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Dynamic derivative strategies☆

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Abstract

We study optimal investment strategies given investor access not only to bond and stock markets but also to the derivatives market. The problem is solved in closed form. Derivatives extend the risk and return tradeoffs associated with stochastic volatility and price jumps. As a means of exposure to volatility risk, derivatives enable non-myopic investors to exploit the time-varying opportunity set; and as a means of exposure to jump risk, they enable investors to disentangle the simultaneous exposure to diffusive and jump risks in the stock market. Calibrating to the S&P 500 index and options markets, we find sizable portfolio improvement from derivatives investing.

JEL classification: G11; G12

Keywords: Asset allocation; Portfolio selection; Derivatives; Stochastic volatility; Jumps

1. Introduction

"Derivatives trading is now the world's biggest business, with an estimated daily turnover of over US$2.5 trillion and an annual growth rate of around 14%."[1] Yet

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despite increasing usage and growing interest, little is known about optimal trading strategies incorporating derivatives. In particular, academic studies on dynamic asset allocation typically exclude derivatives from the investment portfolio. In a complete market setting, of course, such an exclusion can very well be justified by the fact that derivative securities are redundant (e.g., Black and Scholes, 1973; Cox and Ross, 1976). When the completeness of the market breaks down, however—either because of infrequent trading or the presence of additional sources of uncertainty—it then becomes suboptimal to exclude derivatives.

The idea that derivatives can complete the market and improve efficiency has long been documented in the literature.[2] Our contribution in this paper is to build on this intuition, and work out an explicit case for a realistic model of market incompleteness with a realistic set of derivatives. In particular, we push the existing intuition one step further by asking: What are the optimal dynamic strategies for an investor who can control not just his holdings in the aggregate stock market and a riskless bond, but also in derivatives? What is the benefit from including derivatives?

We address these questions by focusing on two specific aspects of market incompleteness that have been well documented in the empirical literature for the aggregate stock market: stochastic volatility and price jumps.[3] Specifically, we adopt an empirically realistic model for the aggregate stock market that incorporates three types of risk factors: diffusive price shocks, price jumps, and volatility risks. Taking this market condition as given, we solve the dynamic asset allocation problem (Merton, 1971) of a power-utility investor whose investment opportunity set includes not only the usual riskless bond and risky stock, but also derivatives on the stock.

What makes derivatives valuable in such a setting of multiple risk factors is that the stock and bond alone cannot provide independent exposure to each and every risk factor. For example, the risky stock by itself can only provide a "package deal" of risk exposures: one unit each to diffusive and jump risks and none to volatility risk. With the help of derivatives, however, this "package deal" can be broken down into its three individual components. For example, an at-the-money option, being highly sensitive to market volatility, provides exposure to volatility risk; a deep out-of-the-money put option, being much more sensitive to

[2] Among others, the spanning role of derivatives has been studied extensively by Ross (1976), Breeden and Litzenberger (1978), Arditti and John (1980), and Green and Jarrow (1987) in static settings, and, more recently, by Bakshi and Madan (2000) in a dynamic setting. In a buy-and-hold environment, Haugh and Lo (2001) use derivatives to mimic the dynamic trading strategy of the underlying stock. Using historical stock data, Merton et al. (1978, 1982) investigate the return characteristics of various option strategies. Carr et al. (2001) consider the optimal portfolio problem in a pure-jump setting by including as many options as the number of jump states. In an information context, Brennan and Cao (1996) analyze the role of derivatives in improving trading opportunities. Ahn et al. (1999) consider the role of options in a portfolio Value-at-Risk setting.

[3] Both aspects have been the object of numerous studies. Among others, Jorion (1989) documented the importance of jumps in the aggregate stock returns. Recent studies documenting the importance of both stochastic volatility and jumps include Andersen et al. (2002), Bates (2000), and Bakshi et al. (1997).

negative jump risk than to diffusive risk, serves to disentangle jump risk from diffusive risk.[4]

The market incompleteness that makes derivatives valuable in our setting also makes the pricing of such derivatives not unique. In particular, using the risk and return information contained in the underlying risky stock, we are unable to assign the market price of volatility risk or the relative pricing of diffusive and jump risks. In other words, when we introduce derivatives to complete the market, say one at-the-money and one out-of-the-money put options, we need to make additional assumptions on the volatility-risk and jump-risk premia implicit in such derivatives. Once such assumptions are made and the derivatives are introduced, the market is complete. Alternatively, we can start with a pricing kernel that supports the risk-and-return tradeoffs implied by these derivatives and the risky stock, and work only with that pricing kernel. These two approaches are equivalent, and the key element that is important for our analysis is the specification of the market prices of the three risk factors.[5]

The dynamic asset allocation problem is solved in closed form. Our results can be interpreted in three steps. First, we solve the investor's optimal wealth dynamics. Second, we find the exposure to the three risk factors that supports the optimal wealth dynamics. Finally, we find the optimal positions in the risky stock and the two derivative securities that achieve the optimal exposure to the risk factors. Instrumental to the final step is the ability of the derivative securities to complete the market, which is formalized in our paper as a non-redundancy condition on the chosen derivatives.

Our first illustrative example is on the role of derivatives as a vehicle to volatility risk. In this setting, the demand for derivatives arises from the need to access volatility risk. As a result, the optimal portfolio weight on the derivative security depends explicitly on how sensitive the chosen derivative is to stock volatility. Our result also shows that there are two economically different sources from which the need to access the volatility risk arises. Acting myopically, the investor participates in the derivatives market simply to take advantage of the risk-and-return tradeoff provided by volatility risk. For instance, if volatility risk is not priced at all, there is no "myopic" incentive to take on derivative positions. On the other hand, negatively priced volatility risk, which is supported by the empirical evidence from the option market (Pan, 2002; Benzoni, 1998; Bakshi and Kapadia, 2003) induces him to sell

[4] Although one can think of derivatives in their most general terms, not all financial contracts can provide such a service. For example, bond derivatives or long-term bonds can only provide access to the risk of the short rate, which is a constant in our setting. Given that the three risk factors are at the level of the aggregate stock market, linear combinations of individual stocks are unlikely to provide independent exposures to such risk factors.

[5] It should be noted that by exogenously specifying the market prices of risk factors, our analysis is of a partial-equilibrium nature. In fact, this is very much the spirit of the asset allocation problem: a small investor takes prices (both risks and returns) as given and finds for himself the optimal trading strategy. By the same token, as we later quantify the improvement for including derivatives, we are addressing the improvement in certainty-equivalent wealth for this very investor, not the welfare improvement of the society as a whole. The latter requires an equilibrium treatment. See, for example, the literature on financial innovation (Allen and Gale, 1994).

volatility by writing options. Acting non-myopically, the investor holds derivatives to further exploit the time-varying nature of the investment opportunity set, which, in our setting, is driven exclusively by stochastic volatility. As the volatility becomes more persistent, this non-myopic demand for derivatives becomes more prominent, and it also changes sharply around the investment horizon that is close to the half-life of the volatility.

To assess the portfolio improvement from participating in the derivatives market, we compare the certainty-equivalent wealth of two utility-maximizing investors with and without access to the derivatives market. To further quantify the gain from taking advantage of derivatives, we calibrate the parameters of the stochastic volatility model to those reported by empirical studies on the S&P 500 index and option markets. Our results show that the improvement from including derivatives is driven mostly by the myopic component. With normal market conditions and a conservative estimate of the volatility-risk premium, the improvement in certainty-equivalent wealth for an investor with relative risk aversion of three is about 14%per year, which becomes higher when the market becomes more volatile.

Our second illustrative example is on the role of derivatives as a vehicle to disentangle jump risk from diffusive risk. In this setting, the relative attractiveness between jump risk and diffusive risk is the economic driving force behind our result. If jump risk is compensated in such a way that the investor finds it as attractive as diffusive risk, then there is no need to disentangle the two risk factors, and, consequently, the demand for derivatives is zero. It is, however, generally not true that the two risk factors are rewarded equally. In fact, the empirical evidence from the option market suggests that, for investors with a reasonable range of risk aversion, jump risk is compensated more highly than diffusive risk (Pan, (2002)). To explain the differential pricing between diffusive and jump risks in equilibrium, Bates (2001) considers an investor with an additional aversion to market crashes, while Liu et al. (2002) consider an investor with uncertainty aversion toward rare events.

Apart from the quantitative difference, jump risk differs from diffusive risk in an important qualitative way. Specifically, in the presence of large, negative price jumps, the investor is reluctant to hold too much jump risk regardless of the premium assigned to it. Intuitively, this is because in contrast to diffusive risk, which can be controlled via continuous trading, the sudden, high-impact nature of jump risk takes away the investor's ability to continuously trade out of a leveraged position to avoid negative wealth. As a result, without access to derivatives, the investor avoids taking too leveraged a position in the risky stock (Liu et al. 2003). The same investor is nevertheless freer to make choices when the worst-case scenarios associated with jump risk can be taken care of by trading derivatives. In our quantitative example, this is done by taking a larger position in the risky stock and buying deep out-of-the-money put options to hedge out the negative jump risk.

The rest of the paper is organized as follows. Section 2 describes the investment environment, including the risky stock and the derivative securities. Section 3 formalizes the investment problem and provides the explicit solution. Section 4 provides an extensive example on the role of derivatives in the presence of volatility risk, while Section 5 focuses on jump risk. Section 6 concludes the paper. Technical details and proofs are provided in the Appendices.

2. The model

2.1. The stock price dynamics

The fundamental securities in this economy are a riskless bond that pays a constant rate of interest r, and a risky stock that represents the aggregate equity market. To capture the empirical features that are important in the time-series data on the aggregate stock market, we assume the following dynamics for the price process S of the risky stock:

$$\mathrm{d}S_t = (r + \eta V_t + \mu(\lambda - \lambda^Q)V_t)S_t\,\mathrm{d}t + \sqrt{V_t}S_t\,\mathrm{d}B_t + \mu S_{t-}(\mathrm{d}N_t - \lambda V_t\,\mathrm{d}t), \tag{1}$$

$$\mathrm{d}V_t = \kappa(\bar{v} - V_t)\mathrm{d}t + \sigma\sqrt{V_t}\left(\rho\,\mathrm{d}B_t + \sqrt{1-\rho^2}\,\mathrm{d}Z_t\right), \tag{2}$$

where B and Z are standard Brownian motions, and N is a pure-jump process. All three random shocks B, Z, and N are assumed to be independent.

