Monotonicity in Asset Returns: New Tests with Applications to the Term Structure, the CAPM and Portfolio Sorts*

Andrew J. Patton
University of Oxford

Allan Timmermann
University of California San Diego

30 April 2009

Abstract

Many theories in finance imply monotonic patterns in expected returns and other financial variables: The liquidity preference hypothesis predicts higher expected returns for bonds with longer times to maturity; the CAPM implies higher expected returns for stocks with higher betas; and standard asset pricing models imply that the pricing kernel is declining in market returns. The full set of implications of monotonicity is generally not exploited in empirical work, however. This paper proposes new and simple ways to test for monotonicity in financial variables and compares the proposed tests with extant alternatives such as t-tests, Bonferroni bounds and multivariate inequality tests through empirical applications and simulations.

JEL Codes: G12, G14

1   Introduction

Finance contains many examples of theories which imply that expected returns should be monotonically decreasing or monotonically increasing in securities' risk or liquidity characteristics. For example, under the liquidity preference hypothesis, expected returns on treasury securities should increase monotonically with their time to maturity; the CAPM implies a monotonically increasing pattern in the expected return of stocks ranked by their market betas. Another fundamental implication of finance theory is that the pricing kernel should be monotonically decreasing in investors' ranking of future states as measured, e.g., by market returns.

The full set of implications of such monotonic patterns is generally not explored, however, in empirical analysis. For example, when testing the CAPM, it is conventional practice to form portfolios of stocks ranked by their beta estimates. A t-test may then be used to consider the mean return differential between the portfolios with the highest and lowest betas. Yet comparing only the average returns on the top and bottom portfolios does not provide a sufficient way to test for a monotonic relation between expected returns and betas. As an illustration, Figure 1 presents average monthly returns on stocks sorted into deciles according to their estimated betas. The mean return on the high-beta stocks exceeds that of the low-beta stocks, but a $t-$test on the top-minus-bottom return differential comes out insignificant. A test that only considers the return difference between the top and bottom ranked securities does not utilize the observation from Figure 1 that none of the declining segments—which go against the CAPM—appears to be particularly large, so the question arises whether the CAPM is in fact refuted by this evidence.

As a second illustration, Figure 2 shows the term premia on T-bills with a maturity between 2 and 11 months. Clearly the overall pattern in the term premium is increasing, and this is confirmed by a $t-$test on the mean differential between the 11- and 2-month bills which comes out significant. However, there are also segments where the term premium appears to be negative—particularly between the 9 and 10-month bills—so the question here is whether there are sufficiently many, and sufficiently large, negative segments to imply a rejection of the liquidity preference hypothesis. Only a test that simultaneously considers the mean returns across all maturities can answer this.

This paper proposes new ways to test for monotonicity in the expected returns of securities sorted by characteristics which theory predicts should earn a systematic premium. Our tests are nonparametric and easy to implement via bootstrap methods. Thus they do not require specifying

the functional form (e.g. linearity) relating the sorting variables to expected returns. This is important because for many economic models the underlying hypothesis is only that expected returns should rise or decline monotonically in one or more security characteristics that proxy for risk exposures or liquidity.

In common with a conventional one-sided $t-$test, our monotonic relation (MR) test holds that expected returns are identical or weakly declining under the null, while under the alternative we maintain a monotonically increasing relation. (Testing for a monotonic decreasing relation can of course be accomplished by simply re-ordering the assets.) Thus a rejection of the null of no relationship in favor of the hypothesized relationship (i.e. a finding of "statistical significance") represents a strong empirical endorsement of the theory. We also develop separate tests based on the sum of 'up' and 'down' moves. These combine information on both the number and magnitude of deviations from a flat pattern and so can help determine the direction of deviations in support of or against the theory.

The converse approach of maintaining a monotonically increasing relation under the null versus no such relationship under the alternative has been developed by Wolak (1987, 1989) and was also adopted by Fama (1984) in the context of a Bonferroni bound test to summarize the outcome of several $t-$tests. Depending on the research question and the economic framework, one may prefer to entertain the presence of a monotonic relationship under the null or under the alternative hypothesis. Richardson *et al.* (1992), for example, used the Wolak test to see if there was evidence against an upward-sloping term structure of interest rates, as predicted by the liquidity preference hypothesis.

Since the MR and Wolak tests use different ways to test the theory, outcomes from such tests are not directly comparable. One drawback of entertaining the hypothesized monotonic relationship under the null is that a confirmation of a theory from a failure to reject the null may simply be due to limited power for the test (due to a short time series of data, or due to noisy data, for example). This turns out to be empirically important as the Wolak test sometimes fails to reject the null in cases where the t-test and the MR test are able to differentiate between theories that find support in the data and those that do not. Conversely, in cases where the MR test has weak power, it may fail to reject the null—and thus fail to support the theory—even for expected return patterns that appear to be monotonic.

Empirically, our tests reveal many interesting findings. For the CAPM example shown in Figure

2

1, the MR test strongly rejects the null in favor of a monotonically increasing relationship between portfolio betas and expected returns. Consistent with this, the Bonferroni bound and Wolak tests fail to reject the null that expected returns increase in betas. Turning to the term structure example in Figure 2, the MR, Bonferroni and Wolak tests all fail to find evidence in support of the liquidity preference hypothesis as the term premia do not appear to be monotonically increasing in the maturity. Moreover, when applied to a range of portfolio sorts considered in the empirical finance literature, we find many examples where the difference in average returns between the top and bottom portfolios is highly significant, but the pattern in average returns across multiple portfolios is non-monotonic. This holds, for example, for decile portfolios sorted on short-term reversal, momentum or firm size.

Our tests are not restricted to monotonic patterns in the expected returns on securities sorted on one or more variables and can be generalized to test for monotonic patterns in risk-adjusted returns or in the factor loadings emerging from asset pricing models. They can also be adopted to test for piece-wise monotonic patterns, as in the case of the U-shaped relationship between fee waivers and mutual fund performance reported by Christoffersen (2001) or the U-shaped pricing kernels considered by Bakshi *et al.* (2009). Finally, using methods for converting conditional moments into unconditional ones along the lines of Boudoukh *et al.* (1999), we show that the approach can be used to conduct conditional tests of monotonicity.

The outline of the paper is as follows. Section 2 describes our new approach to testing for monotonic patterns in expected returns on securities ranked by one or more variables and compares it with extant methods. Section 3 uses the various methodologies to analyze a range of return series from the empirical finance literature. Section 4 uses Monte Carlo simulations to shed light on the behavior of the tests under a set of controlled experiments. Finally, Section 5 concludes.

2   Testing Monotonicity

This section first provides some examples from finance to motivate monotonicity tests. We next introduce the monotonic relationship test and compare it with extant alternatives such as a student $t-$test based on top-minus-bottom return differentials, the multivariate inequality test proposed by Wolak (1989) and the Bonferroni bound.

2.1 Monotonicity Tests in Finance

One of the most basic implications of financial theory is that the pricing kernel should be monotonically decreasing in investors' ordering of future states (Shive and Shumway (2009)). In empirical work, this implication is typically tested by studying pricing kernels as a function of market returns. Using options data, Jackwerth (2000) finds that this prediction is supported by the data prior to the October 1987 crash, where risk aversion functions were monotonically declining. However, it appears to no longer hold in post-crash data. Rosenberg and Engle (2002) also find evidence of a region with increasing marginal utility for small positive returns. These papers do not formally test monotonicity of the pricing kernel, however.

The practice of looking for monotonic patterns in expected returns on portfolios of stocks sorted by observables such as firm size or book-to-market ratio can be motivated by the fact that, although such variables are clearly not risk factors themselves, they may serve as proxies for unobserved risk exposures. For example, Berk, Green and Naik (1999) develop a model of firms' optimal investment choices where expected returns depend on a single risk factor. Estimating the true betas with regard to this risk factor requires knowing the covariance of each investment project in addition to the entire stock of ongoing projects—a task that is likely to prove infeasible. However, expected returns can be re-written in terms of observable variables such as the book-to-market ratio and size which become sufficient statistics for the risk of existing assets. Hence expected returns on portfolios of stocks sorted on these variables should be monotonically increasing in book-to-market value and monotonically decreasing in size. Similar conclusions are drawn from the asset pricing model developed by Carlson et al. (2004).

As a second illustration, in a model of momentum effects where growth rate risk rises with growth rates and has a positive price, Johnson (2002) shows that expected returns should be monotonically increasing in securities' past returns and uses decile portfolios to study this implication.

If factor loadings on risk factors are either observed or possible to estimate without much error, then tests based on the linear asset pricing model may be preferable on efficiency grounds. However, in situations where the nature (functional form) of the relationship between expected returns and some observable variable used to rank or sort is unknown, the linear regression approach may be subject to misspecification biases. Hence there is inherently a trade-off between the efficiency of

regression models that assume linearity, but make use of the full data, versus tests based on portfolio sorts that do not rely on this assumption. Tests of monotonicity between expected portfolio returns and observable stock characteristics such as book-to-market value or size offer a fairly robust way to test asset pricing models, although they should be viewed as joint tests of the hypothesis that the sorting variable proxies for exposure to the unobserved risk factor and the validity of the underlying asset pricing model.

2.2 Monotonicity and Inequality Tests

The problem of testing for the presence or absence of a monotonic pattern in expected returns can be transformed into tests of inequality restrictions on estimated parameters. Consider a simple example where decile portfolios have been formed by sorting stocks based on their past estimated market betas. Letting $r_{1,t}, ..., r_{10,t}$ be the associated returns on the decile portfolios listed in ascending order, the CAPM implies that the expected returns on these portfolios are increasing:

$$E\left[r_{10,t}\right] > E\left[r_{9,t}\right] > \cdots > E\left[r_{1,t}\right]. \tag{1}$$

If we define $\Delta_i \equiv E\left[r_{i,t}\right] - E\left[r_{i-1,t}\right]$, for $i = 2, ..., 10$, this implication can be re-written as

$$\Delta_i > 0 \ \ \text{for } i = 2, ..., 10. \tag{2}$$

Alternatively, consider a test of the liquidity premium hypothesis (LPH), as in Richardson, *et al.* (1992) and Boudoukh, *et al.* (1999). If we define the term premium as $E\left[r_t^{(\tau_i)} - r_t^{(1)}\right]$, where $r_t^{(\tau_i)}$ is the one-period return on a bond with maturity τ_i, the simplest form of the LPH implies

$$E\left[r_t^{(\tau_i)} - r_t^{(1)}\right] > E\left[r_t^{(\tau_j)} - r_t^{(1)}\right] \ \ \text{for all } \tau_i \geq \tau_j. \tag{3}$$

That is, term premia are increasing with maturity. If we define $\Delta_i \equiv E\left[r_t^{(\tau_i)} - r_t^{(1)}\right] - E\left[r_t^{(\tau_{i-1})} - r_t^{(1)}\right]$, then this prediction can be re-written as

$$\Delta_i > 0 \ \ \text{for } i = 2, ..., N. \tag{4}$$

We next propose a new and simple non-parametric approach that tests directly for the presence of a monotonic relation between expected returns and the underlying sorting variable(s) but does not otherwise require that this relationship be specified or known. This can be a great advantage in situations where standard distributions are unreliable guides for the test statistics, difficult to

compute, or simply unknown (Ang and Chen (2007)). Effectively our test allows us to examine whether there can exist a monotonic mapping from an observable characteristic used to sort stocks or bonds and their expected returns.

2.3 Testing for a Monotonic Relationship: A New Approach

Suppose we are considering the ranking of expected returns on $N+1$ securities. We take the number of portfolios, $N + 1$, as given and then show how a test can be conducted that accounts for the relationship between the complete set of portfolios (not just the top and bottom) and their expected returns. Denoting the expected returns by $\boldsymbol{\mu} = (\mu_0, \mu_1, ..., \mu_N)'$, and defining the associated return differentials as $\Delta_i = \mu_i - \mu_{i-1}$, we can use the link between monotonicity and inequality tests discussed above to consider tests on the parameter $\boldsymbol{\Delta} \equiv [\Delta_1, ..., \Delta_N]'$.

The approach proposed in this paper specifies a flat or weakly decreasing pattern under the null hypothesis, and a strictly increasing pattern under the alternative, without requiring any maintained assumptions on $\boldsymbol{\Delta}$:[1]

$$H_0 \quad : \quad \boldsymbol{\Delta} \leq \mathbf{0} \tag{5}$$
$$\text{vs.} \quad H_1 \quad : \quad \boldsymbol{\Delta} > \mathbf{0}.$$

Under this approach, the test is designed so that the alternative hypothesis is the one that the researcher hopes to prove, and in such cases it is sometimes called the "research hypothesis", see, e.g., Casella and Berger (1990). A theoretical prediction of a monotonic relationship is therefore only confirmed if there is sufficient evidence in the data to support it. This is parallel to the standard empirical practice of testing the significance of the coefficient of a variable hypothesized to have a non-zero effect in a regression.

