Economic Implications of Bull and Bear Regimes in UK Stock and Bond Returns*

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Abstract

This paper presents evidence of persistent 'bull' and 'bear' regimes in UK stock and bond returns and considers their economic implications from the perspective of an investor's portfolio allocation. We find that the perceived state probability has a large effect on the optimal asset allocation, particularly at short investment horizons. If ignored, the presence of such regimes gives rise to substantial welfare costs. Parameter estimation uncertainty, while clearly important, does not overturn the conclusion that predictability in the return distribution linked to the presence of bull and bear states has a significant effect on investors' strategic asset allocation.

Key words: Strategic Asset Allocation, Regime Switching, Bull and Bear Markets, Model Specification.

Short form title: Bull and Bear Regimes in UK Asset Returns.

JEL classification codes: G110, C510.

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Returns in financial markets are difficult to predict and the absence of predictability served historically as one of the corner stones of financial economics. This proposition was largely supported by empirical studies. As recent as in the mid-seventies, the consensus among researchers was that, to a good approximation, returns in stock, bond and foreign exchange markets were unpredictable and prices were well characterized by a random walk. Following a string of papers that documented limited predictability of returns across different predictor variables, sample periods and asset classes, the earlier consensus has largely been replaced by a view that − although predictability may be over-stated because of data-snooping effects and small sample distortions − returns are predictable, particularly at longer horizons.

Predictability of returns does not, on its own, reject the notion that financial markets are efficient. In fact, since the predictable component in asset returns tends to be very small and uncertain, it is important to carefully consider how useful predictability really is to risk averse investors. Only recently have the economic implications of return predictability been explored by authors such as Barberis (2000), Brandt (1999), Campbell and Viceira (1999; 2001) and Kandel and Stambaugh (1996). These studies find that, faced with time-varying investment opportunities, it is optimal for investors to vary their portfolio weights both as a function of a set of predictor variables and as a function of their investment horizon. Although predictability of returns is generally weak from a statistical perspective, it is often found to have large effects on optimal portfolio holdings.

So far the literature has almost invariably explored the asset allocation implications of return predictability in the context of simple linear models designed to characterize predictability in the conditional mean of returns. However, for asset allocation purposes it is important to go beyond this and correctly model the full probability distribution of returns. Unless investors have very restrictive preferences such as mean-variance utility, the calculation of expected utility and the derivation of optimal portfolio weights will reflect the higher order moments of the probability distribution of returns.

This paper finds strong evidence of underlying economic regimes in the process driving U.K. stock and bond returns and considers their economic implications. We identify three states that can broadly be interpreted as a high-volatility bear state with large, negative mean returns, a 'normal' state with returns closer to their historical averages and a bull state with high mean returns on stocks and bonds. Specification tests that consider the full probability distribution of asset returns strongly reject single-state, linear models against the three-state alternative.

Since the risk-return trade-off on U.K. stocks and bonds varies substantially across the bull and bear states, their presence has the potential to significantly affect investors' optimal asset allocation. We consider the effect of regimes by studying an investor's decision between a broad portfolio of stocks (the FTSE All Share index), 15-year government bonds, and T-bills. The presence of regimes gives rise to a wide variety of investment shapes linking optimal asset holdings to the investment horizon and generates very sensible patterns in the optimal asset allocation.

Our paper makes several contributions to the existing literature. While bull and bear markets are part of financial folklore, their asset allocation implications have not previously been studied, certainly not in the rigorous framework that we consider.[1] Our model has a rich set of implications for optimal asset holdings as a function of the underlying state probabilities. Optimal asset allocations are found to be strongly affected by investors' beliefs about the underlying state. A buy-and-hold investor who perceives a high probability of being in the bear state will invest very little in stocks and bonds in the short run. This investor will hold more in stocks at longer investment horizons as the likelihood of shifting to the normal or bull state grows. In contrast, in the persistent bull state, investors hold less in stocks - but more in bonds - the longer their investment horizon. This is because there is only a small chance of leaving the bull state in the short run, while this probability grows as the investment horizon expands. While clearly important, parameter estimation uncertainty does not change our conclusion that asset allocations are substantially different under the three state model compared to a single-state model.