This model incorporates, in addition to the usual diffusive price shock B, two risk factors that are important in characterizing the aggregate stock market: stochastic volatility and price jumps. Specifically, the instantaneous variance process V is a stochastic process with long-run mean $\bar{v} > 0$, mean-reversion rate $\kappa > 0$, and volatility coefficient $\sigma \geqslant 0$. This formulation of stochastic volatility (Heston, 1993), allows the diffusive price shock B to enter the volatility dynamics via the constant coefficient $\rho \in (-1, 1)$, introducing correlation between the price and volatility shocks — a feature that is important in the data.

The random arrival of jump events is dictated by the pure-jump process N with stochastic arrival intensity $\{\lambda V_t : t \geqslant 0\}$ for constant $\lambda \geqslant 0$. Intuitively, the conditional probability at time t of another jump before $t + \Delta t$ is, for some small Δt, approximately $\lambda V_t \Delta t$. This formulation (Bates, 2000), has the intuitive interpretation that price jumps are more likely to occur during volatile markets. Following Cox and Ross (1976),we adopt deterministic jump amplitudes. That is, conditional on a jump arrival, the stock price jumps by a constant multiple of $\mu > -1$, with the limiting case of -1 representing the situation of total ruin. As becomes clear later, this specification of deterministic jump amplitude simplifies our analysis in the sense that only one additional derivative security is needed to complete the market with respect to the jump component. This formulation, though simple, is capable of capturing the sudden and high-impact nature of jumps that cannot be produced by diffusions. More generally, one could introduce random jumps with multiple outcomes and use multiple derivatives to help complete the market.

Finally, η and λ^Q are constant coefficients capturing the two components of the equity premium: one for diffusive risk B, the other for jump risk N. More detailed discussions on these two parameters will be provided in the next section as we introduce the pricing kernel for this economy.

2.2. The derivative securities and the pricing kernel

In addition to investing in the risky stock and the riskless bond, the investor is also given the chance to include derivatives in his portfolio. The relevant derivative securities are those that serve to expand the dimension of risk-and-return tradeoffs for the investor. More specifically for our setting, such derivatives are those that provide differential exposures to the three fundamental risk factors in the economy.

For concreteness, we consider the class of derivatives whose time-t price O_t depends on the underlying stock price S_t and the stock volatility V_t through $O_t = g(S_t, V_t)$, for some function g. Although more complicated derivatives can be adopted in our setting, this class of derivatives provides the clearest intuition possible, and includes most of the exchange-traded derivatives. Letting τ be its time to expiration, this particular derivative is defined by its payoff structure at the time of expiration. For example, a derivative with a linear payoff structure $g(S_\tau, V_\tau) = S_\tau$ is the stock itself, and it must be that $g(S_t, V_t) = S_t$ at any time $t < \tau$. On the other hand, for some strike price $K > 0$, a derivative with the non-linear payoff structure $g(S_\tau, V_\tau) = (S_\tau - K)^+$ is a European-style call option, while that with $g(S_\tau, V_\tau) = (K - S_\tau)^+$ is a European-style put option. Unlike our earlier example of the linear contract, the pricing relation $g(S_\tau, V_\tau)$ at $t < \tau$ is not uniquely defined in these two cases from the information contained in the risky stock only. In other words, by including multiple sources of risks in a non-trivial way, the market is incomplete with respect to the risky stock and riskless bond.

The market can be completed once we introduce enough non-redundant derivatives $O_t^{(i)} = g^{(i)}(S_t, V_t)$ for $i = 1, 2, \ldots, N$. Alternatively, we can introduce a specific pricing kernel to price all of the risk factors in this economy, and consequently any derivative securities. These two approaches are equivalent. That is, the particular specification of the N derivatives that complete the market is linked uniquely to a pricing kernel $\{\pi_t, 0 \leqslant t \leqslant T\}$ such that

$$O_t^{(i)} = \frac{1}{\pi_t} E_t[\pi_{\tau_i} g^{(i)}(S_{\tau_i}, V_{\tau_i})], \tag{3}$$

for any $t \leqslant \tau_i$, where τ_i is the time to expiration for the i-th derivative security.

In this paper, we choose the latter approach and start with the following parametric pricing kernel:

$$\mathrm{d}\pi_t = -\pi_t\left(r\,\mathrm{d}t + \eta\sqrt{V_t}\,\mathrm{d}B_t + \xi\sqrt{V_t}\,\mathrm{d}Z_t\right) + \left(\frac{\lambda^Q}{\lambda} - 1\right)\pi_{t-}(\mathrm{d}N_t - \lambda V_t \mathrm{d}t), \tag{4}$$

where $\pi_0 = 1$ and the constant coefficients η, ξ, and λ^Q/λ, respectively, control the premiums for the diffusive price risk B, the additional volatility risk Z, and the jump risk N. Consistent with this pricing kernel is the following parametric specification of the price dynamics for the i-th derivative security:

$$\mathrm{d}O_t^{(i)} = rO_i^{(i)}\,\mathrm{d}t + (g_s^{(i)}S_t + \sigma\rho g_v^{(i)})(\eta V_t\mathrm{d}t + \sqrt{V_t}\,\mathrm{d}B_t) + \sigma\sqrt{1-\rho^2}g_v^{(i)}(\xi V_t\,\mathrm{d}t$$
$$+ \sqrt{V_t}\mathrm{d}Z_t) + \Delta g^{(i)}((\lambda - \lambda^Q)V_t\,\mathrm{d}t + \mathrm{d}N_t - \lambda V_t\,\mathrm{d}t), \quad (5)$$

where $g_s^{(i)}$ and $g_v^{(i)}$ measure the sensitivity of the i-th derivative price to infinitesimal changes in the stock price and volatility, respectively, and where $\Delta g^{(i)}$ measures the change in the derivative price for each jump in the underlying stock price. Specifically,

$$g_s^{(i)} = \left.\frac{\partial g^{(i)}(s,v)}{\partial s}\right|_{(S_t,V_t)};\, g_v^{(i)} = \left.\frac{\partial g^{(i)}(s,v)}{\partial v}\right|_{(S_t,V_t)};$$
$$\Delta g^{(i)} = g^{(i)}((1+\mu)S_t, V_t) - g^{(i)}(S_t, V_t). \quad (6)$$

A derivative with non-zero g_s provides exposure to the diffusive price shock B; a derivative with non-zero g_v provides exposure to additional volatility risk Z; and a derivative with non-zero Δg provides exposure to jump risk N. To complete the market with respect to these three risk factors, one needs at least three securities. For example, one can start with the risky stock, which provides simultaneous exposure to the diffusive price shock B and the jump risk N: $g_s = \Delta g/\Delta S = 1$. To separate exposure to jump risk from that to the diffusive price shock, the investor can add out-of-the-money put options to his portfolio, which provide more exposure to jump risk than diffusive risk: $|\Delta g/\Delta S| \geqslant |g_s|$. Finally, for exposure to additional volatility risk Z, the investor can add at-the-money options, which provide $g_v > 0$.

In essence, the role of the derivative securities here is to provide separate exposures to the fundamental risk factors. It is important to point out that not all financial contracts can achieve such a goal. For example, bond derivatives are infeasible because they can only provide exposure to the constant riskfree rate. Other individual stocks are generally infeasible because our risky stock represents the aggregate equity market, which is a linear combination of the individual stocks.[6]

In addition to providing exposures to the risk factors, the derivatives also pick up the associated returns. This risk-and-return tradeoff is controlled by the specific parametric form of the pricing kernel π, or equivalently by the particular price dynamic specified for the derivatives. To be more specific, from either (4) or (5) we can see that the constant η controls the premium for the diffusive price risk B, the constant ξ controls that for the additional volatility risk Z, and the constant ratio λ^Q/λ controls that for jump risk.[7]

[6] Of course, one can think of an extreme case where one group of individual stocks contributes exclusively to the diffusive risk or the jump risk at the aggregate level, but not both. It is even more unlikely that an individual stock that is linear in nature could provide exposure to the volatility risk at the aggregate level.

[7] It should be noted that $\lambda^Q \geqslant 0$, and $\lambda^Q = 0$ if and only if $\lambda = 0$.

Apart from analytical tractability,[8] this specific parametric form has the advantage of having three parameters η, ξ, and λ^Q/λ to separately price the three risk factors in the economy. This flexibility is in fact supported empirically. Using joint time-series data on the risky stock (the S&P 500 index) and European-style options (the S&P 500 index options), recent studies have documented the importance of the risk premia implicit options, particularly those associated with volatility and jump risks (Chernov and Ghysels, 2000; Pan, 2002; Benzoni, 1998; Bakshi and Kapadia, 2003). Consistent with these findings, Coval and Shumway (2001) report expected option returns that cannot be explained by the risk-and-return tradeoff associated with the usual diffusive price shock B. Collectively, these empirical studies on the options market suggest that additional risk factors such as volatility risk and jump risk are priced in the option market. Given our focus on the optimal investment decision associated with derivatives, it is all the more important for us to choose a parametric form that accommodates the empirically documented risk and return tradeoff associated with options on the aggregate market.

Although our approach is partial equilibrium in nature, our pricing kernel can also be related to those derived from equilibrium studies. For the special case of constant volatility, our specific pricing kernel can be mapped to the equilibrium result of Naik and Lee (1990). Letting γ be the relative risk-aversion coefficient of the representative agent, the coefficient for the diffusive-risk premium is $\eta = \gamma$, and the coefficient for the jump-risk premium is $\lambda^Q/\lambda = (1+\mu)^{-\gamma}$. In the presence of adverse jump risk ($\mu < 0$), the investor fears that jumps are more likely to occur ($\lambda^Q > \lambda$), consequently requiring a positive premium for holding jump risk. It is important to notice that the market prices of both risk factors are controlled by one parameter: the risk-aversion coefficient γ of the representative agent. The empirical evidence from the option market, however, seems to suggest that jump risk is priced quite differently from diffusive risk. To accommodate this difference, a recent paper by Bates (2001) introduces a representative agent with an additional crash aversion coefficient Y. Mapping his equilibrium result to our parametric pricing kernel, we have $\eta = \gamma$, and $\lambda^Q/\lambda = (1+\mu)^{-\gamma} \exp(Y)$. The usual risk aversion coefficient γ contributes to the market price of diffusive risk, while the crash aversion contributes an additional layer to the market price of jump risk.