The null and alternative hypotheses in (5) can be re-written as:

$$H_0 \quad : \quad \boldsymbol{\Delta} \leq \mathbf{0} \tag{6}$$
$$H_1 \quad : \quad \min_{i=1,..,N} \Delta_i > 0.$$

To see this, note that if the *smallest* value of $\Delta_i > 0$, then we must have that $\Delta_i > 0$ for *all* $i = 1, ..., N$. This motivates the following choice of test statistic:

$$J_T = \min_{i=1,..,N} \hat{\Delta}_i, \tag{7}$$

---

[1]Equalities and inequalities are interpreted as applying element-by-element for vectors.

where $\hat{\Delta}_i$ is based on the sample analogs $\hat{\Delta}_i = \hat{\mu}_i - \hat{\mu}_{i-1}$, $\hat{\mu}_i \equiv \frac{1}{T} \sum_{t=1}^{T} r_{it}$ and $\{r_{it}\}_{t=1}^{T}$ is the time series of returns on the ith security.

In Section 2.5 we discuss how we obtain appropriate critical values for this test statistic using a bootstrap procedure. We shall refer to the tests associated with hypotheses such as those in (6) as Monotonic Relationship (MR) tests. Testing that expected returns are monotonically decreasing can be done simply by reordering the assets.

Note that in (7) we consider all "adjacent" pairs of security returns, while we could also consider all possible pairwise comparisons, $E[r_{i,t}] - E[r_{j,t}]$ for all $i > j$. The latter approach increases the number of parameter constraints, and the size of the vector $\mathbf{\Delta}$, from N to $N(N+1)/2$. The adjacent pairs are sufficient for monotonicity to hold, but it is possible that considering all possible comparisons leads to empirical gains. We compare the adjacent pairs test to the "all pairs" test in our empirical analysis and Monte Carlo simulations. With the parameter $\mathbf{\Delta}$ suitably modified, the theory presented below holds in both cases.

2.3.1 Diagnostic Tests for Monotonicity

The proposed MR test for monotonicity is useful for detecting the presence or absence of a monotonic relationship between expected returns and some economic variable. However researchers may also be interested in other aspects of this relationship, and so we propose two statistics that provide further information. Suppose the MR test fails to reject the null in favor of a monotonically increasing relationship, yet visual inspection suggests that the relationship is "mostly" increasing. A useful test, then, would be one that determines whether the negatively-sloped parts of the pattern are significantly different from zero, which may help explain why the MR test does not reject the null. To that end, consider the following null and alternative:

$$
\begin{aligned}
H_0 &: \quad \mathbf{\Delta} = \mathbf{0} \\
\text{vs.} \quad H_1^- &: \quad \sum_{i=1}^{N} |\Delta_i|\, \mathbf{1}\left\{\Delta_i < 0\right\},
\end{aligned}
$$

where the indicator $\mathbf{1}\left\{\Delta_i < 0\right\}$ is one if $\Delta_i < 0$, and otherwise is zero. Here the null is a flat pattern (no relationship) and the alternative is that at least some parts of the pattern are strictly negative. By summing over all negative deviations, this statistic accounts for both the frequency

7

and magnitude of deviations from a flat pattern. The natural test statistic is:

$$J_T^- = \sum_{i=1}^{N} \left| \hat{\Delta}_i \right| \mathbf{1} \left\{ \hat{\Delta}_i < 0 \right\}. \tag{8}$$

As for the MR test, this "Down" test statistic does not have a standard limiting distribution under the null hypothesis, but critical values can again be obtained using a bootstrap approach, as described in Section 2.5.

The corresponding version of the test for cumulative evidence of an increasing pattern is:

$$
\begin{aligned}
H_0 &: \quad \boldsymbol{\Delta} = \mathbf{0} \\
\text{vs.} \quad H_1^+ &: \quad \sum_{i=1}^{N} |\Delta_i| \, \mathbf{1} \left\{ \Delta_i > 0 \right\},
\end{aligned}
$$

suggesting the "Up" test statistic

$$J_T^+ = \sum_{i=1}^{N} \left| \hat{\Delta}_i \right| \mathbf{1} \left\{ \hat{\Delta}_i > 0 \right\}. \tag{9}$$

Combining the results of the "Down" and "Up" tests in (8) and (9) leads to four possible outcomes: (i) neither 'Up', nor 'Down' test rejects, in which case we have no evidence of any relationship between expected returns and the sorting variable, suggesting a flat pattern; (ii) 'Up' test rejects, but 'Down' test fails to reject, in which case we have evidence consistent with a weakly increasing relationship; (iii) 'Down' test rejects, but 'Up' test fails to reject, in which case we have evidence consistent with a weakly decreasing relationship; (iv) 'Up' and 'Down' test both reject, in which case we have evidence of a non-monotonic relationship with up and down segments that are both significant.

When used in conjunction with the MR test, the "Down" and "Up" tests can be helpful in diagnosing the reasons for a rejection of—or a failure to reject—the underlying theory. In addition, these tests may be more powerful for some patterns of deviations from the hypothesized theory that lead to weak power for the MR test, e.g., many small deviations. The "Down" and "Up" tests should not be viewed as formal tests of monotonicity, however, and we recommend that they be used as diagnostic tests in conjunction with the MR test.

2.4 Extant Tests and the Choice of Null and Alternative Hypotheses

The MR test is closely related to earlier work on multivariate inequality tests by Bartholomew (1961), Kudo (1963), Perlman (1969), Gourieroux, et al. (1982) and Wolak (1987, 1989). Wolak

(1989) proposed a test that entertains (weak) monotonicity under the null hypothesis, and specifies the alternative as non-monotonic:

$$H_0 \quad : \quad \boldsymbol{\Delta} \geq \mathbf{0} \tag{10}$$

$$\text{vs.} \quad H_1 \quad : \quad \boldsymbol{\Delta} \text{ unrestricted.}$$

Here the theoretical prediction of monotonicity is contained in the null hypothesis and is only rejected if the data contain sufficient evidence against it. The test statistic in this approach is based on a comparison of an unconstrained estimate of $\boldsymbol{\Delta}$ with an estimate obtained by imposing weak monotonicity. Wolak (1987, 1989) shows that these test statistics have a distribution under the null that is a weighted sum of chi-squared variables, $\sum_{i=1}^{N} \omega(N, i) \chi^2(i)$, where $\omega(N, i)$ are the weights and $\chi^2(i)$ is a chi-squared variable with i degrees of freedom. Critical values are generally not known in closed form, but a set of approximate values can be calculated through Monte Carlo simulation. This procedure is computationally intensive and difficult to implement in the presence of large numbers of inequalities. As a result, the test has only found limited use in finance. Richardson *et al.* (1992) applied the method in Wolak (1989) to test for monotonicity of the term premium, and in our empirical work below we present the results of Wolak's test for comparison.

It is worth noting an important difference between the MR approach in equation (6) and that of Wolak (1989) in equation (10). In Wolak's framework, the null hypothesis is that there is a weakly monotonic relationship between expected returns and the sorting variable, while the alternative hypothesis contains the case of no such monotonic relationship. Depending on the research question and the economic framework, one may prefer to entertain the presence of a monotonic relationship under the null or under the alternative. One potential drawback of entertaining the hypothesized monotonic relationship under the null is that limited power (due to a short time series of data, or due to noisy data) will make it difficult to reject the null hypothesis and thus difficult to have much confidence in a confirmation of a theory from a failure to reject the null.[2] The MR approach, on the other hand, contains the monotonic relationship under the alternative, and thus a rejection of the null of *no* relationship in favor of the hypothesized relationship represents a strong empirical endorsement of the theory. Conversely, in cases where the MR test has weak power, it may fail to reject the null and so incorrectly fail to support the theory entertained under the alternative

---

[2]Furthermore, the null equation (10) includes the case of *no* relationship (when $\boldsymbol{\Delta} = \mathbf{0}$) and so a failure to reject the null could actually be the result of the absence of a relationship between expected returns and the sorting variable.

hypothesis. In such cases the 'Up' and 'Down' tests come in conveniently as they can help to diagnose if the problem is indeed lack of power.

Since the setup of the null and alternative hypothesis under the MR test in equation (5) is the mirror image of that under the Wolak test in (10), one cannot draw universally valid conclusions about which approach is 'best'. Rather, which test to use depends on the research question at hand. The MR test is more appropriate to use when the relevant question is "Does the data support the theory?" Conversely, the Wolak setup is more appropriate for a researcher interested in finding out if there is significant evidence in the data against some theory. In cases where this distinction is not clear, one could even consider inspecting both types of tests. There is strong support for the theory if the MR test rejects while the Wolak test fails to reject. Conversely, if the Wolak test rejects while the MR test fails to reject, this constitutes strong evidence against theory. Cases where both tests fail to reject constitute weak confirmation of the theory and could be due to the MR test having weak power. Finally, if both tests reject, they disagree about the evidence. We should note that we do not find a single case with this latter outcome in any of our empirical tests.

The MR test has greater *apparent* similarity to the setup of the multivariate one-sided tests considered by Bartholomew (1961), Kudo (1963), Perlman (1969), Gourieroux, *et al.* (1982) and labeled 'EI' in Wolak (1989):

$$H_0 \quad : \quad \boldsymbol{\Delta} = \mathbf{0}$$
$$\text{vs.} \quad H_1 \quad : \quad \boldsymbol{\Delta} \geq \mathbf{0}, \tag{11}$$

with at least one inequality strict, under the *maintained* hypothesis $H_m : \boldsymbol{\Delta} \geq \mathbf{0}$. The test statistic in this approach is based purely on an estimate of $\boldsymbol{\Delta}$ obtained by imposing the maintained assumption.[3] The main drawback of this framework, if one wishes to test for a monotonic relationship, is that if the true relationship is non-monotonic, then the behavior of the test is unknown, as the maintained hypothesis is then violated. In a Monte Carlo study of this test (available upon request) we found that it performed well when the maintained hypothesis was satisfied. However when this hypothesis is violated the finite-sample size of the test tends to be very high, likely due to the fact

---

[3]Kudo characterized the weights analytically in cases with up to four constraints under the assumption that the covariance matrix of the parameter estimator is known; Gourieroux *et al.* (1982) proposed simulation methods to compute critical values when the covariance matrix is unknown; and Kodde and Palm (1986) derived lower and upper bounds on the critical values for the test which avoids the need for simulations.

that this test is not designed to work when the maintained hypothesis of weak monotonicity is violated. This leads the test to overreject and so we do not consider this test further here.

Lastly, a naïve approach to testing the hypotheses in equation (10) would be to conduct a set of pair-wise $t-$tests to see if Δ_i is positive for each $i = 1, .., N$. Unfortunately, it is not clear how to summarize information from these N tests into a single number since the test statistics are likely to be correlated and their joint distribution is unknown. To deal with this problem, Fama (1984) proposed using a Bonferroni bound. This method analyzes whether the smallest t-statistic on $\hat{\Delta}_i$, $i = 1, .., N,$ falls below the lower-tail critical value obtained by using a bound on the probability of a Type I error. The technique is simple to implement but tends to be a conservative test of the null hypothesis. This is confirmed in a Monte Carlo study reported in Section 4.

2.5 A Bootstrap Approach to the MR Test

Under standard conditions, provided in detail in the appendix, the estimated parameter $\hat{\boldsymbol{\Delta}} = \left[\hat{\Delta}_1, ..., \hat{\Delta}_N \right]'$ will asymptotically follow a normal distribution, i.e., in large samples $(T \to \infty)$,

$$\sqrt{T} \left(\left[\hat{\Delta}_1, ..., \hat{\Delta}_N \right]' - [\Delta_1, ..., \Delta_N]' \right) \overset{a}{\sim} N(\mathbf{0}, \Omega). \tag{12}$$

Using this result would require knowledge, or estimation, of the full set of $N(N+1)/2$ parameters of the covariance matrix for the sample moments, Ω. These parameters influence the distribution of the test statistic even though we are not otherwise interested in them. Unfortunately, when the set of assets involved in the test grows large, the number of covariance parameters increases significantly and it can be difficult to estimate these parameters with much precision.