Our analysis further allows us to compute the expected utility cost if regimes are ignored. This is a natural metric in an economic assessment of our findings and allows us to map the potential economic gains from considering regime shifts. Gains appear to be large enough to be relevant to long-term investors such as pension funds and even to investors with shorter investment horizons.

Evidence of three states remains strong when the return model is extended to include the lagged dividend yield as a predictor. In this extended model we find that both regime shifts

[1] The only published paper that we are aware of that considers the effect of regimes on optimal asset allocation is Ang and Bekaert (2002a). However, their paper studies international asset allocation and uses a very different methodology for model selection and asset allocation than that presented here.

and predictability from the dividend yield significantly affect the optimal asset allocation. Regimes appear to capture a mean-reverting component operating at a short- to medium-term horizon, while the dividend yield captures a slow, mean-reverting component in the return distribution that mostly affects long-run asset allocations.

Regular rebalancing in the optimal portfolio holdings decreases the sensitivity of the asset allocation with respect to the investment horizon. This is to be expected since rebalancing means that the current position can be adjusted if there is a change in the underlying state. Frequent rebalancing therefore makes asset holdings more responsive to the current state. If investors find themselves in the bear state, under rebalancing they will reduce stock and bond holdings to zero in the knowledge that they can raise them in a subsequent period. A long-term investor deprived of this possibility will not reduce risky asset holdings as aggressively and will instead base these on the returns expected over the full investment horizon.

The outline of the paper is as follows. Section 2 presents estimation results for the regime switching model fitted to U.K. stock and bond returns and provides a range of results from specification tests applied to the probability distribution of returns. Section 3 introduces the optimal asset allocation problem and reports empirical asset allocation results under a buy-and-hold investment scheme. Section 4 extends our empirical results to include predictability from the dividend yield. Section 5 considers rebalancing while Section 6 analyzes the economic significance and robustness of regimes. Section 7 concludes.

1. **Regimes in U.K. stock and bond returns**

The possibility of predicting asset returns has fascinated generations of researchers in economics and finance and this question has spawned numerous studies. One strand of the literature has adopted linear models and concentrated on documenting which state variables predict the conditional mean of stock returns. Examples of this approach include Campbell and Shiller (1988) and Fama and French (1988). For U.K. stock returns Clare, Thomas and Wickens (1994) find that the gilt-equity yield ratio has some predictive power, while Black and Fraser (1995) find that default- and term-premium variables have predictive power over returns. Pesaran and Timmermann (2000) extend this list to find predictability of U.K. stock returns based on a variety of macroeconomic and financial variables. Flavin and Wickens (2001) examine strategic asset allocation among U.K. equities, long government bonds, and T-bills in a mean-variance portfolio framework when the joint process of excess returns and a macroeconomic variable (inflation) follows a multivariate VAR with GARCH effects.

Another strand of the literature on return predictability has broadened the scope by investigating the presence of regime dynamics in asset returns. Ang and Bekaert (2002b), Driffill and Sola (1994), Gray (1996) and Hamilton (1988) study regimes in interest rates while Ang and Bekaert (2002a), Perez-Quiros and Timmermann (2000) and Turner, Starz and Nelson (1989) consider regimes in stock returns.

1.1. *The model*

We will consider a framework that incorporates both the possibility of regimes as well as return predictability arising from other variables. As a starting point, suppose that the return or excess return on stocks and bonds, $\mathbf{R}_t = (R_t^s \; R_t^b)$, follows an autoregressive process with mean, variance and autoregressive parameters that can vary across k regimes driven by a latent state variable, S_t:

$$\mathbf{R}_t = \boldsymbol{\mu}_{S_t} + \sum_{j=1}^{p} \mathbf{A}_{j,S_t} \mathbf{R}_{t-j} + \boldsymbol{\varepsilon}_t. \tag{1}$$

S_t takes integer values between 1 and k, $\boldsymbol{\mu}_{S_t}$ is the intercept in state S_t, $\mathbf{A}_{j,S_t}$ is a 2×2 matrix of autoregressive coefficients at lag j in state S_t, and $\boldsymbol{\varepsilon}_t \sim N(0, \Sigma_{S_t})$ is the vector of return innovations which has mean zero and state-specific covariance matrix Σ_{S_t} and is assumed to be normally distributed. A linear model is obtained as a special case when $k = 1$.

