p^c)(1-\beta_s^c)^2 - \pi_a\pi_z \ge 0, \tag{A148}$$

where g is minimized at $\beta_s^c = w_4 \triangleq 1-\pi_a/[c_p(\pi_a+4\pi_z\lambda_p^c(1-\lambda_p^c))]$. Defining $\beta_g \triangleq \sup\{\beta_s^c\colon g(\beta_s^c)=0\}$, it follows that, under these conditions, $w_2 \le \beta_g \le w_3$ is satisfied, where β_g satisfies

$$\begin{aligned}\beta_g &= \frac{1}{c_p^2(\pi_a+\pi_z)(\pi_a+4\lambda_p^c\pi_z(1-\lambda_p^c))}\\ &\quad\cdot\left\{c_p(\pi_a+\pi_z)[4c_p\pi_z\lambda_p^c(1-\lambda_p^c)-\pi_a(1-c_p)] - \sqrt{\pi_a^2\pi_z(\pi_a+\pi_z)c_p^2(1-2\lambda_p^c)^2}\right\}.\end{aligned} \tag{A149}$$

Because $w_4 \le w_2$, we can conclude that $p^* = p_{ii}^*$ for $\beta_s^c \le \beta_g$ and $p^* = p_i^*$ for $\beta_s^c > \beta_g$. Suppose $\beta_g > 0$. If $\beta_s^c \le \beta_g$, substituting (A145) and (A146) into (11) yields

$$W_{ii}(\mathbf{\Lambda}) = -C(\beta_s^c) + \frac{1}{4(\pi_a+\pi_z)(1-\beta_s^c)\alpha} - \frac{1}{16(\pi_a+\pi_z)^2(1-\beta_s^c)^2\alpha^2} - \frac{3\lambda_z^c\pi_z(2\pi_a+\pi_z(2+3\lambda_z^c))}{256(\pi_a+\pi_z)^5(1-\beta)^3\alpha^3} + O\left(\frac{1}{\alpha^4}\right), \tag{A150}$$

from which it follows that $\lambda_z^c = 0$. Further, by (A150), it follows that $\beta_s^c = 0$ when $C'(0) > 0$ and

$$\beta_s^c = \frac{1}{4(\pi_a+\pi_z)C''(0)\alpha} + \frac{1-C''(0)}{8(\pi_a+\pi_z)^2C''(0)^2\alpha^2} + O\left(\frac{1}{\alpha^3}\right), \tag{A151}$$

when $C'(0)=0$. Thus, $\beta_s^c \le \beta_g$ is indeed satisfied for sufficiently large α. Substituting (A151) into (A150), we obtain

$$W_{ii}(\mathbf{\Lambda}) = \frac{1}{4(\pi_a+\pi_z)\alpha} + \frac{1-2C''(0)}{32(\pi_a+\pi_z)^2 C''(0)\alpha^2} + O\left(\frac{1}{\alpha^3}\right). \tag{A152}$$

On the other hand, if $\beta_s^c > \beta_g$, then substituting (A142) and (A143) into (11) yields

$$W_i(\mathbf{\Lambda}) = -C(\beta_s^c) + \frac{1}{\alpha}\left(\frac{c_p^2(1-\beta_s^c)(1-\lambda_p^c)\lambda_p^c}{\pi_a} + \frac{(1-c_p(1-\beta_s^c))^2}{4\pi_z(1-\beta_s^c)}\right) + O\left(\frac{1}{\alpha^2}\right). \tag{A153}$$

Because $W_i(\mathbf{\Lambda} \mid \beta_s^c > \beta_g) < 0$, by (A152) and (A153), the optimal welfare when $\beta_g > 0$ is given by $W_{ii}(\mathbf{\Lambda})$. Suppose $\beta_g \le 0$ such that, by (A153), we obtain $\beta_s^c = 0$ when $C'(0) > 0$ and

$$\beta_s^c = \frac{\pi_a(1-c_p^2) - 4\pi_z\lambda_p^c c_p^2(1-\lambda_p^c)}{4\pi_a\pi_z C''(0)\alpha} + O\left(\frac{1}{\alpha^2}\right), \tag{A154}$$

when $C'(0)=0$. Upon substitution of (A154) into (A153) yields

$$W_i(\mathbf{\Lambda}) = \frac{1}{\alpha}\left(\frac{c_p^2\lambda_p^c(1-\lambda_p^c)}{\pi_a} + \frac{(1-c_p)^2}{4\pi_z}\right) + O\left(\frac{1}{\alpha^2}\right). \tag{A155}$$

Because, by (A149), β_g is decreasing in λ_p^c when $\lambda_p^c \le 1/2$, for $\beta_g < 0$ to be satisfied, a necessary condition is $c_p < \pi_a/(\pi_a+\pi_z)$ under sufficiently large α. By (A153), we obtain

$$\lambda_p^c = \frac{1}{2} - \frac{c_p}{16\pi_a\alpha} + O\left(\frac{1}{\alpha^2}\right). \tag{A156}$$

Substituting (A156) into (A155), we obtain

$$W_i(\mathbf{\Lambda}) = \frac{\pi_a(1-c_p)^2 + \pi_z c_p^2}{4\pi_a\pi_z\alpha} + O\left(\frac{1}{\alpha^2}\right). \tag{A157}$$

Comparing (A152) and (A157), we obtain $W_i \ge W_{ii}$, hence $W(\mathbf{\Lambda}^*) = W_i$ for $c_p < \pi_a/(\pi_a+\pi_z)$ and $W(\mathbf{\Lambda}^*) = W_{ii}$ for $c_p \ge \pi_a/(\pi_a+\pi_z)$. Suppose instead that $\lambda_p^c > 1/2$ such that $w_1 \le w_3 \le w_2$. In this case, $p^* = p_{ii}^*$ for $\beta_s^c \le w_3$, $p^* = \bar{p}$ for $w_3 < \beta_s^c \le w_2$, and $p^* = p_i^*$ for $\beta_s^c > w_2$. Because the maximal value of welfare for both Region I and II can already be achieved under $\lambda_p^c \le 1/2$, only the case where $p^* = \bar{p}$ remains. It is straightforward to show that the optimal welfare in this boundary region is bounded above by (A152) and (A157), which we omit for brevity. By Proposition 5, part (ii)(b) of Proposition 8, and comparing (A101) and (A157), part (a) of the proposition follows. For part (b) again, by part (ii)(b) of Proposition 8, the result is immediate because $W(\mathbf{\Lambda}^*) = W_{ii}$. This completes the proof. □