In this respect, we can think of our parametric approach to the pricing kernel as a reduced-form approach. For the purpose of understanding the economic sources of the risk and return, a structural approach such as that of Naik and Lee (1990) and Bates (2001) is required. For the purpose of obtaining the optimal derivative strategies with given market conditions, however, such a reduced-form approach is in fact sufficient and has been adopted in the asset allocation literature. Finally, to verify that the parametric pricing kernel π is a valid pricing kernel, which rules out arbitrage opportunities involving the riskless bond, the risky stock, and any

[8] For a European-style option with maturity τ_i and strike price K_i we have $g^{(i)} = c(S_t, V_t; K_i, \tau_i)$, where the explicit functional form of c can be derived via transform analysis. For this specific case, the original solution is given by Bates (2000). See also Heston (1993) and Duffie et al. (2000).

derivative securities, one can apply Ito's lemma and show that $\pi_t \exp(-rt)$, $\pi_t S_t$, and $\pi_t O_t^{(i)}$ are local martingales.[9]

3. The investment problem and the solution

The investor starts with positive wealth W_0. Given the opportunity to invest in the riskless asset, the risky stock and the derivative securities, he chooses, at each time t, $0 \leqslant t \leqslant T$, to invest a fraction ϕ_t of his wealth in the stock S_t, and fractions $\psi_t^{(1)}$ and $\psi_t^{(2)}$ in the two derivative securities $O_t^{(1)}$ and $O_t^{(2)}$, respectively. The investment objective is to maximize the expected utility of his terminal wealth W_T,

$$\max_{\{\phi_t, \psi_t, 0 \leqslant t \leqslant T\}} E\left(\frac{W_T^{1-\gamma}}{1-\gamma}\right), \tag{7}$$

where $\gamma > 0$ is the relative risk-aversion coefficient of the investor, and where the wealth process satisfies the self-financing condition

$$\begin{aligned} \mathrm{d}W_t = {} & rW_t \mathrm{d}t + \theta_t^B W_t\left(\eta V_t \mathrm{d}t + \sqrt{V_t}\mathrm{d}B_t\right) + \theta_t^Z W_t\left(\xi V_t \mathrm{d}t + \sqrt{V_t}\mathrm{d}Z_t\right) \\ & + \theta_{t-}^N W_{t-}\mu\left((\lambda - \lambda^Q)V_t \mathrm{d}t + \mathrm{d}N_t - \lambda V_t \mathrm{d}t\right), \end{aligned} \tag{8}$$

where θ_t^B, θ_t^Z, and θ_t^N are defined, for given portfolio weights ϕ_t and ψ_t on the stock and the derivatives, by

$$\begin{gathered} \theta_t^B = \phi_t + \sum_{i=1}^{2} \psi_t^{(i)} \left(\frac{g_s^{(i)} S_t}{O_t^{(i)}} + \sigma\rho \frac{g_v^{(i)}}{O_t^{(i)}}\right); \theta_t^Z = \sigma\sqrt{1-\rho^2} \sum_{i=1}^{2} \psi_t^{(i)} \frac{g_v^{(i)}}{O_t^{(i)}}; \\ \theta_t^N = \phi_t + \sum_{i=1}^{2} \psi_t^{(i)} \frac{\Delta g^{(i)}}{\mu O_t^{(i)}}. \end{gathered} \tag{9}$$

Effectively, by taking positions ϕ_t and ψ_t on the risky assets, the investor invests θ^B in the diffusive price shock B, θ^Z in the additional volatility risk Z, and θ^N in the jump risk N. For example, a portfolio position ϕ_t in the risky stock provides equal exposures to both the diffusive and jump risks in stock prices. Similarly, a portfolio position ψ_t in the derivative security provides exposure to the volatility risk Z via a non-zero g_v, exposure to the diffusive price shock B via a non-zero g_s, and exposure to the jump risk via a non-zero Δg.

Except for adding derivative securities to the investor's opportunity set, the investment problem in (7) and (8) is the standard Merton (1971) problem. Before solving for this problem, we should point out that the maturities of the chosen derivatives do not have to match the investment horizon T. For example, it might be hard for an investor with a ten-year investment horizon to find an option with a matching maturity. He might choose to invest in options with a much shorter time to expiration, say LEAPS, which typically expire in one or two years, and switch or roll

[9] See, for example, Appendix B.2 of Pan (2000).

over to other derivatives in the future. For the purpose of choosing the optimal portfolio weights at time t, what matters is the choice of derivative securities O_t at that time, not the future choice of derivatives. This is true as long as, at each point in time in the future, there exist non-redundant derivative securities to complete the market.

We now proceed to solve the investment problem in (7) using the stochastic control approach. Alternatively, our problem can be solved using the Martingale approach of Cox and Huang (1989). (We will come back later and interpret the solution from the angle of the Martingale approach.) Following Merton (1971), we define the indirect utility function by

$$J(t,w,v) = \max_{\{\phi_s,\psi_s,t\leqslant s\leqslant T\}} E\left(\frac{W_T^{1-\gamma}}{1-\gamma}\middle| W_t = w, V_t = v\right), \tag{10}$$

which, by the principle of optimal stochastic control, satisfies the following Hamilton–Jacobi–Bellman (HJB) equation,

$$\begin{aligned}\max_{\phi_t,\psi_t}\Big\{&J_t + W_t J_W\big(r_t + \theta_t^B \eta V_t + \theta_t^Z \xi V_t - \theta_t^N \mu\lambda^Q V_t\big)\\ &+\tfrac{1}{2}W_t^2 J_{WW} V_t\big((\theta_t^B)^2 + (\theta_t^Z)^2\big)\\ &+\lambda V_t \Delta J + \kappa(\bar{v} - V_t)J_V + \frac{1}{2}\sigma V_t J_{VV} + \sigma V_t W_t J_{WV}\Big(\rho\theta_t^{\mathrm{B}} + \sqrt{1-\rho^2}\theta_t^Z\Big)\Big\} = 0\,,\end{aligned} \tag{11}$$

where $\Delta J = J(t, W_t(1+\theta^N\mu), V_t) - J(t, W_t, V_t)$ denotes the jump in the indirect utility function J for given jumps in the stock price, and where J_t, J_W, and J_V denote the derivatives of $J(t,W,V)$ with respect to t, W and V respectively, and similar notations for higher derivatives.

To solve the HJB equation, we notice that it depends explicitly on the portfolio weights θ^B, θ^Z, and θ^N, which, as defined in (9), are linear transformations of the portfolio weights ϕ and ψ on the risky assets. Taking advantage of this structure, we first solve the optimal positions on the risk factors B, Z, and N, and then transform them back via the linear relation (9) to the optimal positions on the risky assets. This transformation is feasible as long as the chosen derivatives are non-redundant in the following sense:

Definition. At any time t, the derivative securities $O_t^{(1)}$ and $O_t^{(2)}$ are non-redundant if

$$\mathscr{D}_t \neq 0 \quad \text{where } \mathscr{D}_t = \left(\frac{\Delta g^{(1)}}{\mu O_t^{(1)}} - \frac{g_s^{(1)} S_t}{O_t^{(1)}}\right)\frac{g_v^{(2)}}{O_t^{(2)}} - \left(\frac{\Delta g^{(2)}}{\mu O_t^{(2)}} - \frac{g_s^{(2)} S_t}{O_t^{(2)}}\right)\frac{g_v^{(1)}}{O_t^{(1)}}. \tag{12}$$

Effectively, the non-redundancy condition in (12) guarantees market completeness with respect to the chosen derivative securities, the risky stock, and the riskless bond. Without access to derivatives, linear positions in the risky stock provide equal exposures to diffusive and jump risks, and none to volatility risk. To complete the market with respect to volatility risk, we need to bring in a risky asset that is sensitive to changes in volatility: $g_v \neq 0$. To complete the market with respect to jump risk, we need a risky asset with different sensitivities to *infinitesimal* and *large* changes in stock prices: $g_s S_t / O_t \neq \Delta g / \mu O_t$. Moreover, (12) also ensures that the two chosen derivative securities are not identical in covering the two risk factors.

Proposition 1. *Assume that there are non-redundant derivatives available for trade at any time $t < T$. Then, for given wealth W_t and volatility V_t, the solution to the HJB equation is given by*

$$J(t, W_t, V_t) = \frac{W_t^{1-\gamma}}{1-\gamma} \exp(\gamma h(T-t) + \gamma H(T-t) V_t), \tag{13}$$

where $h(\cdot)$ and $H(\cdot)$ are time-dependent coefficients that are independent of the state variables. That is, for any $0 \leqslant \tau \leqslant T$,

$$\begin{aligned} h(\tau) &= \frac{2\kappa\bar{v}}{\sigma^2} \ln\left(\frac{2k_2 \exp((k_1+k_2)\tau/2)}{2k_2 + (k_1+k_2)(\exp(k_2\tau)-1)}\right) + \frac{1-\gamma}{\gamma} r\tau, \\ H(\tau) &= \frac{\exp(k_2\tau)-1}{2k_2 + (k_1+k_2)(\exp(k_2\tau)-1)} \delta, \end{aligned} \tag{14}$$

where

$$\delta = \frac{1-\gamma}{\gamma^2}(\eta^2 + \xi^2) + 2\lambda^Q \left[\left(\frac{\lambda}{\lambda^Q}\right)^{1/\gamma} + \frac{1}{\gamma}\left(1 - \frac{\lambda}{\lambda^Q}\right) - 1 \right],$$

$$k_1 = \kappa - \frac{1-\gamma}{\gamma}\left(\eta\rho + \xi\sqrt{1-\rho^2}\right)\sigma; \ k_2 = \sqrt{k_1^2 - \delta\sigma^2}.$$

The optimal portfolio weights on the risk factors B, Z, and N are given by

$$\begin{aligned} &\theta_t^{*B} = \frac{\eta}{\gamma} + \sigma\rho H(T-t); \ \theta_t^{*Z} = \frac{\xi}{\gamma} + \sigma\sqrt{1-\rho^2} H(T-t); \\ &\theta_t^{*N} = \frac{1}{\mu}\left(\left(\frac{\lambda}{\lambda^Q}\right)^{1/\gamma} - 1\right). \end{aligned} \tag{15}$$