Completing the model for returns requires specifying the process followed by the state variable, S_t. Following Hamilton (1989), we assume that S_t is driven by a first order, homogeneous Markov process with transition probability matrix $\mathbf{P}$

$$\mathbf{P}_{[i,j]} = \Pr(S_t = j | S_{t-1} = i) = p_{ij}, \quad i, j = 1, .., k. \tag{2}$$

The underlying state, S_t, is allowed to be unobserved. Predictability in this model is not simply confined to mean returns but also shows up in the form of time-variations in the conditional variance (volatility clustering), skew and kurtosis of returns, c.f. Timmermann (2000). This is likely to be important. For example, time-variations in the covariance matrix are likely to have first-order effects on the optimal asset allocation, c.f. Flavin and Wickens (1998; 2001).

To encompass multivariate models that incorporate a wider set of predictors, $\mathbf{z}_t$, (1) can be generalized to

$$\mathbf{y}_t = \boldsymbol{\mu}_{s_t} + \sum_{j=1}^{p} \mathbf{A}^*_{j,s_t} \mathbf{y}_{t-j} + \boldsymbol{\varepsilon}_t. \tag{3}$$

where $\mathbf{y}_t = (\mathbf{R}_t \ \mathbf{z}_t)'$ and $\boldsymbol{\varepsilon}_t \sim N(\mathbf{0}, \Omega_{s_t})$. This model can capture richer patterns of predictability, combining nonlinear regime switching with linear predictability. The investor's information set at time t, $\Im_t$, is assumed to comprise the history of returns extended by the predictor variables, $\Im_t = \{\mathbf{R}_j \ \mathbf{z}_j\}_{j=1}^t$. Estimation of the parameters of the model, $\boldsymbol{\theta} = (\boldsymbol{\mu}_{s_t}, \mathbf{A}^*_{j,s_t}, \Omega_{s_t}, \mathbf{P})$ proceeds by maximizing the likelihood function through the EM algorithm.

1.2. *Data*

Our data consists of monthly returns on the FTSE All Share stock market index, inclusive of dividends, returns on 15-year government bonds and returns on 1-month T-bills. We use this data to model the return on stocks in excess of the T-bill rate. Section 4 of our analysis also uses the dividend yield on the FTSE All Share portfolio computed as dividends over the preceding 12 months divided by the current stock price. The sample period is 1976:1 - 2000:12, a total of 300 monthly observations. All data was obtained from Datastream.

1.3. *Model selection and specification tests*

Determining the number of states, k, and lags, p, in (1) can pose considerable difficulties, yet is clearly important to understanding the properties of the return process. To select a model specification we adopted an approach that closely reflects the economic objectives of the exercise. Since we will be using the estimated models for asset allocation purposes, it is important to verify that they adequately capture the return distribution and are not misspecified. For this purpose we use the probability integral transform considered by Rosenblatt (1952) and recently used in economic analysis by Diebold et al (1998).