As shown in (7), we are interested in studying the minimum value of a multivariate vector of estimated parameters that is asymptotically normally distributed. Unfortunately, there are no tabulated critical values for such minimum values—precisely because these would depend on the entire covariance matrix, Ω. Furthermore, the asymptotic distribution may not provide reliable guidance to the finite sample behavior of the resulting tests.

To deal with the problem of not knowing the parameters of the covariance matrix or the critical values of the test statistic, we follow recent studies on financial time series such as Sullivan, *et al.* (1999) and Kosowski *et al.* (2006) and use a bootstrap methodology. As pointed out by White (2000), a major advantage of this approach is that it does not require estimating Ω directly. To see how the approach works in practice, let $\{r_{it},\ t = 1, ..., T; i = 0, 1, ..., N\}$ be the original set of

11

returns data recorded for $N + 1$ assets over T time periods. We first use the stationary bootstrap of Politis and Romano (1994) to randomly draw (with replacement) a new sample of returns $\{\tilde{r}_{i\tau(t)}^{(b)}, \tau(1), ..., \tau(T); i = 0, 1, ..., N\}$, where $\tau(t)$ is the new time index which is a random draw from the original set $\{1, .., T\}$. This randomized time index, $\tau(t)$, is common across portfolios in order to preserve any cross-sectional dependencies in returns. Finally, b is an indicator for the bootstrap number which runs from $b = 1$ to B. The number of bootstrap replications, B, is chosen to be sufficiently large that the results do not depend on Monte Carlo errors. Time-series dependencies in returns are accounted for by drawing returns data in blocks whose starting point and length are both random. Following Politis and Romano (1994), the block length is drawn from a geometric distribution, with a parameter that controls the average length of each block.

To implement the MR test, we need to obtain the bootstrap distribution of the parameter estimate $\hat{\boldsymbol{\Delta}}$ under the null hypothesis. The null in equation (6) is composite, and so following White (2000), we choose the point in the null space least favorable to the alternative, namely $\boldsymbol{\Delta} = \boldsymbol{0}$.[4] The null is imposed by subtracting the estimated parameter $\hat{\boldsymbol{\Delta}}$ from the parameter estimate obtained on the bootstrapped return series, $\hat{\boldsymbol{\Delta}}^{(b)}$. We then count the number of times where a pattern at least as unfavorable (i.e. yielding at least as large a value of J_T) against the null as that observed in the real data emerges. When divided by the total number of bootstraps, B, this gives the p-value for the test and allows us to conduct inference:[5]

$$
\begin{aligned}
J_T^{(b)} &= \min_{i=1,..,N} \left(\hat{\Delta}_i^{(b)} - \hat{\Delta}_i \right), \quad b = 1, 2, ..., B. \\
\hat{p} &= \frac{1}{B} \sum_{b=1}^{B} \mathbf{1} \left\{ J_T^{(b)} > J_T \right\}
\end{aligned}
\tag{13}
$$

When the bootstrap p-value is less than 0.05, we conclude that we have significant evidence against the null in favor of a monotonic increasing relationship. We implement a "studentized" version of this bootstrap, as advocated by Hansen (2005) and Romano and Wolf (2005). This eliminates the impact of cross-sectional heteroskedasticity in the portfolio returns, a feature that is prominent for some securities and may lead to gains in power.

Theorem 1, given in the Appendix, provides a formal justification for the application of the

---

[4] Analogously, in a simple one-sided test of a single parameter, $H_0 : \beta \leq 0$ vs. $H_1 : \beta > 0$, the point least favorable to the alternative under the null is zero.

[5] Matlab code to implement the tests proposed in this paper is available from http://www.economics.ox.ac.uk/members/andrew.patton/code.html.

bootstrap to our problem. In words, under a standard set of moment and mixing conditions on returns, the appropriately scaled vector of mean returns converges to a multivariate normal distribution. Moreover, inference about the minimum of a draw from this distribution can be conducted by means of the stationary bootstrap provided that the average block length grows with the sample size but at a slower rate.

2.6 Two-way Sorts

Expected returns on financial securities are commonly modeled as depending on multiple risk or liquidity factors. In this section we show that the MR test is easily generalized to cover tests of monotonicity of expected returns based on two-way sorts.

Suppose that the outcome of the two-way sort is reported in an $(N+1) \times (N+1)$ table with sorts according to one variable ordered across rows and sorts by the other variable listed down the columns. We are interested in testing the hypothesis that expected returns decrease along both the columns and rows. The proposition of no systematic relationship—which we seek to reject—is entertained under the null. To formalize the test, let the expected value of the return on the row i, column j security be denoted μ_{ij}:

$$H_0 : \mu_{i,j} \leq \mu_{i-1,j}, \ \mu_{i,j} \leq \mu_{i,j-1} \ \text{ for all } i, j. \tag{14}$$

The alternative hypothesis is that expected returns increase in both the row and column index:

$$H_1 : \mu_{i,j} > \mu_{i-1,j}, \ \mu_{i,j} > \mu_{i,j-1} \ \text{ for all } i, j. \tag{15}$$

Defining row $\Delta_{ij}^r = \mu_{i,j} - \mu_{i-1,j}$ and column $\Delta_{ij}^c = \mu_{i,j} - \mu_{i,j-1}$ differentials in expected returns, we can restate these hypotheses as

$$
\begin{aligned}
H_0 \ &: \ \Delta_{ij}^r \leq 0, \Delta_{ij}^c \leq 0, \ \text{ for all } i, j \\
\text{vs.} \quad H_1 \ &: \ \Delta_{ij}^r > 0 \text{ and } \Delta_{ij}^c > 0, \ \text{ for all } i, j,
\end{aligned}
\tag{16}
$$

or, equivalently,

$$H_1 : \min_{i,j=1,..,N} \{\Delta_{ij}^r, \Delta_{ij}^c\} > 0. \tag{17}$$

In parallel with the test for the one-way sort in (7), this gives rise to a test statistic

$$J_T = \min_{i,j=1,..,N} \{\hat{\Delta}_{ij}^r, \hat{\Delta}_{ij}^c\}. \tag{18}$$

The alternative hypothesis gives rise to $2N(N-1)$ non-redundant inequalities. For a 5×5 sort, this means 40 inequalities are implied by the theory of a monotonic relationship in expected returns along both row and column dimensions, whereas for a 10×10 sort 180 inequalities are implied. This shows both how potentially complicated and how rich the full set of relations implied by monotonicity can be when applied to returns sorted by two variables. If all pairs of returns are compared (not just the adjacent ones), we get $(\frac{1}{2}N(N+1))^2 - N^2$ inequalities, which for a 5×5 table yields 200 inequalities and for a 10×10 table yields 2,925 inequalities.[6]

2.7 Monotonic Patterns in Risk-Adjusted Returns or Factor Loadings

The MR methodology can be extended to test for monotonic patterns in parameters other than the unconditional mean. For example, in a performance persistence study one might be interested in testing that risk-adjusted returns, obtained via a maintained asset pricing model, are monotonically increasing (or decreasing) in past performance. Alternatively, a corporate finance model may imply that the sensitivity of returns (or sales, or free cash flow) to a credit constraint factor is monotonically decreasing in firm size. These examples, and our original specification above, are nested in the more general framework with K risk factors, $\mathbf{F}_t = (F_{1t}, ..., F_{Kt})'$:

$$
\begin{aligned}
r_{it} &= \boldsymbol{\beta}_i' \mathbf{F}_t + e_{it}, \quad i = 0, 1, ..., N \\
\boldsymbol{\beta}_i &\equiv [\beta_{1i}, ..., \beta_{Ki}]',
\end{aligned}
\tag{19}
$$

with the associated hypotheses on the jth parameter in the above regression:

$$
\begin{aligned}
H_0 &: \quad \beta_{jN} \leq \beta_{jN-1} \leq ... \leq \beta_{j0} , \\
\text{vs.} \quad H_1 &: \quad \beta_{jN} > \beta_{jN-1} > ... > \beta_{j0} \quad (1 \leq j \leq K).
\end{aligned}
\tag{20}
$$

Our framework in the previous sections corresponds to regressing each portfolio return onto a constant and so emerges when $K = 1$ and $F_{1t} = 1$ for all t. A test for monotonic risk-adjusted returns could be conducted by regressing returns onto a constant and a set of risk factors (for example, the Fama-French three-factor model) and then testing that the intercept (the "alpha") from that regression is monotonically increasing. A test for monotonically increasing or decreasing

---

[6] These results are easily generalized to cases where the number of rows and columns differs. For an $N \times K$ table, there will be $2NK - K - N$ inequalities to test. Our results also generalize to sorts on three or more variables. For a D-dimensional sort, with N securities in each direction, the total number of inequalities amounts to $DN^{D-1}(N-1)$.

factor sensitivity can be obtained by regressing returns on a constant and the factor of interest, and possibly other "control" factors, and then testing that the coefficient on the factor of interest is monotonically increasing or decreasing.

In this general case, the bootstrap test is obtained by estimating the regression on the boot-strapped data:

$$\tilde{r}_{i\tau(t)}^{(b)} = \boldsymbol{\beta}_i^{(b)\prime} \mathbf{F}_{i\tau(t)}^{(b)} + e_{i\tau(t)}^{(b)}, \quad i = 0, 1, ..., N. \tag{21}$$

Note that the explanatory variables in this regression are also shuffled using the same time index as the returns. For each bootstrap sample an estimate of the coefficient vector is obtained. The null hypothesis is imposed by subtracting the corresponding estimate from the original data. From the re-centered bootstrapped estimates, $\hat{\boldsymbol{\beta}}_i^{(b)} - \hat{\boldsymbol{\beta}}_i$, the test statistic for the bootstrap sample can be computed:

$$J_{j,T}^{(b)} \equiv \min_{i=1,...,N} \quad \left[\left(\hat{\beta}_{j,i}^{(b)} - \hat{\beta}_{j,i} \right) - \left(\hat{\beta}_{j,i-1}^{(b)} - \hat{\beta}_{j,i-1} \right) \right] \tag{22}$$

By generating a large number of bootstrap samples the empirical distribution of $J_{j,T}^{(b)}$ can be used to compute an estimate of the p-value for the null hypothesis, as in the simpler case presented in Section 2.5. The theorem in the Appendix is for this more general regression case, and is based on the work of White (2000) and Politis and Romano (1994).

2.8   Conditional Tests

Asset pricing models often take the form of conditional moment restrictions and so it is of interest to see how our tests can be generalized to this setting. Following Boudoukh *et al.* (1999), such a generalization is easily achieved by using the methods for converting conditional moment restrictions into unconditional moment restrictions commonly used in empirical finance.

To see how this works, let z_t be some instrument used to convert an unconditional moment condition into a conditional one. This instrument could take the form of an indicator variable that captures specific periods of interest corresponding to some condition being satisfied (e.g., the economy being in a recession) but could take other forms as well. The first step of a conditional version of our test would consist of pre-multiplying the set of returns, r_{it}, by z_t. In a second step, the test is conducted on the unconditional moments of the modified data $\tilde{r}_{it} = r_{it} \times z_t$ along the lines proposed above.

15

This type of test is relevant in a variety of settings. For example, Lettau and Ludvigson (2001) argue that value stocks are riskier than growth stocks because their returns are more highly correlated with consumption growth when risk aversion (proxied by their "cay" variable) is low. Similarly, Zhang (2005) argues that risk increases monotonically with book-to-market, but only in bad states. Using proxies for risk aversion or bad states, one could thus use the properly modified MR test to carry out conditional tests on the slope coefficients capturing risk exposure.

3   Empirical Results

Having introduced the various tests in the previous section, we next revisit a range of examples from the finance literature. We compare the outcome of tests based on our new monotonic relationship (MR) test or the "Up" and "Down" tests to a standard $t-$test, the Wolak (1989) test and the Bonferroni bound.

We first consider empirical tests of the CAPM. An investor believing in this model would hold strong priors that expected stock returns and subsequent estimates of betas should be uniformly increasing in past estimates of betas, and so the CAPM is well suited to illustrate our methodology. We next consider the liquidity preference hypothesis, which conjectures that expected returns on treasury securities rise monotonically with the time to maturity. Finally, we extend our analysis to a range of portfolio sorts previously considered in the empirical finance literature. In all cases we use 1000 bootstrap replications for the bootstrap tests and we choose the average block length to be ten months, which seems appropriate for returns data that display limited time-series dependencies at the monthly horizon. Finally, we use 1000 Monte Carlo simulations to obtain the weight vector, $\omega(N, i)$, used to compute critical values in Wolak's (1989) test.