Transforming the θ^*'s to the optimal portfolio weights on the risky assets, ϕ_t^* for the stock and ψ_t^* for derivatives, we have

$$\phi_t^* = \theta_t^{*B} - \sum_{i=1}^{2} \psi_t^{*(i)} \left(\frac{g_s^{(i)} S_t}{O_t^{(i)}} + \sigma\rho \frac{g_v^{(i)}}{O_t^{(i)}} \right),$$

$$\psi^{*(1)} = \frac{1}{\mathscr{D}_t}\left[\frac{g_v^{(2)}}{O_t^{(2)}} \left(\theta_t^{*N} - \theta_t^{*B} - \frac{\theta_t^{*Z}\rho}{\sqrt{1-\rho^2}} \right) - \left(\frac{\Delta g^{(2)}}{\mu O_t^{(2)}} - \frac{g_s^{(2)} S_t}{O_t^{(2)}} \right) \frac{\theta_t^{*Z}}{\sigma\sqrt{1-\rho^2}} \right], \tag{16}$$

$$\psi^{*(2)} = \frac{1}{\mathscr{D}_t}\left[\left(\frac{\Delta g^{(1)}}{\mu O_t^{(1)}} - \frac{g_s^{(1)} S_t}{O_t^{(1)}} \right) \frac{\theta_t^{*Z}}{\sigma\sqrt{1-\rho^2}} - \frac{g_v^{(1)}}{O_t^{(1)}} \left(\theta_t^{*N} - \theta_t^{*B} - \frac{\theta_t^{*Z}\rho}{\sqrt{1-\rho^2}} \right) \right].$$

Proof. See Appendix A.

To further deliver the intuition behind the result in Proposition 1,we examine our result from the angle of the Martingale approach. Given that the market is complete after introducing the derivative securities (or equivalently, the pricing kernel π), the terminal wealth W_T^* associated with the optimal portfolio strategy can be solved directly from

$$\max_{W_T} E_0\left(\frac{W_T^{1-\gamma}}{1-\gamma}\right) \quad \text{subject to } E_0(\pi_T W_T) = W_0. \tag{17}$$

Solving this constrained optimization problem explicitly, and using the fact that, at any time $t<T$, $W_t^* = E(\pi_T W_T^*)/\pi_t$, we can show that the optimal wealth dynamics $\{W_t^*, 0 \leqslant t \leqslant T\}$ follow that specified in (8), with θ^B, θ^Z and θ^N replaced by the optimal solution given by (15) in Proposition 1.

From this perspective, our results can be viewed in three steps. First, solve for the optimal wealth dynamics. Second, find the optimal exposures θ^{*B}, θ^{*Z}, and θ^{*N} to the fundamental risk factors to support the optimal wealth dynamics. Finally, find the optimal positions ϕ^*, $\psi^{*(1)}$, and $\psi^{*(2)}$ on the risky stock and the two derivative securities to achieve the optimal exposure on the risk factors. The mapping in this last step is only feasible when the market is complete with respect to the three securities S, $O^{(1)}$ and $O^{(2)}$ — that is, when the non-redundancy condition (12) is satisfied. To further illustrate our results, we consider two examples in the next two sections, one on volatility risk and the other on jump risk.

4. Example I: derivatives and volatility risk

This section focuses on the role of derivative securities as a vehicle to stochastic volatility. For this, we specialize in an economy with volatility risk but no jump risk. Specifically, we turn off the jump component in (1) and (2) by letting $\mu = 0$ and $\lambda = \lambda^Q = 0$.

In such a setting, only one derivative security with non-zero sensitivity to volatility risk is needed to help complete the market. Denoting this derivative security by O_t, we can readily use the result of Proposition 1 to derive the optimal portfolio weights:

$$\phi_t^* = \frac{\eta}{\gamma} - \frac{\xi\rho}{\gamma\sqrt{1-\rho^2}} - \psi_t^* \frac{g_s S_t}{O_t}, \tag{18}$$

$$\psi_t^* = \left(\frac{\xi}{\gamma\sigma\sqrt{1-\rho^2}} + H(T-t)\right)\frac{O_t}{g_v}, \tag{19}$$

where ϕ_t^* and ψ_t^* denote the optimal positions in the risky stock and the derivative security, respectively, and where H is as defined in (14) with the simplifying restriction of no jumps.

4.1. The demand for derivatives

The optimal derivative position ψ^* in (19) is inversely proportional to g_v/O_t, which measures the volatility exposure for each dollar invested in the derivative security. Intuitively, the demand for derivatives arises in this setting from the need to access volatility risk. The more "volatility exposure per dollar" a derivative security provides, the more effective it is as a vehicle to volatility risk. Hence a smaller portion of the investor's wealth needs to be invested in this derivative security. By contrast, financial contracts with lower sensitivities to aggregate market volatility are less effective for the same purpose. Of course, the extreme case will be those linear securities (e.g., individual stocks) that provide zero exposure to volatility risk.

The demand for derivatives — or the need for volatility exposures — arises for two economically different reasons. First, a myopic investor finds the derivative security attractive because, as a vehicle to volatility risk, it could potentially expand the dimension of risk-and-return tradeoffs. This myopic demand for derivatives is reflected in the first term of ψ^*_t. For example, negatively priced volatility risk ($\xi < 0$) makes short positions in volatility attractive, inducing investors to sell derivatives with positive "volatility exposure per dollar." Similarly, a positive volatility–risk premium ($\xi > 0$) induces opposite trading strategies. Moreover, the less risk-averse investor is more aggressive in taking advantage of the risk and return tradeoff through investing in derivatives.

Second, for an investor who acts non-myopically, there is a benefit in derivative investments even when the myopic demand diminishes with a zero volatility–risk premium ($\xi = 0$). This non-myopic demand for derivatives is reflected in the second term of ψ^*_t. Without any loss of generality, consider an option whose volatility exposure is positive ($g_v > 0$). In our setting, the Sharpe ratio of the option return is driven exclusively by stochastic volatility. In fact, it is proportional to volatility. This implies that a higher realized option return at one instant is associated with a higher Sharpe ratio (better risk-return tradeoff) for the next-instant option return. That is, a good outcome is more likely to be followed by another good outcome. By the same token, a bad outcome in the option return predicts a sequence of less attractive future risk-return tradeoffs. An investor with relative risk aversion $\gamma < 1$ is particularly averse to sequences of negative outcomes because his utility is unbounded from below. On the other hand, an investor with $\gamma > 1$ benefits from sequences of positive outcomes because his utility is unbounded from above. As a result, they act quite differently in response to this temporal uncertainty. The one with $\gamma < 1$ takes a short position in volatility so as to hedge against temporal uncertainty, while the one with $\gamma < 1$ takes a long position in volatility so as to speculate on the temporal uncertainty. Indeed, it is easy to verify that $H(T-t)$, which is the driving force of this non-myopic term, is strictly positive for investors with $\gamma < 1$ strictly negative for investors with $\gamma > 1$, and zero for the log-utility investor.[10]

[10] One way to show this is by taking advantage of the ordinary differential equation (A.1) for $H(\cdot)$ with the additional constraints of no jumps. Given the initial condition $H(0) = 0$, it is easy to see that the driving force for the sign of H is the constant term which has the same sign as $1-\gamma$.

4.2. The demand for stock

Given that volatility risk exposure is taken care of by the derivative holdings, the "net" demand for stock should simply be linked to the risk-and-return tradeoff associated with price risk. Focusing on the first term of ϕ_t^* in (18), this is indeed true. Specifically, demand for stock is proportional to the attractiveness of the stock and inversely proportional to the investor's risk aversion.

The interaction between the derivative security and its underlying stock, however, complicates the optimal demand for stocks. For example, by holding a call option, one effectively invests a fraction g_s—typically referred to as the "delta" of the option — on the underlying stock. The last term in ϕ^* is there to correct this "delta" effect. In addition, there is also a "correlation" effect that originates from the negative correlation between volatility and price shocks, typically referred to as the leverage effect (Black, 1976). Specifically, a short position in the volatility automatically involves long positions in the price shock, and equivalently, the underlying stock. The second term in ϕ^* is there to correct this "correlation" effect.

4.3. Empirical properties of the optimal strategies

To examine the empirical properties of our results, we fix a set of base-case parameters for our current model, using results from existing empirical studies.[11] Specifically, for one-factor volatility risk, we set the long-run mean at $\bar{v} = (0.13)^2$, the rate of mean reversion at $\kappa = 5$, and the volatility coefficient at $\sigma = 0.25$. The correlation between price and volatility risks is set at $\rho = -0.40$.

Important for our analysis is how the risk factors are priced. Given the well-established empirical property of the equity risk premium, calibrating the market price of the Brownian shocks B is straightforward. Specifically, setting $\eta = 4$ and coupling it with the base-case value of $\bar{v} = (0.13)^2$ for the long-run mean of volatility, we have an average equity risk premium of 6.76% per year.

The properties of the market price of volatility risk, however, are not as well established. In part because volatility is not a directly tradable asset, there is less consensus on reasonable values for its market price. Empirically, however, there is strong support that volatility risk is priced. For example, using joint time-series data on the S&P 500 index and options, Chernov and Ghysels (2000), Pan (2002), Benzoni (1998), and Bakshi and Kapadia (2003) report that volatility risk is negatively priced. That is, short positions in volatility are compensated with a positive premium. Similarly, Coval and Shumway (2001) report large negative returns generated by positions that are long on volatility.

[11] The empirical properties of the Heston (1993) model have been extensively examined using either the time-series data on the S&P 500 index alone (Andersen et al., 2002; Eraker et al., 2003), or the joint time-series data on the S&P 500 index and options (Chernov and Ghysels, 2000; Pan, 2002). Because of different sample periods and/or empirical approaches in these studies, the exact model estimates may differ from one paper to another. Our chosen model parameters are in the generally agreed region, with the exception of those reported by Chernov and Ghysels (2000).

Given that volatility risk at the aggregate level is generally related to economic activity (Officer, 1973; Schwert, 1989), it is quite plausible that it is priced. At an intuitive level, a negative volatility-risk premium could be supported by the fact that aggregate market volatility is typically high during recessions. A short position in volatility, which loses value when volatility becomes high during recessions, is therefore relatively more risky than a long position in volatility, requiring an additional risk premium.