The probability integral transform or $z-$score is the probability of observing a value smaller than or equal to the realization r_{t+1} of returns under the null that the model is correctly specified. Under the k-state mixture of normals, this is given by

$$
\begin{aligned}
\Pr(R_{t+1} \leq r_{t+1}|\Im_t) &= \sum_{i=1}^{k} \Pr(R_{t+1} \leq r_{t+1}|s_{t+1}=i, \Im_t) \Pr(s_{t+1}=i|\Im_t) \\
&= \sum_{i=1}^{k} \Phi\left(\sigma_i^{-1}(r_{t+1} - \mu_i - \sum_{j=1}^{p} a_{j,i} r_{t+1-j})\right) \Pr(s_{t+1}=i|\Im_t) \\
&\equiv z_{t+1}. \qquad (4)
\end{aligned}
$$

Here R_t is the excess asset return and $\Phi(\cdot)$ is the cumulative density function of a standard normal variable. Provided that our model is correctly specified, z_{t+1} should be independently and identically distributed (IID) on the interval $[0, 1]$, with a uniform distribution c.f. Rosenblatt (1952). Based on this idea, Berkowitz (2001) proposes a likelihood-ratio test that inverts Φ to get a transformed $z-$score

$$
z^*_{t+1} = \Phi^{-1}(z_{t+1}). \qquad (5)
$$

Under the null of a correctly specified model, z^* should be IID and normally distributed ($IIN(0,1)$). This suggests conducting specification tests based on moment conditions such as

$$
\begin{aligned}
E[z^*_{t+1}] &= 0, \\
Var[z^*_{t+1}] &= 1, \\
Cov[z^*_{t+1}, z^*_t] &= 0, \\
Cov[(z^*_{t+1})^2, (z^*_t)^2] &= 0, \qquad (6) \\
Skewness[z^*_{t+1}] &= 0, \\
Kurtosis[z^*_{t+1}] &= 3.
\end{aligned}
$$

We use a likelihood ratio test that focuses on a few salient moments of the return distribution. Suppose the log-likelihood function is evaluated under the null that $z^*_{t+1} \sim IIN(0,1)$:

$$
L_{IIN(0,1)} \equiv -\frac{T}{2}\ln(2\pi) - \sum_{t=1}^{T} \frac{(z^*_t)^2}{2}, \qquad (7)
$$

where T is the sample size. Under the alternative of a misspecified model, the log-likelihood function incorporates deviations from the null, $z_{t+1}^* \sim IIN(0,1)$:

$$z_{t+1}^* = \mu + \sum_{j=1}^{p} \sum_{i=1}^{l} \rho_{ji} (z_{t+1-i}^*)^j + \sigma e_{t+1}, \tag{8}$$

where $e_{t+1} \sim IIN(0,1)$. The null of a correct return model implies $p \times l + 2$ restrictions — i.e., $\mu = \rho_{ji} = 0$ ($j = 1, ..., p$ and $i = 1, ..., l$) and $\sigma = 1$ — in equation (8). Let $L(\hat{\mu}, \{\hat{\rho}_{ji}\}_{j=1}^{p}{}_{i=1}^{l}, \hat{\sigma})$ be the maximized log-likelihood obtained from (8). To test that the forecasting model (1) is correctly specified, we use the following test statistic

$$LR = -2 \left[L_{IIN(0,1)} - L(\hat{\mu}, \{\hat{\rho}_{ji}\}_{j=1}^{p}{}_{i=1}^{l}, \hat{\sigma}) \right] \sim \chi^2_{p \times l+2}. \tag{9}$$

In addition to the standard Jarque-Bera (1980) test, we focus on three likelihood ratio tests proposed by Berkowitz (2001):

1. A test with $p = l = 0$ that only restricts the mean and variance of the transformed $z-$scores: $LR_2 = -2 \left[L_{IID\ N(0,1)} \text{ - } L(\hat{\mu}, \hat{\sigma}) \right] \sim \chi^2_2$;

2. A test with $p = l = 1$ that also restricts the transformed $z-$scores to be serially uncorrelated: $LR_3 = -2 \left[L_{IID\ N(0,1)} \text{ - } L(\hat{\mu}, \hat{\rho}_{11}, \hat{\sigma}) \right] \sim \chi^2_3$;

3. A test with $p = l = 2$ that restricts the transformed $z-$scores and their squares to be serially uncorrelated: $LR_6 = -2 \left[L_{IID\ N(0,1)} \text{ - } L(\hat{\mu}, \{\hat{\rho}_{ji}\}_{j=1}^{2}{}_{i=1}^{2}, \hat{\sigma}) \right] \sim \chi^2_6$.