3.1   Portfolio Sorts on CAPM Beta: Expected Returns

We now present the results of tests for a relationship between ex-ante estimates of CAPM beta and subsequent returns, using the same data as in Ang, Chen and Xing (2006), which runs from July 1963 to December 2001.[7] At the beginning of each month stocks are sorted into deciles on the basis of their beta estimated using one year of daily data, value-weighted portfolios are formed, and returns on these portfolios in the subsequent month is recorded. If the CAPM holds, we would

---

[7]We thank the authors for providing us with this data.

expect to see a monotonically increasing pattern in average returns going from the low-beta to the high-beta portfolio.

A plot of the average returns on these portfolios is presented in Figure 1, and the results of tests for a relationship between historical beta and subsequent returns are presented in Table 1. Although the high-beta portfolio has a larger mean return than the low-beta portfolio, the spread is not significant and generates a t-statistic of only 0.34. The MR test, on the other hand, does reject the null of no relationship between past beta and expected returns in favor of a strictly increasing relationship, with a p-value of 0.04.

One possible reason for the ability of our test to detect a uniform relationship between beta and expected returns is that it considers all portfolio returns jointly. Another reason stems from the rising pattern in the standard deviation of beta decile portfolio returns: low beta portfolios have much lower standard deviation than high beta portfolios. By using the "studentized" version of our MR test statistic we are able to more efficiently estimate the pattern in these expected portfolio returns in a way that accounts for cross-sectional heteroskedasticity. Interestingly, neither the "Up" or "Down" test rejects the null individually, suggesting that in this case, the MR test—which looks at the largest deviation—has higher power than tests that consider the sum of signed deviations.

The Wolak and Bonferroni tests do not reject the null of a weakly increasing relationship between betas and expected returns and so are consistent with the conclusion from our MR test.

3.2   Portfolio Sorts on CAPM Beta: Post-ranked Betas

As an illustration of the methods in Section 2.7, we next examine whether the post-ranked betas of portfolios ranked by their *ex-ante* beta estimates from the previous section are monotonically increasing across portfolios. Failure of this property would suggest that past beta estimates have little predictive content over future betas, perhaps due to instability, thus making them inadequate for the purpose of testing the CAPM. For this reason it is common to check monotonicity of the post-ranked betas, see, e.g. Fama and French (1992).

As above, at the beginning of each month stocks are sorted into deciles on the basis of their beta calculated using one year of daily data, value-weighted portfolios are formed, and returns on these portfolios in the subsequent month is recorded. If betas are stable over time and estimated without too much error, we would expect to see a monotonically increasing pattern in post-ranked betas. We compute the post-ranked betas using the realized monthly returns on the decile portfolios ranked

17

by ex-ante betas. Specifically, denoting the return on the ith (ex-ante sorted) decile portfolio as r_{it}, we estimate the following regressions:

$$r_{it} = \alpha_i + \beta_i r_{mt} + e_{it}, \ \ i = 1, 2, .., 10$$

and, using the theory discussed in Section 2.7, test the hypothesis

$$H_0 \ \ : \ \ \beta_{10} \leq \beta_9 \leq \cdots \leq \beta_1$$
$$\text{vs.} \ \ H_1 \ \ : \ \ \beta_{10} > \beta_9 > \cdots > \beta_1.$$

Panel C of Table 1 presents the results. As might be expected, the post-ranked beta estimates vary substantially from a value of 0.60 for the stocks with the lowest historical beta estimates to a value of 1.54 for the stocks with the highest historical beta estimates. This difference in betas is large and significant with a p-value of 0.00. Moreover, the MR test in panel D confirms that the pattern is indeed monotonically increasing across the portfolios: the null of no relationship is strongly rejected in favor of a monotonically increasing relationship, with a p-value of 0.003.

3.3 Testing Monotonicity of the Term Premium

In a series of papers, Fama (1984), McCulloch (1987) and Richardson *et al.* (1992) explored the implication of the liquidity preference hypothesis that expected returns on treasury securities should be higher, the longer their time to maturity. As is clear from equation (3), this fits directly with our framework.

To test this theory, Fama (1984) used a Bonferroni bound based on individual $t-$tests applied to term premia on T-bills with a maturity up to 12 months. He found evidence against monotonicity of the term premium as 9-month bills earned a higher premium than bills with longer maturity, particularly as compared with 10-month bills. McCulloch (1987) argued that this finding was explained by the unusual behavior of the bid-ask spread of 9-month bills during 1964-72. Subsequently, Richardson *et al.* (1992) analyzed monotonicity in the term structure using the Wolak test applied to bills with a maturity ranging from 2 to 11 months. For the period 1964-1990, they found that the Wolak test strongly rejects the null of a monotonically increasing pattern. However, this rejection appeared to be confined to the 1964-72 period as monotonicity was not rejected in subsamples covering the period 1973-1990.

We revisit the liquidity preference hypothesis by inspecting term premia on T-bills over the period 1964-2001, the longest available sample from the CRSP monthly treasuries files. Like Richardson *et al.* (1992), we restrict our analysis to maturities between 2 and 11 months. Table 2 presents the results. Panel A shows the average term premia as a function of maturity. Over the full sample the spread in term premia between 11- and 2-month bills is 0.05% per month or 0.6% per annum. Panel B shows that the associated $t-$statistic equals 2.42 and hence is statistically significant. Turning to the tests for monotonicity, both the Wolak and Bonferroni tests reject the null of an increasing term structure, while the MR test fails to find evidence in favor of a monotonically increasing term structure. These results are all consistent with the presence of some declining segments in the term structure. Figure 2 shows that, consistent with the earlier studies, the culprit appears to be the high term premium on 9-month bills.

When conducted separately on the subsample 1964-72, very similar conclusions emerge. In this subsample, a monotonically rising term structure is clearly rejected by both the Wolak and Bonferroni tests and the MR test also finds no evidence to support uniformly increasing term premia. In sharp contrast, for the period 1973-2001, the Bonferroni and Wolak tests both fail to reject the null of an increasing term premium. The inability of the MR test to reject the null against an increasing term premium may simply reflect low power of this test in this example. In support of this interpretation, notice that the "Up" test finds significant evidence of segments with a strictly increasing term premium, while conversely the "Down" test fails to find significant evidence of decreasing segments.

3.4   One-way Portfolio sorts: Further Evidence

It is common practice to inspect mean return patterns based on portfolios sorted on firm or security characteristics. For many sorts, the implications of theory are not as clear-cut as in the case of the CAPM and so should be viewed as joint tests that the sorting variable proxies for exposure to some unobserved risk factor and the validity of the underlying asset pricing model.

To illustrate how our approach can be used in this context, we consider returns on a range of portfolios sorted on firm characteristics such as market equity (size), book-to-market ratio, cashflow-price ratio, earnings-price ratio and the dividend yield or past returns over the previous month (short-term reversal) 12 months (momentum) or 60 months (long-term reversal). Data on value-weighted portfolio returns are obtained from Ken French's web site at Dartmouth College

and comprise stocks listed on NYSE, AMEX and NASDAQ. The findings on cross-sectional return patterns in portfolios sorted on various firm characteristics reported by Fama and French (1992) were based on data starting in July 1963 and we keep this date as our starting point. However, we also consider the earliest starting point for each series which is 1926 or 1927 except for the portfolios sorted on long-term reversal which begin in 1931 and the portfolios sorted on the earnings-price or cashflow-price ratios which begin in 1951. In all cases the data ends in December 2006. We focus our discussion on the more recent sample 1963-2006.

Panel A of Table 3 reports estimates of expected returns for the decile portfolios sorted on the eight variables listed in the columns. We preserve the order of the portfolios reported by Ken French. This means that we are interested in testing an increasing relationship between portfolio rank and expected returns for the portfolios sorted on book-to-market, cashflow-price, earnings-price, dividend yield and momentum. Conversely we are interested in testing for a decreasing relationship for the portfolios sorted on size, short-term reversal and long-term reversal.

For the portfolios sorted on the book-to-market, cash flow-price or earnings-price ratios or long term reversal there are either no reversals of the monotonic pattern, or few and smaller ones compared with those observed for the other sorts. There are larger reversals in average returns for the size-sorted portfolios (three reversals of up to five basis points per month) as well as for the portfolios sorted on dividend yield (five reversals of up to nine basis points), momentum (a 10 basis point decrease) and short-term reversal (an increase of 14 basis points). Hence, for some portfolio sorts a monotonic pattern is observed in average returns while for other portfolio sorts non-monotonic patterns of varying degrees arise. Moreover, there are large differences in the magnitude of the top-bottom spreads which range from a minimum of seven basis points per month for the portfolios sorted on the dividend yield to nearly 150 basis points for the portfolios sorted on momentum. A key question is clearly how strong the deviations from a monotonic pattern must be in order for us to reject the null hypothesis and establish monotonicity in expected returns.

To answer this question, Panel B presents test results for the eight portfolio sorts. The first row reports the t-statistic for testing the significance of the difference in expected returns between the top and bottom portfolios. These range from 0.30 (for portfolios sorted on the dividend yield) to 5.7 in the case of the momentum-sorted portfolios. The associated $p-$values show that the portfolios sorted on the dividend yield fail to produce a statistically significant top-bottom spread. Moreover, the spread in average returns on the size-sorted portfolios is borderline insignificant with a p-value

of 0.06. The remaining portfolios generate significant spreads.

To see if the mean return patterns are consistent with monotonicity in expected returns, the third row in panel B reports the bootstrapped p-values associated with the MR test. This test fails to find a monotonic relationship between expected returns and portfolios ranked by short term reversal (p-value of 0.26), momentum (p-value of 0.29), size (p-value of 0.27) or the dividend yield (p-value of 0.34). Only for the portfolios sorted on long term reversal and the book-to-market, cash flow-price and earnings-price ratios do we continue to find strong evidence of a monotonic pattern in expected returns. The fourth row in panel B shows that the MR tests based on comparing only the adjacent portfolios versus comparing all possible pairs always lead to the same conclusions.

The MR test, of course, accounts for the effects of random sampling variation. This has important implications. For example, for the portfolios sorted on the earnings-price ratio where three reversals appear in the ordering of average returns, the test still rejects very strongly because these reversals are small in magnitude (less than two basis point per decile portfolio) relative to the sampling variability of the average returns. In contrast, the relationship between expected returns and the dividend yield or momentum are insignificant. This is to be expected given the large reversals in the mean return patterns observed for the portfolios sorted on this variable.

For all eight portfolio sorts, Wolak's (1989) test fails to reject the null of a (weakly) monotonic relationship. Moreover, this conclusion is supported by the multivariate inequality test based on the Bonferroni bound which always equals one. Even the conventional t-test found no evidence of a monotonic pattern for the portfolios sorted on the dividend yield and so this evidence illustrates the difficulty that may arise in interpreting the Wolak and Bonferroni test: Failure to find evidence against a weakly monotonic pattern in the sorted portfolio returns may simply reflect weak power.

Similar results are obtained in the longer samples listed in Panels C and D, although the MR test now also finds evidence in support of monotonicity for the portfolios sorted on size, momentum and short-term reversal, but now fails to reject the null for the portfolios sorted on long-term reversal. As before, in each case the Bonferroni and Wolak tests fail to reject the null.

3.5 Two-way Portfolio Sorts

As an illustration of how our methodology can be extended, we next consider two-way sorts that combine portfolios sorted on firm size with portfolios sorted on either the book-to-market ratio or momentum. In both cases we study 5×5 portfolio sorts. Results, reported in Table 4, are for the

period from July 1963 to December 2006. Inspecting the mean returns, there is some evidence of a non-monotonic pattern across size-sorted portfolios for the quintiles with a low book-to-market ratio tracking growth stocks (panel A) or stocks with poor past performance (panel B).

Because the two-way sorts involve a large number of inequalities, it is useful to decompose the overall (joint) test into a series of conditional tests that help identify the economic source of the results. Table 4 therefore uses the MR test to examine patterns in expected returns keeping one sorting variable (e.g., size) constant while varying another (e.g., the book-to-market ratio) or vice versa. For example, the penultimate row in each panel presents the p-values from the tests for a monotonic relationship between the portfolio sorting variable in the row (size) and expected returns, conditional on keeping the column portfolio fixed. Hence the p-value of a test for the size effect, conditional on being in the top book-to-market quintile (value stocks) is 0.031, while it is 0.687 for the bottom book-to-market portfolio (growth stocks). The final row in each panel presents the p-value from a joint test for a monotonic relationship between the sorting variable in the row (size), computed across all column portfolios. Similarly, the penultimate column presents results from tests for a monotonic relationship between the portfolio sorting variable in the columns and expected returns, conditional on the row portfolio, representing firm size.