Instead of calibrating the volatility-risk premium coefficient ξ to the existing empirical results, however, we will allow this coefficient to vary in our analysis so as to get a better understanding of how different levels and signs of the volatility risk premium could a effect the optimal investment decision.

Using this set of base-case parameters, particularly the risk-and-return tradeoff implied by the data, we now provide some quantitative examples of optimal investments in the S&P 500 index and options. To make the intuition as clean as possible, we focus on "delta-neutral" securities. Specifically, we consider the following "delta-neutral" straddle:

$$O_t = g(S_t, V_t; K, \tau) = c(S_t, V_t; K, \tau) + p(S_t, V_t; K, \tau), \tag{20}$$

where c and p are pricing formulas for call and put options with the same strike price K and time to expiration τ. The explicit formulation of c and p is provided in Appendix B.1. For a given stock price S_t, market volatility V_t, and time to expiration τ, the strike price K is selected so that the call option has a delta of 0.5, and, by put/call parity, the put option has a delta of -0.5, making the straddle delta-neutral.[12]

Fixing the riskfree rate at 5%, and picking a delta-neutral straddle with 0.1 year to expiration, Fig. 1 provides optimal portfolio weights under different scenarios. The top right panel examines the optimal portfolio allocation with varying volatility-risk premia. Qualitatively, this result is similar to our analysis in Section 4.1. Quantitatively, however, it indicates that the demand for derivatives is driven mainly by the myopic component. In particular, when the volatility-risk premium is set to zero ($\xi = 0$), the non-myopic demand for straddles is only 2% of the total wealth for an investor with relative risk aversion $\gamma = 3$ and investment horizon $T = 5$ years. In contrast, when we set $\xi = -6$, which is a conservative estimate for the volatility-risk premium, the optimal portfolio weight in the delta-neutral straddle increases to 54% for the same investor.

The quantitative effect of the non-myopic component can best be seen by varying the investment horizon (bottom left panel) or the volatility persistence (bottom right panel). Consider an investor with $\gamma = 3$, who would like to hedge against temporal uncertainty by taking short positions in volatility. The bottom left panel shows that as we increase the investment horizon, this intertemporal hedging demand increases. And, quite intuitively, the change is most noticeable around the region close to the

[12] Although "delta-neutral" positions can be constructed in numerous ways, we choose the "delta-neutral" straddle mainly because it is made of call and put options that are typically very close to the money. In particular, we intentionally avoid deep out-of-the-money options in our quantitative examples because they are most subject to concerns of option liquidity and jump risk, two important issues that are not accommodated formally in this section.

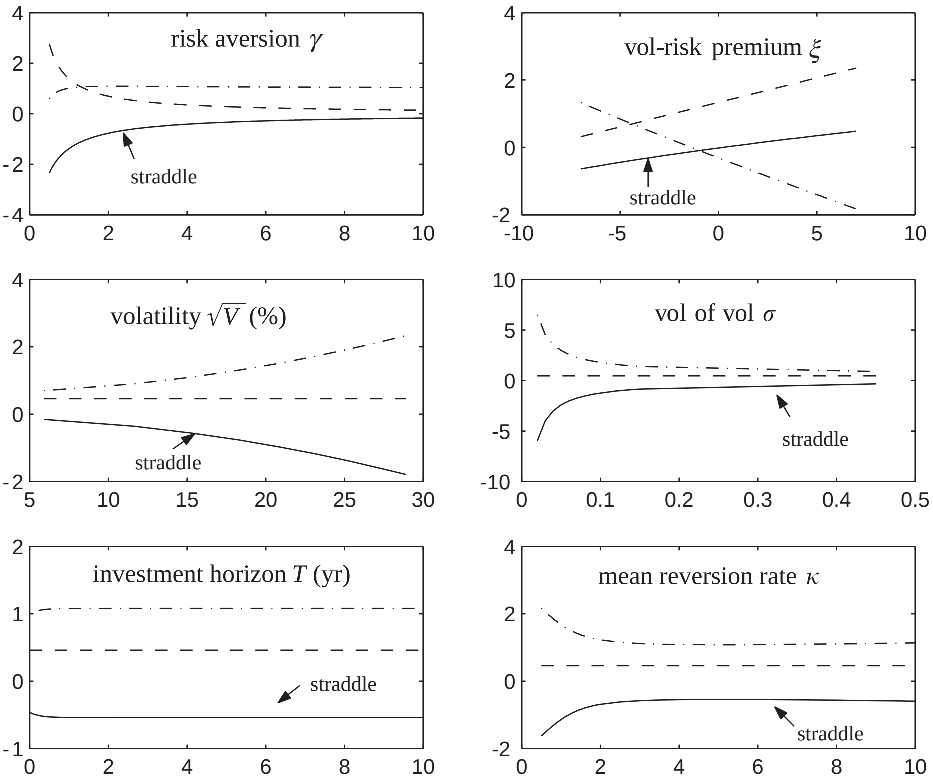

Fig. 1. Optimal portfolio weights. The y-axes are the optimal weight ψ^* on the "delta-neutral" straddle (solid line), ϕ^* on the risky stock (dashed line), and $1 - \psi^* - \phi^*$ on the riskfree bank account (dashed–dot line). The base-case parameters are as described in Section 2, and the volatility-risk premium coefficient is fixed at $\xi = -6$. The base-case investor has risk aversion $\gamma = 3$ and investment horizon $T = 5$ years. The riskfree rate is fixed at $r = 5\%$, and the base-case market volatility is fixed at $\sqrt{V} = 15\%$.

half-life of volatility risk. Similarly, the bottom right panel shows that as we decrease the persistent level of the volatility by increasing the mean-reversion rate κ, there is less benefit in taking advantage of the intertemporal persistence, hence a reduction in intertemporal hedging demand.

As the market becomes more volatile, the costs of the straddle (O_t) increase, but the volatility sensitivity (g_v) of such straddles decreases. In effect, the delta-neutral straddles provide less "volatility exposure per dollar" as market volatility increases. To achieve the optimal volatility exposure, more needs to be invested in the straddle, hence the increase in $|\psi^*|$ with the market volatility $\sqrt{V}$. As the volatility of volatility increases, the risk-and-return tradeoff on volatility risk becomes less attractive, hence the decrease in magnitude of the straddle position with increasing "vol of vol" σ. Finally, the optimal strategy with varying risk aversion γ is as expected: less risk-averse investors are more aggressive in their investment strategies.

4.4. Portfolio improvement

Suppose that market volatility is V_0 at time 0, and consider an investor with initial wealth W_0 and investment horizon T years, who takes advantage of the derivatives market. By Proposition 1, his optimal expected utility is

$$J(0, W_0, V_0) = \frac{W_0^{1-\gamma}}{1-\gamma} \exp(\gamma h(T) + \gamma H(T) V_0),$$

where h and H are as defined in (14) with the simplifying constraint of no jumps. It should be noted that the optimal expected utility is independent of the specific derivative contract chosen by the investor. This is quite intuitive, because in our setting the market is complete in the presence of the derivative security. Letting $\mathscr{W}^*$ be the investor's certainty-equivalent wealth, defined by $\mathscr{W}^{*1-\gamma}/(1-\gamma) = J(0, W_0, V_0)$, we have

$$\mathscr{W}^* = W_0 \exp\left(\frac{\gamma}{1-\gamma}[h(T) + H(T)V_0]\right). \tag{21}$$

The indirect utility for an investor with no access to the derivatives market is provided in Appendix B. Let $\mathscr{W}^{\text{no-op}}$ be the associated certainty-equivalent wealth. To quantify the portfolio improvement from including derivatives, we adopt the following measure:[13]

$$\mathscr{R}^{\mathscr{W}} = \frac{\ln \mathscr{W}^* - \ln \mathscr{W}^{\text{no-op}}}{T}. \tag{22}$$

Effectively, $\mathscr{R}^{\mathscr{W}}$ measures the portfolio improvement in terms of the annualized, continuously compounded return in certainty-equivalent wealth. The following Proposition summarizes the results.

Proposition 2. *For a power-utility investor with risk aversion coefficient* $\gamma > 0$ *and investment horizon* T, *the improvement from including derivatives is*

$$\mathscr{R}^{\mathscr{W}} = \frac{\gamma}{1-\gamma}\left(\frac{h(T) - h^{\text{no-op}}(T)}{T} + \frac{H(T) - H^{\text{no-op}}(T)}{T} V_0\right), \tag{23}$$

where V_0 *is the initial market volatility, and* h^{no-op} *and* H^{no-op} *are defined in Eq.* (*B.5*) *in Appendix B. For an investor with* $\gamma \neq 1$, *the portfolio improvement from including derivatives is strictly positive. For an investor with log-utility, the improvement is strictly positive if* $\xi \neq 0$, *and zero otherwise.*

Proof. See Appendix B.

The improvement from including derivatives is closely linked to the demand for derivatives. For a myopic investor with log-utility, the demand for derivatives arises from the need to exploit the risk-and-return tradeoff provided by volatility risk.

[13] The indirect utility of the "no-option "investor can be derived using the results from Liu (1998). For the completeness of the paper, it is provided in Appendix B.