Table 1 reports tests for a range of regime-switching models, including the special case of a linear, single-state model. We apply the tests to the density forecasts of (excess) stock and bond returns, maintaining $p = 0$ since there was no evidence of serial correlation in returns but consider a range of values for the number of states, $k = 1$, 2, 3, and 4. The single-state model is strongly rejected by the Jarque-Bera test, indicating that residuals are skewed and/or fat-tailed. In contrast, the two- and three-state models pass all tests although the former is close to being rejected at the 10% level by the normality test, suggesting that it does not fully fit the predictive distribution of asset returns. This analysis suggests that at least two states and probably three states are needed to model asset returns. Consistent with this we found that a single state model was systematically rejected in likelihood ratio tests. To further support our choice of model we considered information criteria that trade off fit against parsimony and found that a three state model was preferred.[2]

1.4. *Interpretation of the states*

Fig. 1 plots smoothed state probabilities, while Table 2 presents parameter estimates for the three-state model fitted to monthly (excess) returns on U.K. stocks and bonds. The first state picks up isolated periods with very high volatility and large negative returns such as October 1987 and August 1998. At 28% per annum, stock returns are particularly volatile in this state. The probability that the market remains in this state is only 25% while exits – which occur with 75% probability – are mostly to state 3. The steady state probability of the first state is only four percent. State 2, in contrast, lasts much longer with an average duration of 11 months. Most of the time since 1994 was spent in this state where asset returns and volatilities are close to their historical averages. In steady state, the probability of state 2 is 60%. State 3 identifies markets with high mean returns and above-normal volatility during the late 1970s, early-to-mid 1980s and a spell during 1994. The duration of this 'bull' state is seven months and its steady state probability is 36%.

Though not visited frequently, the bear state is important in determining both the expected value and the risk of stock and bond returns and it clearly helps to capture outliers in the excess return distribution that cannot be accommodated by a Gaussian model.

The parameters capturing mean returns appear to vary substantially across the three regimes. However, they are not all statistically significant so we tested whether they are identical across regimes:

$$\mathbf{r}_t = \boldsymbol{\mu} + \boldsymbol{\varepsilon}_t \quad \boldsymbol{\varepsilon}_t \sim N(\mathbf{0}, \Sigma_{s_t}). \tag{10}$$

These restrictions were tested through a likelihood ratio test:

$$LR = 2\left[-1590.12 - (-1600.59)\right] = 20.94.$$

The associated p-value is less than one percent, so state-independence of mean returns is strongly rejected.

[2]Detailed results are available upon request.

The estimated correlations between stock and bond returns appear to vary substantially across regimes, ranging from -0.45 in the bear state to 0.55 in the bull state. This could again have important asset allocation implications so we estimated a restricted model where correlations were fixed across states, but volatilities were allowed vary across states:

$$\begin{aligned} \mathbf{R}_t &= \boldsymbol{\mu}_{S_t} + \boldsymbol{\varepsilon}_t, \qquad \boldsymbol{\varepsilon}_t \sim N(\mathbf{0}, \Sigma_{s_t}), \\ \mathbf{e}_1' \Sigma_{s_t} \mathbf{e}_2' &= \phi \times \sqrt{\mathbf{e}_1' \Sigma_{s_t} \mathbf{e}_1'} \times \sqrt{\mathbf{e}_2' \Sigma_{s_t} \mathbf{e}_2'}, \end{aligned} \tag{11}$$

where $\mathbf{e}_j$ is a 2×1 column vector with a one in its j-th position and zeros everywhere else, and ϕ is the constant correlation parameter.[3] This is a restricted version of the three-state model from Table 2 with two covariance constraints imposed. The resulting likelihood ratio test is

$$LR = 2\left[-1590.12 - (-1594.41)\right] = 8.58,$$

which implies a p-value of 0.014, suggesting strong evidence of regime-dependence in the correlation between stock and bond returns. In the bear state stock and bond returns do not move closely together, while correlations are positive and significant in the normal and bull states.