The results in Table 4 shed new light on the earlier findings. For example, panel A reveals that the value effect is quite strong among all size portfolios (with p-values ranging from 0.004 to 0.057). Hence the statistical evidence appears to support the conclusion in Fama and French (2006) that there is a value effect even among large stocks, although the spread in the top-minus-bottom portfolios' average returns is much wider for the smallest stocks than for the largest stocks. Conversely, the size effect is only significant among value firms; for the other book-to-market sorted portfolios the size effect is non-monotonic. Overall, the joint test fails to find evidence in support of a size and book-to-market effect.

Both the Wolak and Bonferroni tests failed to reject the null of monotonic patterns in both the size and book-to-market dimensions, with $p-$values close to one. Again this highlights the different conclusions that can emerge depending on whether monotonicity is entertained under the null or alternative hypothesis.

The results for the size and momentum two-way sorts provide strong support for a momentum effect among the four quintiles with the smallest stocks, but fails to find a momentum effect for the largest stocks, for which the MR test records a $p-$value of 0.552. Similarly, there is evidence

of a size effect only for the three portfolios with the strongest past performance, but not among the two loser portfolios. Hence, it is not surprising that the tests for separate size and momentum effects both fail to reject, as does the overall joint test ($p-$value of 0.545).

Consistent with the results from the MR test, the Wolak test rejected the null that expected returns follow a monotonic pattern in both size and momentum. In contrast, the more conservative Bonferroni test failed to reject the null hypothesis of weak monotonicity.

To summarize, two-way sorts can be used to diagnose why empirical evidence may fail to support a hypothesized pattern in expected returns. Here our findings suggest that the size effect in expected returns is absent from growth firms and among loser stocks. They also suggest that momentum effects are strong for small and medium-sized firms but not among the largest quintile of stocks.

4   Performance of the Tests: A Simulation Study

The hypothesis tests proposed here are non-standard. Moreover, unlike the standard $t-$test for equal expected returns, there are no optimality results or closed-form distributions against which test statistics such as those in equations (7) or (18) can be compared and from which critical values can be computed. Since we are effectively in uncharted territory, we next undertake a series of Monte Carlo simulation experiments that offer insights into the finite-sample behavior of the proposed tests.

4.1   Monte Carlo Setup

The first set of scenarios covers situations where the hypothesized theory is valid and there is a monotonic relationship between portfolio rank and the portfolios' true expected returns. We would like the MR tests to reject the null of no systematic relationship in this situation (while the Wolak and Bonferroni tests should not reject) and the more often they reject, the more powerful they are.

Experiment I assumes monotonically increasing expected returns with identically sized increments between adjacent decile portfolios. Experiment II lets the expected return increase by 80% of the total from portfolio 1 through portfolio 5, and then increase by the remaining 20% of the total across the remaining 5 portfolios. Experiment III assumes a single large increase in the expected return from decile 1 to decile two, equal to 50% of the total increase, and then spreads the remain-

ing 50% of the increase across the 9 remaining portfolios. These three patterns are illustrated in the first column of Figure 3 and all have in common that the theory of a monotonic relation holds.

The second set of scenarios covers situations where the theory fails to hold and there is in fact a non-monotonic relationship between portfolio ranks and expected returns, so the MR test should not reject, while the Wolak and Bonferroni tests should reject.

Experiments IV-VIII all break the monotonic pattern in expected returns in some way: Experiment IV assumes an increasing but non-monotonic pattern with declines in expected returns for every second decile. Experiment V assumes a rising, then declining pattern in the expected return for a net gain in the expected return from the first to the tenth portfolio. The next two experiments assume a pattern where expected returns first rise and then decline so the expected return of the first and tenth deciles are identical, with the pattern being symmetric for Experiment VI, and being smoothly increasing then flat and finally sharply decreasing for Experiment VII. Finally, Experiment VIII assumes a mostly flat, jagged pattern in expected returns.

Each pattern is multiplied by a step size which varies from a single basis point per month to two, five and ten basis point differentials in the expected returns.

To ensure that our experiments are computationally feasible and involve both a sufficiently large number of Monte Carlo draws of the original returns and a sufficient number of bootstrap iterations for each of these draws, we focus on a one-dimensional monotonic pattern with $N = 10$ assets. We present results based on two sets of assumptions: The first is based on a Normality assumption, while the second set of results are based on more realistic data, where we use the bootstrap to reshuffle the true returns on the size-sorted decile portfolios and use these as our Monte Carlo simulation data. We draw 2500 bootstrap samples of the original returns. As in our empirical work, for each simulated data set we then employ $B = 1000$ replications of the stationary bootstrap of Politis and Romano (1994) and use 1000 Monte Carlo simulations to get the weights required for Wolak's (1989) test.

4.2 Analytical results under Normality

In order to obtain simple analytical results, we first make the assumption that the estimated differences in portfolio returns are independently and normally distributed:

$$\Delta\hat{\mu}_i \sim N\left(\Delta\mu_i, \frac{1}{T}\sigma_i^2\right), \text{ for } i = 2, ..., N \tag{23}$$

$$Corr\left[\Delta\hat{\mu}_i, \Delta\hat{\mu}_j\right] = 0 \text{ for all } i \neq j. \tag{24}$$

This setup allows us to present formulas for the power of the tests and establish intuition for which results to expect. Under the assumptions in equations (23) and (24), we can derive the power of the t-test analytically. First, note that the t-test is based on the difference between the mean returns of the Nth and the first portfolios, which is given by

$$\hat{\mu}_N - \hat{\mu}_1 = \sum_{i=2}^{N}\Delta\hat{\mu}_i \sim N\left(\sum_{i=2}^{N}\Delta\mu_i, \frac{1}{T}\sum_{i=2}^{N}\sigma_i^2\right).$$

Assuming that the variances are known, the t-statistic will thus be

$$tstat \equiv \frac{\sqrt{T}\left(\hat{\mu}_N - \hat{\mu}_1\right)}{\sqrt{\sum_{i=2}^{N}\sigma_i^2}} \sim N\left(\sqrt{T}\frac{\mu_N - \mu_1}{\sqrt{\sum_{i=2}^{N}\sigma_i^2}}, 1\right). \tag{25}$$

Under the null we have $\mu_N = \mu_1$ and so the t-statistic has the usual $N(0,1)$ distribution. Under the alternative hypothesis that $\mu_N > \mu_1$ the t-statistic will, as usual, diverge as $T \to \infty$. For finite T, the probability of rejecting the null hypothesis using a one-sided test with a 5% critical value is then

$$\Pr\left[tstat > 1.645\right] = \Phi\left(\sqrt{T}\frac{\mu_N - \mu_1}{\sqrt{\sum_{i=2}^{N}\sigma_i^2}} - 1.645\right), \tag{26}$$

where $\Phi(\cdot)$ is the *cdf* of a standard Normal distribution. Given a sample size, T, the vector of differences in expected returns $\Delta\boldsymbol{\mu} \equiv [\Delta\mu_2, ..., \Delta\mu_N]'$ and the vector of associated standard deviations $\boldsymbol{\sigma} \equiv [\sigma_2, ..., \sigma_N]'$ we can directly compute the power of the t-test.

For the MR test, the power is obtained as follows. Recall our test statistic:

$$J_T \equiv \min_{i=2,...,N} \Delta\hat{\mu}_i.$$

To obtain the distribution of this statistic under the null, we use 100,000 simulated draws to compute critical values, denoted $J_T^*(\boldsymbol{\sigma})$. The power of our test is then simply

$$\Pr\left[J_T(\Delta\boldsymbol{\mu}, \boldsymbol{\sigma}) > J_T^*(\boldsymbol{\sigma})\right].$$

25

We compute this power using 1000 simulated draws and set $T = 966$, which is the number of monthly returns on the "size" portfolios in the long sample in Table 3.

Results for this benchmark case are presented in the first column of Table 5 labeled "normal simulation". For experiments I, II and III the power of the t-test converges to one as the step size grows from 1 to 10 basis points. The probability of rejecting the null also approaches one for experiment IV, which assumes a non-monotonic but increasing pattern of expected returns.

The MR test has somewhat lower power than the t-test for the three experiments in which both tests should reject the null (experiments I, II and III). Compensating for the reduction in power, we observe that the probability that the bootstrap rejects the null in experiments IV–VIII—which would constitute a Type I error as these experiments do not have a monotonic pattern—goes to zero as the step size grows and never much exceeds the nominal size of the test. Thus the MR test is very unlikely to falsely reject the null hypothesis. In contrast, the t-test frequently rejects the null under experiments IV and V when the step size is comparable to that observed for the majority of portfolio sorts in the empirical analysis, i.e. 5-10 basis points. Of course, the t-test is not "wrong"; however it has a limited scope since it only compares the top and bottom portfolios and thus fails to detect non-monotonic patterns in the full portfolio sorts.

4.3 Bootstrap simulation results

The second set of columns in Table 5 present the results from the simulation based on bootstrap draws of monthly returns on the size sorted decile portfolios, so again $T = 966$. Broadly stated, the results from the t-test and MR test from these simulations are comparable to those obtained under the Normality assumption described previously. We also present the "Up" and "Down" tests presented in Section 2.3.1 along with the Wolak (1989) test and the Bonferroni bound test. Recall that Wolak's test and the Bonferroni-based test have a weakly monotonic relationship under the null hypothesis, and a non-monotonic relationship under the alternative. Thus, in contrast with the MR tests, these tests should *not* reject the null for returns generated under experiments I-III, while they should reject the null hypothesis under experiments IV-VIII.

The first panel, with step size set to zero, shows that the t, Bonferroni, Wolak, Up, Down, MR and MR$^{\text{all}}$ tests have roughly the correct size when there is genuinely no relationship between expected returns and portfolio rank, although most of the tests slightly over-reject the null hypothesis. This is not an unusual finding and mirrors those in simulation studies of the finite-sample size

of asset pricing tests, see e.g. Campbell, Lo and MacKinlay (1997), section 5.4.

Under experiment I, the t-test rejects slightly more frequently than the MR test. When the expected return differential increases by a single basis point per month for each decile portfolio, approximately 11-13% of the simulations correctly reject and this increases to around 20% under the two basis point differential. Under the five basis point return differential—which from Table 3 appears to be empirically relevant for many of the portfolio sorts—the rejection rate is close to 50%. Finally, under the largest step size with a 10 basis point return differential per portfolio, the rejection rate is above 80%. The Wolak and Bonferroni tests should not reject the null under experiment I and this is indeed what we find.

In experiments II and III, the expected return pattern is monotonic but non-linear, and both the t and MR tests should again reject the null hypothesis. The t-test is of course unaffected by the presence of a kink in the expected returns, as it only reflects the difference in expected returns between portfolio 1 and portfolio 10. Since the MR test focuses on the minimum difference $\min_i \Delta\mu_i$ (when looking for an increasing pattern) it is the smallest step size that affects power, and thus we expect the MR test to have lower power to detect patterns like Experiments II and II than those like Experiment I. This is indeed what we find. For a step size equal to 5 basis points, for example, the power of the MR test is 30% in experiments II and III, compared with 47% in experiment I.

The column in Table 4 labeled 'MR$^{\text{all}}$' shows the result of using all possible pair-wise inequalities in the test. For a one-way sort with $N = 10$, this entails comparing 45 rather than 9 pairs of portfolio returns. There appears to be a small gain in power from including the full set of inequalities, although this may in part reflect that this approach leads to a slightly oversized test.

Turning to the second set of experiments involving a non-monotonic relationship between expected returns and portfolio ranks, the MR tests very rarely reject, whereas the standard $t-$test does so frequently. For example, the $t-$test rejects 77% of the time in experiment IV with the largest step size. These are cases where we do not want a test to reject if the theory implies a monotonic relationship between portfolio rank and expected returns. For these experiments we expect Wolak's test and the Bonferroni bound test to reject the null hypothesis of a weakly monotonic relationship: for step sizes less than 5 basis points neither of these tests exhibit much power, but for step sizes of 5 and particularly 10 basis points these two tests do detect the non-monotonic relationship, with Wolak's test in most cases having considerably better power than the Bonferroni bound test. Comparing experiments VI and VII, we see that the power of both the Wolak and

Bonferroni bound test is much greater in the presence of a single large deviation from the null compared with many small deviations that add up to the same 'total' deviation.