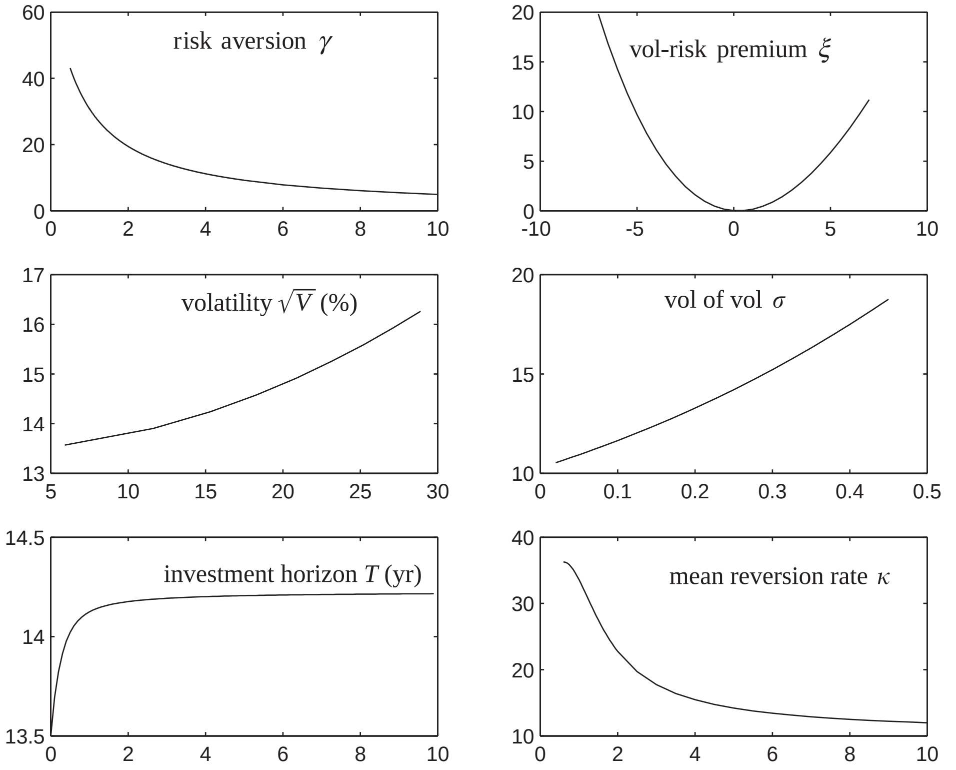

Fig. 2. Portfolio improvement from including derivatives. The y-axes are the improvement measure $\mathcal{R}^W$, defined by (22) in terms of returns over certainty-equivalent wealth. The base-case parameters are as described in Section 2, and the volatility-risk premium coefficient is fixed at $\xi = -6$. The base-case investor has risk aversion $\gamma = 3$ and investment horizon $T = 5$ years. The riskfree rate is fixed at $r = 5\%$, and the base-case market volatility is fixed at $\sqrt{V} = 15\%$.

When the volatility-risk premium is set to zero ($\xi = 0$), there is no myopic demand for derivatives. Consequently, there is no benefit from including derivatives. There is, however, still non-myopic demand for derivatives. Hence the portfolio improvement for a non-myopic investor is strictly positive regardless of the value of ξ.

To provide a quantitative assessment of the portfolio improvement, we again use the base-case parameters described in Section 4.3. The results are summarized in Fig. 2. Focusing first on the top right panel, we see that the portfolio improvement is very sensitive to how volatility risk is priced. Under normal market conditions with a conservative estimate[14] of the volatility–risk premium $\xi = -6$, our results show that

[14] For example, Coval and Shumway (2001) report that zero-beta at-the-money straddle positions produce average losses of approximately 3% per week. This number roughly corresponds to $\xi = -12$. Using volatility-risk premium to explain the premium implicit in option prices, Pan (2002) reports a total volatility-risk premium that translates to $\xi = -23$. This level of volatility-risk premium, however, could be overstated due to the absence of jump and jump-risk premium in the model. In fact, after introducing

the portfolio improvement from including derivatives is about 14.2% per year in certainty-equivalent wealth for an investor with risk aversion $\gamma = 3$. As the investor becomes less risk averse and more aggressive in taking advantage of the derivatives market, the improvement from including derivatives becomes even higher (top left panel).

We can further evaluate the relative importance of the myopic and non-myopic components of portfolio improvement by setting $\xi = 0$. The portfolio improvement from non-myopic trading in derivatives is as low as 0.02% per year. This is consistent with our earlier result: the demand for derivatives is driven mostly by the myopic component. The non-myopic component of the portfolio improvement is further examined in the bottom panels of Fig. 2 as we vary the investment horizon and the persistence of volatility. Intuitively, as the investment horizon T increases, or as the volatility shock becomes more persistent, the benefit of the derivative security as a hedge against temporal uncertainty becomes more pronounced. Hence there is an increase in portfolio improvement. Finally, from the middle two panels, we can also see that when market volatility $\sqrt{V}$ increases, or when the volatility of volatility increases, there is more to be gained from investing in the derivatives market.

5. Example II: derivatives and jumps

In this section, we examine the role of derivative securities in the presence of jump risk. For this, we specialize in an economy with jump risk but no volatility risk. That is, setting $V_0 = \bar{v}$ and $\sigma = 0$, we have $V_t = \bar{v}$ at any time t.

The risky stock is now affected by two types of risk factors: the diffusive price shock with constant volatility $\sqrt{\bar{v}}$, and the pure jump with Poisson arrival $\lambda\bar{v}$ and deterministic jump size μ. In the absence of either risk factor, derivative securities are redundant since the market can be completed by dynamic trading of the stock and bond (Black and Scholes, 1973; Cox and Ross, 1976). In their simultaneous presence, however, one more derivative is needed to complete the market. Applying Proposition 1, the optimal portfolio weights ϕ on the risky stock and ψ on the derivative are

$$\phi_t^* = \frac{\eta}{\gamma} - \psi_t^* \frac{g_s S_t}{O_t}, \tag{24}$$

$$\psi_t^* = \left(\frac{\Delta g}{\mu O_t} - \frac{g_s S_t}{O_t}\right)^{-1} \left(\frac{1}{\mu}\left[\left(\frac{\lambda}{\lambda^Q}\right)^{1/\gamma} - 1\right] - \frac{\eta}{\gamma}\right). \tag{25}$$

Throughout this section, we will compare this set of results with that of Liu et al. (2003), who study the optimal portfolio problem under the same dynamic setting for an investor without access to derivatives.

(*footnote continued*)
jumps and estimating jump-risk premium simultaneously with volatility-risk premium, Pan (2002) reports a volatility-risk premium that translates to $\xi = -10$.

5.1. The demand for derivatives

Evident in our solution is the role of derivative securities in separating jump risk from diffusive price risk. Specifically, the optimal demand ψ^* for the derivative security is inversely proportional to its ability to disentangle the two — the more effective it is in providing separate exposure, the less is needed to be invested in this derivative security. Deep out-of-the-money put options are examples of derivatives with high sensitivities to large price drops but low sensitivities to small price movements. In contrast, if a financial contract is equally sensitive to infinitesimal price movements and large price movements,

$$\frac{\partial g}{\partial S} = \frac{\Delta g}{\Delta S},$$

then it is not effective at all in disentangling the two risk factors. Linear financial contracts including the risky stock are such examples.

Economically, the ultimate driving force for holding derivatives is the risk-and-return tradeoff involved, which brings us to the second term in the optimal derivative position ψ^*. A derivative might be able to disentangle the two risk factors, but the need for such a disentanglement diminishes if the investor finds the two risk factors equally attractive. Recall that the premia for the two risk factors are controlled, respectively, by λ^Q/λ and η. Suppose that the relative value of the two coefficients is set so that

$$\frac{\lambda^Q}{\lambda} = \left(1 + \mu\frac{\eta}{\gamma}\right)^{-\gamma}. \tag{26}$$

From (25), we can see that under such a constraint, the optimal derivative position ψ^* is zero for an investor with risk-aversion coefficient γ. In other words, if the two risk factors are equally attractive to this investor, his desire to disentangle them diminishes, and therefore so does his demand ψ^* for the derivative security.

Empirically, however, it is generally not true that the two risk factors are rewarded equally. Specifically, the empirical evidence from the option market suggests that for a reasonable range of risk aversion γ, the coefficient λ^Q/λ is much higher than that implied by (26). If so, then derivatives — with their ability to disentangle the two risk factors — can be used by the investor to load more on jump risk. In contrast, if jump risk is not being compensated at all, then derivatives can be used by the investor to carve out his exposure to jump risk. Later in Section 5.3, we allow the coefficient λ^Q/λ for the jump-risk premium to vary, and examine the impact on the optimal derivative position.

Finally, to further emphasize the important role played by derivatives in disentangling the two risk factors, let's focus again on the "equally attractive" condition (26). One important observation is that, for a given diffusive-risk premium, one cannot always find the appropriate jump-risk premium to make jump risk equally attractive. In particular, for (26) to hold, it must be that $1 + \mu\eta/\gamma > 0$, which can be easily violated when $\eta/\gamma > 1$ and μ is negative and large. This reflects the qualitative difference between the two risk factors: in the presence of large, negative jumps, the investor is reluctant to hold too much of jump risk regardless of the premium (λ^Q/λ) assigned to it. This is because in contrast to diffusive risk, which can be controlled via continuous trading, the sudden, high-impact nature of jump risks takes away the investor's ability to continuously trade out of a leveraged position to avoid negative wealth. As a result, the investor needs to prepare for the worst-case scenario associated with jump risk so that his wealth remains positive when the jump arrives.

5.2. The demand for stock

To understand how having access to derivatives might change the investor's demand ϕ^* for the risky stock, let's compare our solution for ϕ^* with that for an investor with no access to the derivatives market (Liu et al., 2003):

$$\phi_t^{*\text{no-op}} = \frac{\eta}{\gamma} + \frac{\lambda\mu}{\gamma}\left[(1+\mu\phi_t^{*\text{no-op}})^{-\gamma} - \frac{\lambda^Q}{\lambda}\right], \tag{27}$$

where $\phi^{*\text{no-op}}$ is the optimal portfolio weight on the risky stock.

By taking a position in the risky stock, an investor is exposed to both diffusive and jump risks. Without access to derivatives, his optimal stock position is generally a compromise between the two risk factors. This tension is evident in the non-linear equation (27) that gives rise to the optimal stock positions $\phi^{*\text{no-op}}$. For example, when diffusive risk becomes more attractive with increasing η, an investor with risk aversion γ would like to increase his exposure to diffusive risk via η/γ . But the second term in (27) pulls him back, because, at the same time, he is also increasing his exposure to jump risk. If the jump-risk premium λ^Q/λ fails to catch up with the diffusive-risk premium, then tension arises. It is only when the investor finds the two risk factors equally attractive in the sense of (26) does this tension go away.

In general, however, the "equally attractive" condition (26) does not hold either empirically or theoretically. As mentioned earlier, for some large and negative jumps, no amount of jump-risk premium λ^Q/λ can compensate for jump risk. This qualitative difference between the two risk factors also manifests itself in the endogenously determined bound on $\phi^{*\text{no-op}}$. Specifically, (27) implies that $1 + \mu\phi^{*\text{no-op}} > 0$. In other words, in the presence of adverse jump risk ($\mu < 0$), the investor cannot afford to take too leveraged a position in the risky stock. The intuition behind this result is the same that makes the "equally attractive" condition untenable for large, negative jumps. That is, when being blindsided by things that he cannot control, the investor adopts investment strategies that prepare for worst-case scenarios.