1.5. *ARCH effects*

A vast empirical literature has presented evidence of autoregressive conditional heteroskedasticity (ARCH) in asset returns, c.f. Bollerslev et al. (1992). Recent work by Flavin and Wickens (1998; 2001) on U.K. data shows that in a single-state framework, ARCH effects have important effects on optimal portfolio weights. It is therefore worthwhile considering whether the preferred three-state model is misspecified or needs to be extended to incorporate such effects. To address this question, we estimated a bivariate Markov switching ARCH model similar to that considered by Hamilton and Lin (1996):

$$\begin{aligned} \mathbf{R}_t &= \boldsymbol{\mu}_{S_t} + \boldsymbol{\varepsilon}_t, \quad \boldsymbol{\varepsilon}_t \sim N(0, \Sigma_{S_t}) \\ \Sigma_{S_t} &= K_{S_t} + \Delta_{S_t} \boldsymbol{\varepsilon}_t' \boldsymbol{\varepsilon}_t \Delta_{S_t}'. \end{aligned} \tag{12}$$

[3]The restriction that conditional correlations are constant but conditional covariances time-varying is similar to the approach of Bollerslev (1990).

Here K_{S_t} is restricted to be symmetric and positive definite and Δ_{S_t} captures regime-dependent effects of past shocks on current volatility. Table 3 presents estimation results for this model. Most of the coefficients capturing ARCH effects in the matrices K_{S_t} and Δ_{S_t} turn out to be statistically insignificant and imprecisely estimated with large standard errors, particularly in the first state. To formally test for ARCH effects, we imposed the restriction $\Delta_{S_t} = \Delta$, $S_t = 1, 2, 3$ and obtained the likelihood ratio test

$$LR = 2\left[-1586.48 - (-1590.12)\right] = 7.28.$$

The associated p-value is 0.839 and the null hypothesis of no ARCH effects fails to be rejected. We therefore maintain the simpler three-state model without ARCH effects. The absence of ARCH effects in our model can be explained by the fact that regime switching can capture volatility clustering through time-variations in the probabilities of (persistent) states with very different levels of volatility, c.f. Timmermann (2000).

2. Optimal asset allocation

Using the three-state model for U.K. stock and bond returns, this section studies the optimal asset allocation for a buy-and-hold investor with constant relative risk aversion preferences. Later sections introduce predictability from the dividend yield and periodic rebalancing. Abstracting from these effects in the initial analysis simplifies the problem considerably and makes our results easier to follow. We first characterize the investor's optimization problem and then present empirical results.

2.1. *The investor's optimal asset allocation problem*

Consider a buy-and-hold investor with unit wealth at time t and power utility defined over the level of wealth T periods from now, W_{t+T}:[4]

$$u(W_{t+T}) = \frac{W_{t+T}^{1-\gamma}}{1-\gamma}, \quad \gamma \neq 0. \tag{13}$$

[4] Our analysis follows much of the existing literature by assuming power utility over final wealth, but the qualitative results appear to be robust to alternative functional forms.

Here $\gamma > 0$ is the coefficient of relative risk aversion while T is the investment horizon. The investor is assumed to maximize expected utility at time t by allocating $\boldsymbol{\omega}_t = (\omega_t^s \ \omega_t^b)$ to stocks and bonds while $1 - \boldsymbol{\omega}_t' \boldsymbol{\iota}_2$ is invested in riskless T-bills, where $\boldsymbol{\iota}_2 = (1\ 1)'$.[5] The problem solved by the investor is

$$\max_{\omega_t} \quad E_t \left[\frac{W_{t+T}^{1-\gamma}}{1-\gamma} \right]$$
$$s.t.\ W_{t+T} = \left\{ (1 - \boldsymbol{\omega}_t' \boldsymbol{\iota}_2) \exp\left(Tr^f\right) + \boldsymbol{\omega}_t' \exp\left(\mathbf{R}_{t:t+T} + Tr^f \boldsymbol{\iota}_2\right) \right\} \tag{14}$$
$$\boldsymbol{\omega}_t \in [0,1] \times [0,1],$$