Because the tests consider different hypotheses, their size and power are not directly comparable: The t-test only compares the top and bottom portfolio; the MR test considers all portfolios and continues to have equality of means as the null and inequality as the alternative; finally, the Wolak and Bonferroni bound tests have weak inequality of expected returns under the null. Due to these differences, the tests embed different trade-offs in terms of size and power. While the t-test is powerful when expected returns are genuinely monotonically rising, this test cannot establish a uniformly monotonic pattern in expected returns across all portfolios and, as shown in experiments IV-VIII, if used for this purpose can yield misleading conclusions. The Wolak test is not subject to this criticism. However, when this test fails to reject, as the empirical results and simulations clearly illustrate, this could simply be due to the test having weak power. Finally, the MR test has weak power for small steps in the direction hypothesized by the theory but appears to have good power for step sizes that match much of the empirical data considered earlier. Moreover, this test does not reject the null when the evidence contradicts the theory as in experiments IV-VIII.[8]

5   Conclusion

Empirical research in finance often seeks to address whether there is a systematic relationship between an asset's expected return and some measure of the asset's risk or liquidity characteristics. In this paper we propose a test that reveals whether a null hypothesis of no systematic relationship can be rejected in favor of a monotonic relationship predicted by economic theory. The test summarizes in a single number whether the relationship is monotonic or not. Moreover, it is non-parametric and does not require making any assumptions about the functional form of the relationship between the variables used to sort securities and the corresponding expected returns. This is a big advantage since monotonicity in expected returns on securities sorted by some variable is preserved under very general conditions, including non-linear mappings between sorting variables and risk factor loadings. Perhaps most importantly, our test is extremely easy to use.

---

[8]In unreported simulation results, we imposed identical pairwise correlations across the test statistics and investigated how the results change when the correlation increased from 0 to 0.5 and 0.9. We found that the power of the MR, Wolak and Bonferroni tests declines, the higher the correlation. The bootstrapped results in Table 5 reflect the empirical correlations in the data, which range from -0.28 to 0.32 and equal 0.10 on average.

We see two principal uses for the new test. First, it can be used as a descriptive statistic for monotonicity in the expected returns of individual securities or portfolios of securities ranked according to one or more sorting variables. Besides providing a single summary statistic for monotonicity, our approach allows researchers to decompose the results to better diagnose the source of a rejection of (or failure to reject) the theory being tested. In general, it is good practice to consider the complete cross-sectional pattern in expected returns on securities sorted by liquidity or risk characteristics and our test makes it easy to do this.

Second, if theoretical considerations suggest a monotonic relationship between the sorting variable and expected returns, then the approach can be used to formally test asset pricing implications such as in the liquidity preference and CAPM examples covered here or in tests of whether the pricing kernel decreases monotonically in market returns. Moreover, when a model implies a particular ranking in the loadings of individual stocks on observed risk or liquidity factors, the monotonicity test can be conducted on the estimated asset betas. Lack of monotonicity in such cases may imply that the conjectured theoretical model is not an adequate description of the data.

Appendix: Theorem 1

Since the theorem below applies not just to sample means or differences in sample means, but also to slope coefficients, we use the general notation $\boldsymbol{\beta}$ as the coefficient of interest. The case of sample means is a special case of the result below, setting $\mathbf{F}_{it} = 1$ for all i, t, i.e., regressing each of the asset returns simply on a constant. In the theorem we also consider the slightly more general case, relative to the discussion in Section 2.7, that the regressors in each equation can differ, so we index the regressors by both i and t rather than just t. This generalization may be of use in cases where different control variables are needed for different assets, for example.

Theorem 1 *Consider a set of regressions, with potentially different regressors in each regression:*

$$r_{it} = \boldsymbol{\beta}_i' \mathbf{F}_{it} + e_{it}, \ \ i = 0, 1, ..., N; \ \ t = 1, 2, ..., T,$$

and define:

$$\mathbf{h}_t \equiv \left[vech \left(\mathbf{F}_{0t} \mathbf{F}_{0t}' \right)', ..., vech \left(\mathbf{F}_{Nt} \mathbf{F}_{Nt}' \right)', r_{0t} \mathbf{F}_{0t}', ..., r_{Nt} \mathbf{F}_{Nt}' \right]'$$

where "vech" is the half-vec operator, see Hamilton (1994) for example. Assume that (i) $\mathbf{h}_t$ is a strictly stationary process, (ii) $E\left[|h_{kt}|^{6+\varepsilon} \right] < \infty$ for some $\varepsilon > 0$ for all k, where h_{kt} is the

k^{th} element of $\mathbf{h}_t$, (iii) $\{\mathbf{h}_t\}$ is α-mixing of size $-3(6+\varepsilon)/\varepsilon$, and (iv) $E\left[\mathbf{F}_{it}\mathbf{F}'_{it}\right]$ is invertible for all $i = 0, 1, .., N$. Let $\hat{\boldsymbol{\beta}}_i$ denote the usual OLS estimator of $\boldsymbol{\beta}_i$. Let $\boldsymbol{\theta}_j \equiv \left[\beta_{j0}, ..., \beta_{jN}\right]$ and $\hat{\boldsymbol{\theta}}_j \equiv \left[\hat{\beta}_{j0}, ..., \hat{\beta}_{jN}\right]$ be the vector of the jth regression coefficient in each of the $N+1$ regressions. Then as $T \to \infty$

$$\sqrt{T}\left(\hat{\boldsymbol{\theta}}_j - \boldsymbol{\theta}_j\right) \;\Rightarrow\; N\left(0, \Omega_j\right)$$

$$\text{and} \quad \min_{i=1,...,N} \sqrt{T}\left\{\left(\hat{\beta}_{j,i} - \hat{\beta}_{j,i-1}\right) - \left(\beta_{j,i} - \beta_{j,i-1}\right)\right\} \;\Rightarrow\; W_j \equiv \min_{i=1,..,N}\left\{Z_i - Z_{i-1}\right\}$$

where $[Z_0, ..., Z_N]' \sim N\left(0, \Omega_j\right)$, and $\Rightarrow$ denotes convergence in distribution. Further, if a_T is the length of the average block in the stationary bootstrap, and $a_T \to \infty$ and $a_T/T \to 0$ as $T \to \infty$, then

$$\sup_z \;\left|P^*\left[\left\|\hat{\boldsymbol{\theta}}^*_j - \hat{\boldsymbol{\theta}}_j\right\| \le z\right] - P\left[\left\|\hat{\boldsymbol{\theta}}_j - \boldsymbol{\theta}_j\right\| \le z\right]\right| \;\xrightarrow{p}\; 0, \;\; as \; T \to \infty$$

$$\text{and} \quad \sup_z \;\left|P^*\left[\min_{i=1,...,N} \sqrt{T}\left\{\left(\hat{\beta}^*_{j,i} - \hat{\beta}^*_{j,i-1}\right) - \left(\hat{\beta}_{j,i} - \hat{\beta}_{j,i-1}\right)\right\} \le z\right] - \right.$$
$$\left. P\left[\min_{i=1,...,N} \sqrt{T}\left\{\left(\hat{\beta}_{j,i} - \hat{\beta}_{j,i-1}\right) - \left(\beta_{j,i} - \beta_{j,i-1}\right)\right\} \le z\right]\right| \;\xrightarrow{p}\; 0, \;\; as \; T \to \infty$$

where $|| \cdot ||$ represents a norm on $\mathbb{R}^{N+1}$, $\xrightarrow{p}$ denotes convergence in probability, and P^* is the probability measure induced by the bootstrap conditional on the original data, so $\hat{\beta}^*_{j,i}$ refers to the bootstrapped OLS estimate of the jth variable and the ith asset.

Proof of Proposition 1. The first part of Theorem 1, on the asymptotic Normality of the $(N+1) \times 1$ vector of jth coefficients in each regression, follows from Theorem 4 in Politis Romano (1994) under the stated conditions. As shown in Proposition 2.2 in White (2000), since the minimum of a vector of (differences in) parameters is a continuous function of the elements of the vector, by the continuous mapping theorem we have that

$$\min_{i=1,...,N} \sqrt{T}\left\{\left(\hat{\beta}_{j,i} - \hat{\beta}_{j,i-1}\right) - \left(\beta_{j,i} - \beta_{j,i-1}\right)\right\} \Rightarrow W_j \equiv \min_{i=1,...,N}\left\{Z_i - Z_{i-1}\right\},$$

where $[Z_0, ..., Z_N]' \sim N\left(\mathbf{0}, \Omega_j\right)$. The final part of Theorem 1, which justifies use of the bootstrap, follows from Corollary 2.6 in White (2000), noting that we do not need any further assumptions than stated in the theorem due to the fact that there are no estimated parameters here. ∎

The "Up" and "Down" tests can be justified using the same reasoning as for the directional accuracy test in Section 4 of White (2000) .

References

[1] Ang, A. and J. Chen, 2007, The CAPM over the Long Run: 1926-2001. Journal of Empirical Finance 14, 1-40.

[2] Ang, A., J. Chen and Y. Xing, 2006, Downside Risk. Review of Financial Studies 19, 1191-1239.

[3] Bakshi, G., D. Madan and G. Panayotov, 2009, Are U-Shaped Pricing Kernels a Viable Concept? An Empirical Appraisal of Expected Returns and a Possible Theoretical Reconciliation. Mimeo, University of Maryland.

[4] Bartholomew, D.J., 1961, A Test of Homogeneity of Means under Restricted Alternatives. Journal of the Royal Statistical Society B 23, 239-281.

[5] Berk, J.B., R. C. Green and V. Naik, 1999, Optimal Investment, Growth Options, and Security Returns. Journal of Finance 54, 1553-1607.

[6] Boudoukh, J., M. Richardson, T. Smith and R.F. Whitelaw, 1999, Ex Ante Bond Returns and the Liquidity Preference Hypothesis. Journal of Finance 54, 1153-1167.

[7] Campbell, J.Y., A.W. Lo and A.C. MacKinlay, 1997, The Econometrics of Financial Markets. Princeton University Press: New Jersey.

[8] Carlson, M., A. Fisher and R. Giammarino, 2004, Corporate Investment and Asset Price Dynamics: Implications for the Cross-section of Returns. Journal of Finance 59, 2577-2603.

[9] Casella, G. and R.L. Berger, 1990, Statistical Inference, Duxbury Press: California.

[10] Christoffersen, S.E.K., 2001, Why Do Money Fund Managers Voluntarily Waive Their Fees? Journal of Finance 56, 1117-1140.

[11] Fama, E.F., 1984, Term Premiums in Bond Returns. Journal of Financial Economics 13, 529-546.

[12] Fama, E.F. and K.R. French, 1992, The Cross-Section of Expected Stock Returns. Journal of Finance 47, 427-465.

[13] Fama, E.F. and K.R. French, 2006, The Value Premium and the CAPM. Journal of Finance 61, 2163-2186.

[14] Gourieroux, C., A. Holly and A. Monfort, 1982, Likelihood Ratio Test, Wald Test, and Kuhn-Tucker Test in Linear Models with Inequality Constraints on the Regression Parameters. Econometrica 50, 63-80.

[15] Hamilton, J.D., 1994, Time Series Analysis. Princeton University Press: New Jersey.

[16] Hansen, P.R., 2005, A Test for Superior Predictive Ability. Journal of Business and Economic Statistics 23, 365-380.

[17] Jackwerth, J.C., 2000, Recovering Risk Aversion from Option Prices and Realized Returns. Review of Financial Studies 13, 433-451.

[18] Johnson, T.C., 2002, Rational Momentum Effects. Journal of Finance 57, 585-608.

[19] Kodde, D.A. and F.C. Palm, 1986, Wald Criteria for Jointly Testing Equality and Inequality Restrictions. Econometrica 54, 1243-1248.

[20] Kosowski, R., A. Timmermann, R. Wermers and H. White, 2006, Can Mutual Fund "Stars" Really Pick Stocks? New Evidence from a Bootstrap Analysis. Journal of Finance 61, 2551-2595.

[21] Kudo, A., 1963, A Multivariate Analogue of the One-sided Test. Biometrika 50, 403-418.

[22] Lettau, M and S. Ludvigson, 2001, Resurrecting the (C)CAPM: A Cross-sectional Test when Risk Premia are Time-Varying. Journal of Political Economy 109, 1238-1287.

[23] McCulloch, J.H., 1987, The Monotonicity of the Term Premium: A Closer Look. Journal of Financial Economics 18, 185-192.

[24] Newey, W. and K.D. West, 1987, A Simple, Positive Semi-Definite Heteroskedasticity and Autocorrelation Consistent Covariance Matrix. Econometrica 55, 703-708.

[25] Perlman, M.D., 1969, One-Sided Testing Problems in Multivariate Analyses, Annals of Mathematical Statistics 40, 549-567.

[26] Politis, D.N. and J.P. Romano, 1994, The Stationary Bootstrap. Journal of the American Statistical Association 89, 1303-1313.