The investor is nevertheless freer to make choices when the worst-case scenarios can be taken care of by trading derivatives. For an investor with access to derivatives, the result in (24) indicates that the optimal position in the risky stock is free of the tension between the two risk factors. Specifically, the first term of ϕ^* is to take advantage of the risk-and-return tradeoff associated with diffusive risk, while the second term is to correct for the "delta" exposure introduced by the derivative security.

5.3. A quantitative analysis on optimal strategies

For the quantitative analysis, we set the riskfree rate at $r = 5\%$ and consider three jump cases: (1) $\mu = -10\%$ jumps once every 10 years; (2) $\mu = -25\%$ jumps once every 50 years; and (3) $\mu = -50\%$ jumps once every 200 years. These jump cases are designed to capture the infrequent, high-impact nature of large events. For each jump case, we adjust the diffusive component of the market volatility $\sqrt{\bar{v}}$ so that total market volatility is always fixed at 15% a year.

For each jump case, we consider a wide range of jump-risk premia λ^Q/λ, starting with zero jump-risk premium: $\lambda^Q/\lambda = 1$. For each fixed level of the jump-risk premium, we always adjust the coefficient η for the diffusive-risk premium so that the total equity risk premium is fixed at 8% a year.

The quantitative analysis is summarized in Table 1. We choose one-month 5% out-of-the-money (OTM) European-style put options as the derivative security for the investor to include in his portfolio. Known to be highly sensitive to large negative jumps in stock prices, such OTM put options are among the most effective exchange-traded derivatives for the purpose of disentangling jump risk from diffusive risk. For an investor with varying degrees of risk aversion γ, Table 1 reports the optimal portfolio weights ϕ^* and ψ^* on the risky stock and the OTM put option, respectively. For comparison, the optimal portfolio weights for the case of no derivatives (stock only) are also reported.

Table 1
Optimal strategies with/without options

Jump Cases		$\mu = -10\%$ Every 10 yr			$\mu = -25\%$ Every 50 yr			$\mu = -50\%$ Every 200 yr		
			Stock & put			Stock & put			Stock & put	
γ	λ^Q/λ	Stock only	ϕ^*	$\psi^*(\%)$	Stock only	ϕ^*	$\psi^*(\%)$	Stock only	ϕ^*	$\psi^*(\%)$
	1	6.74	9.34	4.33	4.00	8.52	2.28	2.00	8.38	1.85
0.5	2	6.74	6.25	−0.67	4.00	7.59	1.40	2.00	7.94	1.54
	5	6.74	1.95	−5.63	4.00	5.88	0.85	2.00	7.10	1.53
	1	1.17	1.56	0.72	1.12	1.42	0.38	0.99	1.40	0.31
3	2	1.17	0.82	−0.66	1.12	1.22	0.12	0.99	1.31	0.21
	5	1.17	−0.44	−3.38	1.12	0.85	−0.34	0.99	1.13	0.09
	1	0.70	0.93	0.43	0.68	0.85	0.23	0.62	0.84	0.18
5	2	0.70	0.48	−0.43	0.68	0.73	0.06	0.62	0.78	0.13
	5	0.70	−0.35	−2.28	0.68	0.50	−0.25	0.62	0.67	0.04
	1	0.35	0.47	0.22	0.34	0.43	0.11	0.32	0.42	0.09
10	2	0.35	0.23	−0.23	0.34	0.36	0.03	0.32	0.39	0.06
	5	0.35	−0.21	−1.24	0.34	0.24	−0.15	0.32	0.33	0.01

To put the results in perspective, recall that for all cases considered in Table 1, total market volatility is always fixed at 15% a year, and the total equity risk premium is always fixed at 8% a year. If there were no jump risk, then options would be redundant and this investor's optimal stock weight would be $0.08/0.15^2/\gamma$. This translates to an optimal stock position of 7.11, 1.19, 0.71, and 0.36, respectively, for an investor with $\gamma = 0.5$, 3, 5, and 10.

The introduction of the jump component in Table 1 affects the optimal stock positions in important ways. As discussed earlier, the stock-only investor becomes relatively more cautious in the presence of jump risk.[15] More importantly, because the stock-only investor has no ability to separate jump exposure from diffusive exposure, his position is indifferent to how jump risk is rewarded relative to diffusive risk: all that matters is the total equity premium, which is fixed at 8% a year.

This, however, is no longer true for the investor who can trade both the risky stock and the put options. In particular, his position now depends on how jump risk is rewarded relative to diffusive risk. If jump risk is not being compensated ($\lambda^Q/\lambda = 1$), the investor views the exposure to jump risk as a nuisance. He sees the risky stock simply as an opportunity to achieve his optimal exposure to diffusive risk. By investing in the risky stock, however, he also exposes himself to negative jump risk. To carve out this exposure, he buys put options. In this sense, the put options are playing their traditional hedging role against negative jump risk.

As we increase λ^Q/λ in Table 1, the jump-risk premium increases. At some point, there is a switch between the relative attractiveness of jump and diffusive risks. This is indeed the outcome for some of the cases in Table 1. That is, instead of buying puts, the investor starts writing put options ($\psi^* < 0$) to earn the high premium associated with jump risk. At the same time, his holding of the risky stock decreases along with the decreasing attractiveness of diffusive risk.[16]

Finally, it is interesting to notice that, for some of the cases in Table 1, this switch in relative attractiveness never happens, regardless of the magnitude of λ^Q/λ. For example, we see that the put option continues to play its hedging role for the last jump case for the investor with $\gamma = 0.5$. Using our earlier discussion of the "equally attractive" condition (26), this implies that the jump magnitude in this case is so large that $1 + \mu\eta/\gamma < 0$ for the given level of η and γ.

5.4. Portfolio improvement

In this section, we compare the certainty-equivalent wealth of an investor with access to the derivatives market with that of a stock-only investor. Suppose that, at time 0, the investor starts with initial wealth of W_0 and has an investment horizon of

[15] In particular, in the presence of the −25% and −50% jumps, the endogenously determined portfolio bound kicks in. Specifically, the associated portfolio weights are determined by imposing the constraint that $1 + \mu\phi \geqslant 0$.

[16] It should be noted that part of the reason for this reduction in stock holding is to correct for the "delta" exposure introduced by writing the put. See the last paragraph in Section 5.2.

T years. With access to derivatives, his certainty-equivalent wealth is

$$\mathscr{W}^{*} = W_0 \exp\left(rT + \left[\frac{\gamma}{2}\left(\frac{\eta}{\gamma}\right)^2 + \frac{\gamma}{1-\gamma}\lambda^Q \left(\left(\frac{\lambda}{\lambda^Q}\right)^{1/\gamma} + \frac{1}{\gamma}\left(1 - \frac{\lambda}{\lambda^Q}\right) - 1 \right) \right] \bar{v} T \right). \tag{28}$$

The indirect utility for this special case can be solved in a couple of ways. One is by a derivation similar to that leading to Proposition 1 with the simplifying condition that $V_t \equiv \bar{v}$. Alternatively, one can take advantage of our existing solution, particularly the ordinary differential equations (A.1) for h and H, and take the limit to the case of constant volatility.

Without access to derivatives, the investor's certainty-equivalent wealth is

$$\mathscr{W}^{*\text{no-op}} = W_0 \exp\left(rT + \left[(\eta - \lambda^Q \mu)\phi^{*} - \frac{\gamma}{2}\phi^{*2} + \frac{\lambda}{1-\gamma}\left((1+\phi^{*}\mu)^{1-\gamma} - 1\right) \right] \bar{v} T \right), \tag{29}$$

where ϕ^{*}, solved from (27), is the optimal stock position of the stock-only investor Liu et al. (2003).

The investor with access to the derivative security cannot do worse than the stock-only investor. Hence $\mathscr{W}^{*} \geqslant \mathscr{W}^{*\text{no-op}}$. The equality holds if the "equally attractive" condition (26) holds, that is, when the investor has no incentive to disentangle his exposures to the two risk factors.

A quantitative analysis of the portfolio improvement from including derivatives is summarized in Table 2. Adopting the notation developed in Section 4.4, we use $\mathscr{R}^{\mathscr{W}}$

Table 2
Portfolio improvement for including derivatives

Jump cases		$\mu = -10\%$ Every 10 yr	$\mu = -25\%$ Every 50 yr	$\mu = -50\%$ Every 200 yr
γ	λ^Q/λ	$\mathscr{R}^{\mathscr{W}}$ (%)	$\mathscr{R}^{\mathscr{W}}$ (%)	$\mathscr{R}^{\mathscr{W}}$ (%)
	1	2.11	8.62	16.74
0.5	2	0.13	5.97	15.14
	5	11.78	1.84	11.28
	1	0.26	0.43	0.71
3	2	0.28	0.06	0.46
	5	7.68	0.46	0.09
	1	0.15	0.24	0.37
5	2	0.19	0.02	0.22
	5	5.12	0.36	0.02
	1	0.08	0.12	0.17
10	2	0.10	0.01	0.09
	5	2.77	0.22	0.003

to measure the improvement in terms of the annualized, continuously compounded return in certainty- equivalent wealth. Table 2 can be best understood by comparing the related optimal strategies in Table 1. When derivatives are used to hedge the exposure to the jump risk, the more aggressive investor benefits more from having access to derivatives. This is because, in the absence of jump risk, the more aggressive investor typically would like to take larger stock positions. The presence of jump risk inhibits too leveraged a position. With the help of derivatives, however, the investor is again freer to choose his optimal exposure to the diffusive risk. For the same reason, the improvement from including derivatives decreases when the jump-risk premium increases and the diffusive-risk premium decreases. For example, in the last jump case, the investor with $\gamma = 0.5$ buys put options to hedge out his jump-risk exposure. His improvement in certainty-equivalent wealth is 16.74% a year when jump risk is not compensated. When λ^Q/λ increases to 5, his improvement in certainty-equivalent wealth decreases to 11.28%.

This, however, is not the case when the relative attractiveness of the two risk factors switches, and the investor starts to use derivatives as a way to obtain positive exposure to jump risk. For example, in the first jump case, the investor with $\gamma = 3$ starts writing put options when λ^Q/λ increases to 2. His improvement in certainty-equivalent wealth is 0.28% a year. When λ^Q/λ increases to 5, however, he writes more put options, and his improvement in certainty-equivalent wealth increases to 7.68% a year.

6. Conclusion

In this paper, we studied the optimal investment strategy of an investor who can access not only the bond and the stock markets, but also the derivatives market. Our results demonstrate the importance of including derivative securities as an integral part of the optimal portfolio decision. The analytical nature of our solutions also helps establish direct links between the demand for derivatives and their economic sources.

As a vehicle to additional risk factors such as stochastic volatility and price jumps in the stock market, derivative securities play an important role in expanding the investor's dimension of risk-and-return tradeoffs. In addition, by providing access to volatility risk, derivatives are used by non-myopic investors to take advantage of the time-varying nature of their opportunity set. Similarly, by providing access to jump risk, derivatives are used by investors to disentangle their simultaneous exposure to diffusive and jump risks in the stock market.

Although our analysis focuses on volatility and jump risks, our intuition can be readily extended to other risk factors that are not accessible through positions in stocks. The risk factor that gives rise to a stochastic predictor is such an example. If, in fact, there are derivatives providing access to such additional risk factors, then demand for the related derivatives will arise from the need to take advantage of the associated risk-and-return tradeoff, as well as the time-varying investment opportunity provided by such risk factors.

Appendix A. Proof of Proposition 1

The proof is a standard application of the stochastic control method. Suppose that the indirect utility function J exists, and is of the conjectured form in (13). Then the first-order condition of the HJB Equation (11) implies that the optimal portfolio weights ϕ^* and ψ^* are indeed as given by (18)and (19), respectively.

Substituting (13), (18), and (19) into the HJB equation (11), one can show that the conjectured form (13) for the indirect utility function J indeed satisfies the HJB equation (11) if the following ordinary differential equations are satisfied:

$$\frac{\mathrm{d}h(t)}{\mathrm{d}t} = \kappa\bar{v}H(t) + \frac{1-\gamma}{\gamma},$$
$$\frac{\mathrm{d}H(t)}{\mathrm{d}t} = \left(-\kappa + \frac{1-\gamma}{\gamma}\left(\eta\rho + \xi\sqrt{1-\rho^2}\right)\sigma\right)H(t) + \frac{\sigma^2}{2}H(t)^2 + \frac{1-\gamma}{2\gamma^2}(\eta^2+\xi^2)$$
$$+ \lambda^Q\left[\left(\frac{\lambda}{\lambda^Q}\right)^{1/\gamma} + \frac{1}{\gamma}\left(1-\frac{\lambda}{\lambda^Q}\right) - 1\right]. \tag{A.1}$$

Using the solutions provided in (14) for H and h, it is a straightforward calculation to verify that this is indeed true. □

Appendix B. Appendix to Section 4

B.1. Option pricing

Option pricing for the stochastic-volatility model adopted in this paper is well established by Heston (1993).Using the notation established in Section 2, and letting $\kappa^* = \kappa - \sigma\left(\rho\eta + \sqrt{1-\rho\xi}\right)$ and $\bar{v}^* = \kappa\bar{v}/\kappa^*$ be the risk-neutral mean-reversion rate and long-run mean, respectively, the time-t prices of European-style call and put options with time τ to expiration and striking at K are

$$C_t = c(S_t, V_t; K, \tau); \;\; P_t = p(S_t, V_t; K, \tau), \tag{B.1}$$

where S_t is the spot price and V_t is the market volatility at time t, and where

$$c(S, V;\; K, \tau) = S\mathscr{P}_1 - \mathrm{e}^{-r\tau}K\mathscr{P}_2,$$

and, by put/call parity, the put pricing formula is

$$p(S, V;\; K, \tau) = \mathrm{e}^{-r\tau}K(1-\mathscr{P}_2) - S(1-\mathscr{P}_1).$$

Very much like the case of Black and Scholes (1973), $\mathscr{P}_1$ measures the probability of the call option expiring in the money, while $\mathscr{P}_2$ is the adjusted probability of the same event.

Specifically,

$$\mathscr{P}_1 = \frac{1}{2} - \frac{1}{\pi}\int_0^{\infty} \frac{\mathrm{d}u}{u}\mathrm{Im}\left(\mathrm{e}^{A(1-\mathrm{i}u)+B(1-\mathrm{i}u)V}\mathrm{e}^{\mathrm{i}u(\ln K - \ln S + r\tau)}\right),$$

$$\mathscr{P}_2 = \frac{1}{2} - \frac{1}{\pi}\int_0^{\infty} \frac{\mathrm{d}u}{u}\mathrm{Im}\left(\mathrm{e}^{A(1-\mathrm{i}u)+B(1-\mathrm{i}u)V}\mathrm{e}^{\mathrm{i}u(\ln K - \ln S + r\tau)}\right), \tag{B.2}$$

where $\mathrm{Im}(\cdot)$ denotes the imaginary component of a complex number, and where, for any $y \in \mathrm{C}$.

$$B(y) = -\frac{a(1-\exp(-qt))}{2q-(q+b)(1-\exp(-qt))},$$

$$A(y) = -\frac{\kappa^* \bar{v}^*}{\sigma^2}\left((q+b)\tau + 2\ln\left[1 - \frac{q+b}{2q}(1-\mathrm{e}^{-q\tau})\right]\right), \tag{B.3}$$

where $b = \sigma\rho y - \kappa^*$, $a = y(1-y) - 2\lambda^Q(\exp(y)(1+\mu) - 1 - y\mu)$ and $q = \sqrt{b^2 + a\sigma^2}$.

Connecting to the notation $O_t = g(S_t, V_t)$ adopted in Section 2, we can see that for a call option, g is simply c, while for a straddle, $g(S_t, V_t) = c(S_t, V_t;\ K, \tau) + p(S_t, V_t;\ K, \tau)$.

B.2. The indirect utility of a no-option investor

A "no-option" investor solves the same investment problem as that in (7) and (8) with the additional constraint that he cannot invest in the derivatives market. That is, $\psi_t \equiv 0$. This problem is solved extensively in Liu (1998). For completeness of the paper, the following summarizes the results that are useful for our analysis of portfolio improvement in Section 4.4.

At any time t, the indirect utility of a "no-option" investor with a year-T investment horizon is

$$J^{\text{no-op}}(W_t, V_t, t) = \frac{W_t^{1-\gamma}}{1-\gamma}\exp(\gamma h^{\text{no-op}}(T-t) + \gamma H^{\text{no-op}}(T-t)V_t), \tag{B.4}$$

where $h^{\text{no}-\text{op}}(\cdot)$ and $H^{\text{no}-\text{op}}(\cdot)$ are time-dependent coefficients that are independent of the state variables:

$$h^{\text{no-op}}(t) = \frac{2\kappa\bar{v}}{\sigma^2(\rho^2 + \gamma(1-\rho^2))}\ln\left(\frac{2k_2\exp((k_1+k_2)t/2)}{2k_2 + (k_1+k_2)(\exp(k_2 t)-1)}\right) + \frac{1-\gamma}{\gamma}rt,$$

$$H^{\text{no-op}}(t) = \frac{\exp(k_2 t) - 1}{2k_2 + (k_1+k_2)(\exp(k_2 t)-1)}\frac{1\text{-}\gamma}{\gamma^2}\eta^2, \tag{B.5}$$

where

$$k_1 = \kappa - \frac{1-\gamma}{\gamma}\eta\sigma\rho;\ k_2 = \sqrt{k_1^2 - \frac{1-\gamma}{\gamma^2}\eta^2\sigma^2(\rho^2 + (1-\rho^2)\gamma)}. \tag{B.6}$$

The certainty-equivalent wealth of such a "no-option" investor with initial wealth W_0 then becomes

$$\mathcal{W}^{\text{no-op}} = W_0 \exp\left(\frac{\gamma}{1-\gamma}[h^{\text{no-op}}(T) + H^{\text{no-op}}(T)V_0]\right). \tag{B.7}$$

B.3. Proof of Proposition 2

The indirect utility of an investor with access to derivatives is given in Proposition 1, while that of an investor without access to derivatives is provided in immediately above. It is then straightforward to verify that the portfolio improvement $\mathcal{R}^w$ is indeed of form (23). To show that the improvement is strictly positive for investors with $\gamma \neq 1$, let $DH(t) = H(t) - H^{\text{no-op}}(t)$, and one can show that

$$DH(t) = \frac{1-\gamma}{2}\exp(-y(t))\int_t^T \exp(-y(s))\left(\frac{\xi}{\gamma} - \sqrt{1-\rho^2}\sigma H^{\text{no-op}}(s)\right)^2 \mathrm{d}s,$$

where

$$y(t) = \int_t^T \left[\kappa + \frac{1-\gamma}{\gamma}\left(\eta\rho + \xi\sqrt{1-\rho^2\sigma}\right) + \frac{\sigma^2}{2}(H(s) + H^{\text{no-op}}(s))\right] \mathrm{d}s,$$

is finite for any $t \leqslant T$. Consequently, $DH(T)/(1-\gamma)$ is strictly positive. Moreover, it is straightforward to show that

$$\frac{Dh(t)}{1-\gamma} = \frac{h(t) - h^{\text{no-op}}(t)}{1-\gamma} = \kappa\bar{v}\int_0^t \frac{DH(s)}{1-\gamma}\,\mathrm{d}s. \tag{B.8}$$

As a result, $Dh(T)/(1-\gamma)$ is also strictly positive, making $\mathcal{W}^* > \mathcal{W}^{\text{no-op}}$ for any $\gamma \neq 1$.

For the log-utility case, the intertemporal hedging demand is zero. That is, $H(t) = 0$ and $H^{\text{no-op}}(t) = 0$ for any t .One can show that

$$\lim_{\gamma\to 1}\frac{H^{\text{no-op}}(t)}{1-\gamma} = \frac{1-\exp(-\kappa t)}{2\kappa}\eta^2; \quad \lim_{\gamma\to 1}\frac{H(t)}{1-\gamma} = \frac{1-\exp(-\kappa t)}{2\kappa}(\eta^2 + \xi^2).$$

Moreover, (B.8) also holds for the case of $\gamma = 1$, making $\mathcal{W}^* > \mathcal{W}^{\text{no-op}}$ when $\xi \neq 0$, and $\mathcal{W}^* = \mathcal{W}^{\text{no-op}}$ when $\xi = 0$.

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