[27] Richardson, M., P. Richardson and T. Smith, 1992, The Monotonicity of the Term Premium: Another Look. Journal of Financial Economics 31, 97-106.

[28] Romano, J.P. and M. Wolf, 2005, Stepwise Multiple Testing as Formalized Data Snooping. Econometrica 73, 1237-1282.

[29] Rosenberg, J.V. and R.F. Engle, 2002, Empirical Pricing Kernels. Journal of Financial Economics 64, 341-372.

[30] Shive, S. and T. Shumway, 2009, Is the Pricing Kernel Monotonic? Manuscript, University of Notre Dame.

[31] Sullivan, R., A. Timmermann and H. White, 1999, Data-snooping, Technical Trading Rules and the Bootstrap. Journal of Finance 54, 1647-1692.

[32] White, H., 2000, A Reality Check for Data Snooping. Econometrica 68, 1097-1126.

[33] Wolak, F.A., 1987, An Exact Test for Multiple Inequality and Equality Constraints in the Linear Regression Model. Journal of the American Statistical Association 82, 782-793.

[34] Wolak, F.A., 1989, Testing Inequality Constraints in Linear Econometric Models. Journal of Econometrics 31, 205-235.

[35] Zhang, L., 2005, The Value Premium. Journal of Finance 60, 67-103.

Table 1: Test statistics for portfolios sorted by historical beta

This table presents results of tests for a monotonic relationship between estimates of CAPM beta and subsequent returns, using the same data as in Ang, Chen and Xing (2006), which runs from July 1963 to December 2001. At the beginning of each month stocks are sorted into deciles on the basis of their ex-ante beta estimates calculated using one year of daily data, value-weighted portfolios are formed, and returns on these portfolios in the subsequent month are recorded. Panel A presents the average returns on each of these portfolios in percent per month along with the monthly standard deviation of portfolio returns. Panel B presents various tests of the monotonicity of average returns across portfolios: Column 1 reports the spread in the estimated expected return between the top and bottom ranked portfolio; column 2 reports the t-statistic for this spread (using Newey-West heteroskedasticity and autocorrelation consistent standard errors), while column 3 shows the associated p-value. Columns 4 and 5 present the p-values from the monotonic relationship (MR) test applied to the decile portfolios, based either on the minimal set of portfolio comparisons, or on all possible comparisons (MR^{all}). Columns 6 and 7 show p-values associated with the 'Up' and 'Down' tests that consider signed deviations from a flat pattern. Columns 8 and 9 report the p-values from tests based on Wolak's (1989) test and a Bonferroni bound. Panel C (next page) shows post-ranked beta estimates for the decile portfolios ranked by ex-ante betas. Finally, panel D presents a t-test for the top-minus-bottom difference in post-ranked betas and p-values from the MR and MR^{all} tests applied to the post-ranked beta estimates.

Panel A: Average returns on CAPM beta decile portfolios

	Past beta									
	Low	2	3	4	5	6	7	8	9	High
Mean	0.414	0.502	0.488	0.537	0.539	0.520	0.486	0.576	0.511	0.510
Std dev	3.534	3.746	3.828	4.024	4.307	4.230	4.417	4.942	5.807	7.506

Panel B: Tests of monotonicity for returns on CAPM beta decile portfolios

top minus bottom	t-test		MR	MR^{all}	Up	Down	Wolak	Bonf.
	t-stat	p-val	p-val	p-val	p-val	p-val	p-val	p-val
0.096	0.339	0.367	0.039	0.040	0.648	0.920	0.958	1.000

Table 1: Test statistics for portfolios sorted by historical beta

See description on the previous page.

Panel C: Post-ranked betas on CAPM beta decile portfolios

				Past beta					
Low	2	3	4	5	6	7	8	9	High
0.600	0.659	0.702	0.774	0.856	0.850	0.904	1.013	1.194	1.539

Panel D: Tests of monotonicity for post-ranked beta estimates

top minus	t-test		MR	MR$^{\text{all}}$
bottom	t-stat	p-val	p-val	p-val
0.938	9.486	0.000	0.003	0.003

Table 2: Test statistics for term premia

This table presents results of tests for a monotonic relationship between term premia on US Treasury bills (relative to the one-month Treasury bill) and the time to maturity, using data from the CRSP monthly treasuries files, over the period January 1964 to December 2001. Panel A reports the average term premia in percent per month for the full sample (1964–2001) and for two sub-samples (1964–1972 and 1973–2001). Panel B reports various tests of the monotonicity of average term premia as a function of time to maturity: the first column presents the average difference between the longest and the shortest term premia, and the second and third columns present the t-statistic (using Newey-West heteroskedasticity and autocorrelation consistent standard errors) and p-value corresponding to this difference. In the fourth and fifth columns we present the p-values from the MR test, based either on the minimal set of portfolio comparisons, or on all possible comparisons. In the sixth and seventh columns we present our p-values from the tests for increasing ("Up") and decreasing ("Down") segments in term premia, and in the final two columns we present p-values from Wolak's (1989) test and a Bonferroni-based test of the null of weak monotonicity against an unconstrained alternative.

Panel A: Average term premia

	Maturity (in months)									
Sample	2	3	4	5	6	7	8	9	10	11
1964–2001	0.027	0.049	0.050	0.064	0.068	0.063	0.080	0.086	0.071	0.077
1964–1972	0.023	0.040	0.038	0.052	0.054	0.052	0.069	0.069	0.018	0.050
1973–2001	0.028	0.052	0.053	0.068	0.072	0.066	0.084	0.092	0.087	0.085

Panel B: Tests of monotonicity of term premia

Sample	top minus bottom	t-test t-stat	t-test p-val	MR p-val	MR^{all} p-val	Up p-val	Down p-val	Wolak p-val	Bonf. p-val
1964–2001	0.050	2.416	0.008	0.953	0.906	0.000	0.369	0.036	0.020
1964–1972	0.026	0.908	0.182	0.983	0.991	0.003	0.375	0.007	0.004
1973–2001	0.057	2.246	0.012	0.633	0.617	0.002	0.474	0.340	0.704

Table 3: Estimates of expected returns for decile portfolios

This table reports mean returns (in percent per month) for stocks sorted into value-weighted decile portfolios. The sorting variables are market equity (ME), book-to-market value (BE-ME), cash flow-price (CF-P), earnings-price (E-P), dividend-price (D-P), momentum (M'tum), short term reversal (ST Rev) and long term reversal (LT Rev). All data series are taken from Ken French's web site at Dartmouth. Panels A and B report results for the period 1963:07 – 2006:12, while Panels C and D report results for the earliest available starting point for each series which is 1926 or 1927 except for the portfolios sorted on long-term reversal which begin in 1931 and the portfolios sorted on the earnings-price or cashflow-price ratios which begin in 1951. Panels B and D report various tests for monotonicity in the average returns on these portfolios described in Section 2. Row 1 reports the t-statistic for this spread (using Newey-West heteroskedasticity and autocorrelation consistent standard errors), and row 2 shows the associated p-value. Rows 3 and 4 present the p-values from the monotonic relationship (MR) test between the portfolio sorting variables and expected returns across all decile portfolios, based either on the minimal set of portfolio comparisons, or on all possible comparisons (MR^{all}). Rows 5 and 6 report the p-values from the bootstrap tests for "Up" and "Down" changes in expected returns across adjacent portfolios. The second-last and last rows report the p-values from Wolak's (1989) test and a test based on Bonferroni bounds.

	ME	BE-ME	CF-P	E-P	D-P	M'tum	ST Rev	LT Rev
Panel A: Average returns, 1963-2006								
Low	1.273	0.824	0.850	0.829	1.002	0.177	1.147	1.395
2	1.206	0.948	0.898	0.845	0.938	0.736	1.281	1.244
3	1.241	0.989	0.975	0.975	1.029	0.862	1.255	1.214
4	1.185	1.013	0.960	0.959	1.006	0.903	1.055	1.104
5	1.209	1.014	1.069	0.943	0.912	0.801	1.020	1.127
6	1.097	1.110	1.030	1.075	1.006	0.899	0.935	1.074
7	1.151	1.188	1.090	1.234	1.045	0.941	0.890	1.070
8	1.095	1.216	1.127	1.229	1.133	1.145	0.941	0.977
9	1.026	1.269	1.329	1.284	1.122	1.238	0.748	0.897
High	0.886	1.396	1.334	1.429	1.074	1.648	0.683	0.882
High-Low	-0.387	0.572	0.485	0.600	0.072	1.472	-0.464	-0.512
Panel B: Tests of monotonicity, 1963-2006								
t-statistic	-1.536	2.544	2.404	2.683	0.295	5.671	-2.364	-2.205
t-test p-value	0.062	0.005	0.008	0.004	0.384	0.000	0.009	0.014
MR p-value	0.274	0.000	0.024	0.008	0.336	0.291	0.258	0.002
MR^{all} p-value	0.237	0.000	0.012	0.021	0.256	0.242	0.170	0.002
Up p-value	0.737	0.045	0.035	0.016	0.353	0.000	0.889	0.999
Down p-value	0.051	1.000	0.994	0.985	0.651	0.954	0.015	0.114
Wolak p-value	0.736	1.000	0.990	0.991	0.810	0.873	0.860	0.998
Bonf. p-value	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000

Table 3: Estimates of expected returns for decile portfolios

See description on the previous page.

	ME	BE-ME	CF-P	E-P	D-P	M'tum	ST Rev	LT Rev
Panel C: Average returns, full sample								
Low	1.520	0.872	0.863	0.852	0.924	0.338	1.493	1.504
2	1.326	0.973	0.939	0.864	0.984	0.732	1.217	1.315
3	1.298	0.977	0.986	1.010	0.934	0.741	1.168	1.286
4	1.249	0.970	1.003	1.010	1.024	0.865	1.039	1.094
5	1.211	1.054	1.116	1.020	0.902	0.869	1.064	1.164
6	1.183	1.101	1.086	1.190	0.991	0.943	1.017	1.044
7	1.148	1.115	1.189	1.265	1.086	1.032	0.972	1.072
8	1.093	1.272	1.212	1.344	1.145	1.163	0.932	1.047
9	1.040	1.306	1.397	1.403	1.109	1.268	0.828	0.922
High	0.907	1.410	1.437	1.549	1.094	1.591	0.487	0.902
High-Low	-0.613	0.538	0.574	0.697	0.171	1.254	-1.007	-0.602
Panel D: Tests of monotonicity, full sample								
t-statistic	-2.350	2.439	3.244	3.665	0.938	5.434	-5.060	-2.486
t-test p-value	0.009	0.007	0.001	0.000	0.174	0.000	0.000	0.006
MR p-value	0.002	0.000	0.018	0.000	0.692	0.002	0.012	0.221
MR$^{\text{all}}$ p-value	0.002	0.000	0.012	0.000	0.492	0.002	0.016	0.158
Up p-value	0.987	0.027	0.007	0.001	0.169	0.000	0.993	0.919
Down p-value	0.024	1.000	0.994	1.000	0.606	0.998	0.003	0.020
Wolak p-value	0.985	0.999	0.995	1.000	0.530	0.999	0.995	0.880
Bonf. p-value	1.000	1.000	1.000	1.000	0.361	1.000	1.000	1.000

Table 4: Conditional and joint monotonicity tests for double-sorted portfolios

This table shows mean returns for stock portfolios using 5×5 two-way sorts. The sorting variables are market equity, which is always listed in the row, and one of either book-to-market value or momentum. The sample period is 1963:07 to 2006:12, and all data series are taken from Ken French's web site at Dartmouth. Portfolios are value-weighted and mean returns are reported in percent per month. The penultimate row in each panel presents the p-values from tests for a monotonic relationship (MR) between the portfolio sorting variable in the row and expected returns, conditional on being in a given column portfolio. For example, the p-value on a test for the size effect, conditional on being in the 'Growth' book-to-market portfolio is 0.687, while it is 0.031 if we condition on being in the 'Value' book-to-market portfolio. The final row in each panel presents the p-value from a joint test for a monotonic relationship between the portfolio sorting variable in the row, computed across all column portfolios. The penultimate and final columns present p-values from tests for a monotonic relationship between the portfolio sorting variable in the column and expected returns, conditional on being in a given row portfolio (size). The bottom-right number in each panel is the p-value for the joint test for a monotonic relationship in both variables.

	Panel A: Market equity $\times$ Book-to-market ratio						
	Book-to-market ratio					MR	Joint MR
	Growth	2	3	4	Value	p-val	p-val
Market equity							
Small	0.711	1.297	1.337	1.546	1.660	0.023	
2	0.878	1.141	1.411	1.458	1.524	0.004	
3	0.889	1.205	1.210	1.334	1.506	0.057	0.000
4	0.998	0.994	1.222	1.334	1.374	0.044	
Big	0.879	0.968	0.982	1.066	1.074	0.023	
MR p-val	0.687	0.401	0.405	0.069	0.031		
Joint MR p-val			0.342				0.083

	Panel B: Market value $\times$ Momentum						
	Momentum					MR	Joint MR
	Losers	2	3	4	Winners	p-val	p-val
Market equity							
Small	0.362	1.154	1.417	1.564	1.973	0.000	
2	0.423	1.034	1.257	1.499	1.777	0.000	
3	0.601	0.979	1.123	1.228	1.728	0.000	0.154
4	0.597	0.992	1.026	1.238	1.583	0.008	
Big	0.645	0.883	0.774	0.975	1.272	0.552	
MR p-val	0.893	0.143	0.001	0.117	0.016		
Joint MR p-val			0.708				0.545

Table 5: Monte Carlo simulation results

This table reports rejection frequencies for simulated data repeatedly drawn from the value-weighted decile portfolios sorted on market equity, using 2500 Monte Carlo simulations and 1000 bootstraps for each simulation. The columns report the proportion of rejections from tests of monotonicity, with nominal size of 0.05. Columns 1 and 2 report the proportion of rejections from a t-test applied to the spread in expected returns between the top and bottom portfolios, and from the monotonic relationship (MR) test, respectively. These results are obtained under the normality assumption discussed in Section 4. Columns 3 to 9 show the corresponding proportions of rejections using bootstrapped portfolio data based on the value-weighted decile portfolios sorted on market equity. Column 3 is a standard t-test applied to the spread in expected returns between the top and bottom portfolios. The fourth and fifth columns report the proportion of rejections using the MR test based on the minimal set of possible inequalities (MR) or all possible inequalities implied by monotonicity (MR^{all}). The sixth and seventh columns report the proportion of rejections of the null of no relationship against the alternative that the sum of positive (Up) or absolute values of negative (Down) differences in expected returns is non-zero. The last two columns (Wolak and Bonf) report the proportion of rejections of the null of a weakly monotonic relationship using Wolak's (1989) test, with critical values based on 1000 simulations per replication, and a Bonferroni bound test.

"Step" refers to the step size in percent per month used in forming the spreads in expected returns on the decile portfolios. Experiments I-III impose a monotonic relation between portfolio rank and expected returns, while experiments IV-VIII impose a non-monotonic pattern. Figure 3 displays the shapes of the assumed patterns. The experiments are identical when the step size is zero and so only the results for Experiment I are presented in that case.

Table 5 is on the following page.

Table 5: Monte Carlo simulation results

See description on the previous page.

| Experiment	Normal simulation		Bootstrap simulation						
	t-test	MR	t-test	MR	MR$^{\text{all}}$	Up	Down	Wolak	Bonf.
					Step=0				
I	0.050	0.050	0.074	0.064	0.090	0.078	0.032	0.036	0.034
					Step=0.01				
I	0.126	0.099	0.134	0.114	0.136	0.142	0.014	0.016	0.021
II	0.126	0.096	0.134	0.100	0.150	0.135	0.017	0.021	0.023
III	0.126	0.093	0.134	0.099	0.121	0.132	0.018	0.022	0.025
IV	0.095	0.067	0.112	0.071	0.093	0.124	0.021	0.038	0.037
V	0.056	0.048	0.082	0.056	0.063	0.083	0.035	0.046	0.044
VI	0.050	0.045	0.074	0.054	0.060	0.073	0.037	0.051	0.046
VII	0.050	0.048	0.074	0.046	0.046	0.075	0.036	0.027	0.036
VIII	0.045	0.047	0.080	0.064	0.066	0.068	0.036	0.036	0.048
					Step=0.02				
I	0.260	0.171	0.221	0.176	0.226	0.251	0.005	0.006	0.010
II	0.260	0.156	0.221	0.150	0.194	0.233	0.007	0.012	0.018
III	0.260	0.152	0.221	0.142	0.169	0.218	0.008	0.011	0.017
IV	0.164	0.064	0.166	0.061	0.078	0.212	0.017	0.055	0.058
V	0.063	0.042	0.087	0.043	0.053	0.091	0.039	0.073	0.062
VI	0.050	0.035	0.074	0.036	0.048	0.076	0.047	0.083	0.063
VII	0.050	0.025	0.074	0.032	0.032	0.096	0.039	0.075	0.121
VIII	0.040	0.038	0.075	0.066	0.066	0.073	0.039	0.050	0.055
					Step=0.05				
I	0.805	0.479	0.594	0.468	0.515	0.690	0.000	0.001	0.001
II	0.805	0.382	0.594	0.302	0.322	0.647	0.000	0.002	0.006
III	0.805	0.377	0.594	0.310	0.392	0.582	0.000	0.002	0.005
IV	0.510	0.014	0.384	0.011	0.019	0.705	0.045	0.213	0.200
V	0.086	0.009	0.104	0.004	0.006	0.180	0.091	0.352	0.220
VI	0.050	0.007	0.074	0.003	0.006	0.114	0.013	0.374	0.222
VII	0.050	0.000	0.074	0.000	0.000	0.232	0.082	0.616	0.758
VIII	0.027	0.070	0.046	0.011	0.016	0.152	0.132	0.193	0.191
					Step=0.10				
I	1.000	0.847	0.966	0.837	0.803	0.984	0.000	0.000	0.000
II	1.000	0.681	0.966	0.571	0.596	0.979	0.000	0.001	0.002
III	1.000	0.729	0.966	0.648	0.703	0.950	0.000	0.000	0.000
IV	0.955	0.000	0.768	0.000	0.000	0.998	0.491	0.806	0.729
V	0.138	0.000	0.146	0.000	0.000	0.648	0.596	0.958	0.814
VI	0.050	0.000	0.074	0.000	0.000	0.401	0.657	0.960	0.814
VII	0.050	0.000	0.074	0.000	0.000	0.548	0.521	1.000	1.000
VIII	0.014	0.000	0.034	0.000	0.000	0.577	0.593	0.752	0.700
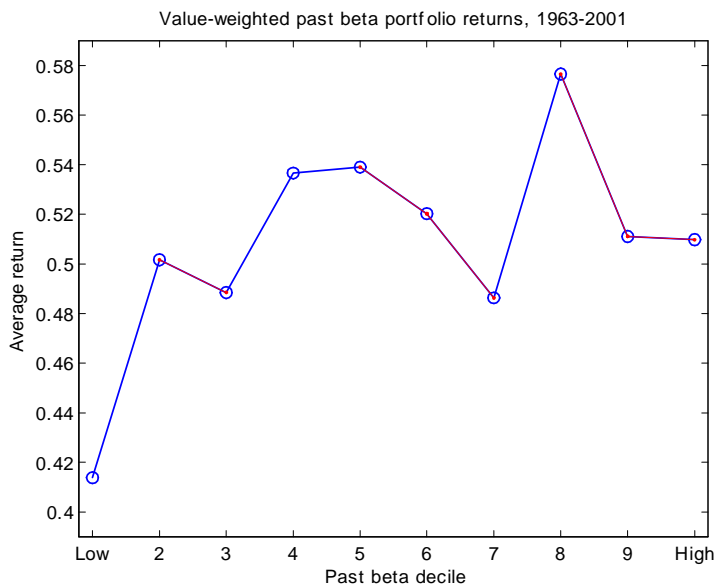
Figure 1: *Average monthly returns on decile portfolios formed on past 12-month CAPM beta, from July 1963 to December 2001.*
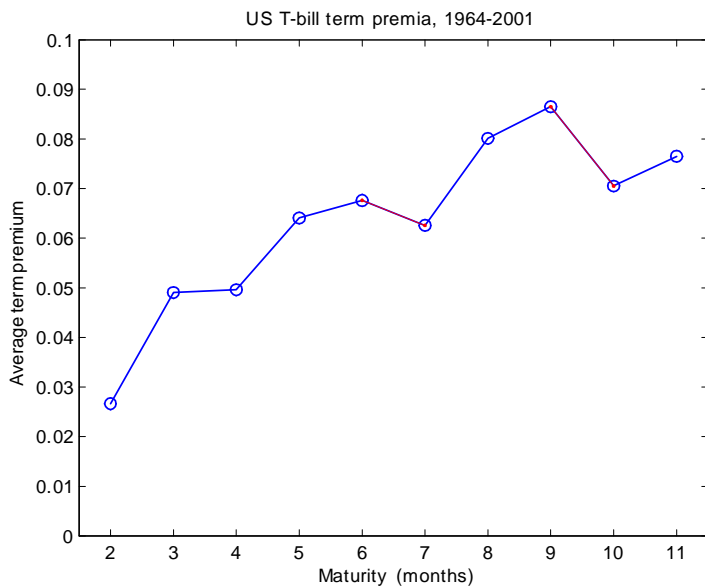

Figure 2: *Average monthly term premia for US T-bills, relative to a T-bill with one month to maturity, over the period January 1964 to December 2001.*
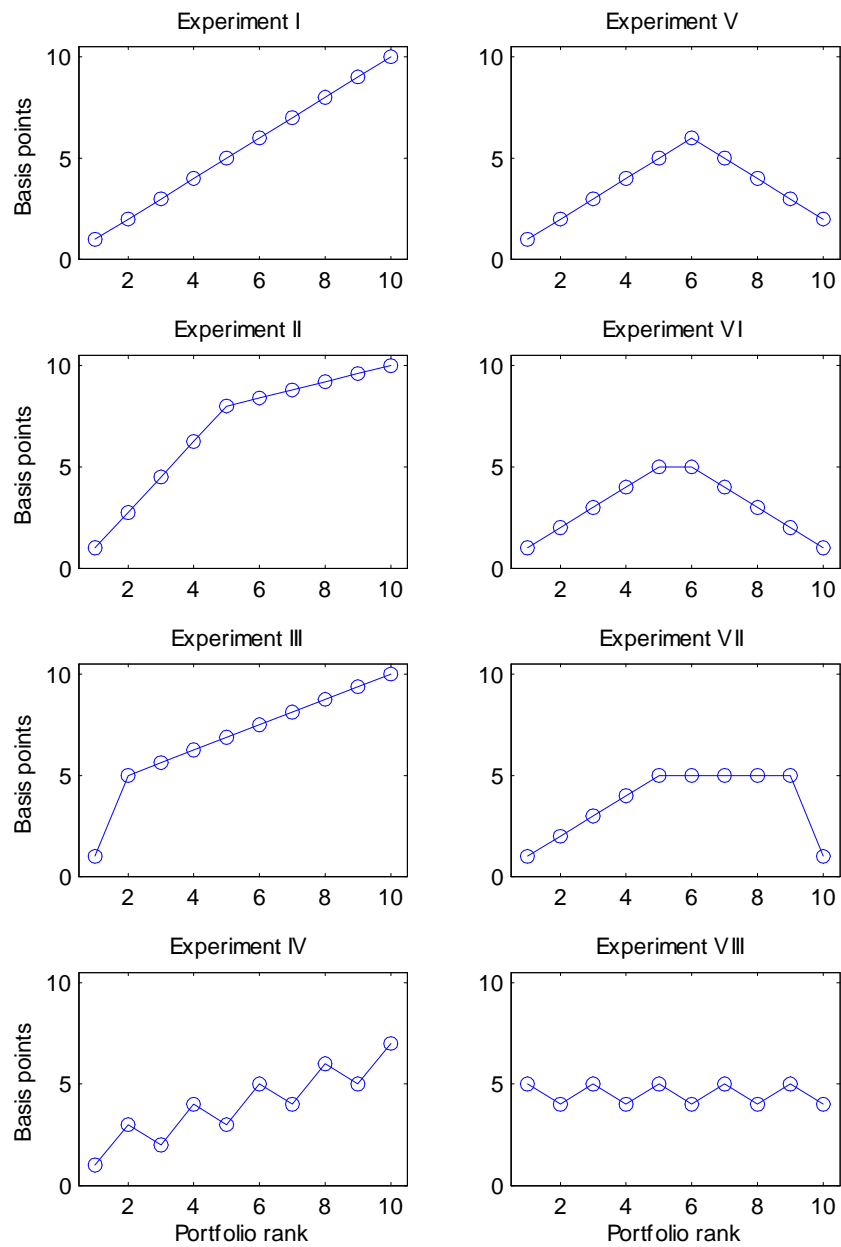
41

Figure 3: *Patterns in expected returns under the eight experiments considered in the Monte Carlo simulations for a step size of 1 basis point.*