where $\mathbf{R}_{t:t+T} \equiv \mathbf{R}_{t+1} + \mathbf{R}_{t+2} + ... + \mathbf{R}_{t+T}$ is the continuously compounded excess returns on stocks and bonds over the $T-$period investment horizon. The constraint $\boldsymbol{\omega}_t \in [0,1] \times [0,1]$ rules out short-selling which is prohibited e.g. for U.K. pension funds.

There is no closed-form solution to the optimal stock holdings under power utility, so we use Monte-Carlo methods to draw N time paths of T monthly excess returns from the regime switching model using the parameter estimates $\hat{\boldsymbol{\theta}}_t = \{\hat{\boldsymbol{\mu}}_t, \hat{\Sigma}_t, \hat{\mathbf{P}}_t\}$. These simulations account for the possibility of stochastic regime switching as governed by the transition matrix $\hat{\mathbf{P}}_t$. For a given value of $\boldsymbol{\omega}_t$, we can approximate the integral in the expected utility functional:

$$\hat{E}_t[U(W_{t+T})|\omega_t] = N^{-1} \sum_{n=1}^{N} \left\{ \frac{\left[(1 - \boldsymbol{\omega}_t' \boldsymbol{\iota}_2) \exp\left(Tr^f\right) + \boldsymbol{\omega}_t' \exp\left(\sum_{i=1}^{T} \mathbf{R}_{t+i,n} + Tr^f \boldsymbol{\iota}_2 \right) \right]^{1-\gamma}}{1-\gamma} \right\}, \tag{15}$$

where $n = 1, ..., N$ tracks the simulation number.[6] A grid search across different values of $\boldsymbol{\omega}_t$ determines the optimal asset holdings.

2.2. *Empirical results*

Investors differ along several dimensions such as their level of risk aversion (γ) and investment horizon (T). In our assessment of the economic significance of regimes in stock and bond returns

[5] Following standard practice we assume that the risk-free rate is known and constant. This assumption means that we do not have to simultaneously model future T-bill rates and stock and bond returns.

[6] A large number of simulations is needed to adequately account for the occurrence of bear regimes with moderate steady-state probability. We found that $N = 40,000$ simulations was sufficient to obtain sufficiently precise and stable solutions.

we start with the investment horizon. Speculators and some mutual funds typically have relatively short investment horizons of up to one year, while pension funds generally have a longer investment horizon of 10-20 years, depending on their liability structure.

Using the last part of our sample (1986-2000), Fig. 2 shows the evolution in the real-time optimal asset allocation at three different investment horizons, namely a short (6 months), medium (36 months) and a long horizon (120 months). For comparison we also show the optimal allocation under no predictability, i.e. assuming that returns are IID. Following studies such as Barberis (2000) and Brandt (1999) we initially fix the coefficient of relative risk aversion at $\gamma = 5$.

To be more realistic and avoid a 'benefit of hindsight' bias, these results do not condition on end-of-sample parameter estimates or state probabilities and instead use the real-time recursive parameter estimates $\{\hat{\boldsymbol{\mu}}_t, \hat{\Sigma}_t, \hat{\mathbf{P}}_t\}$ and state probabilities $\{\hat{\boldsymbol{\pi}}_t\}$, $t = 1986{:}01, ..., 2000{:}12$. This explains why the optimal stock holdings change over time even under the IID model.

Optimal stock holdings are far more volatile at the short horizon than at the longer horizons. As the six-month horizon equity holdings fluctuate between 15% and 60%. As the investment horizon grows, the investor pays less attention to the current state and the optimal stock holdings become much smoother. Following a decline from 70% to 40% after the October 1987 stock market crash, investors with a horizon of at least three years would have steadily invested around 40% of their wealth in stocks. Compared to the IID model, the three-state asset allocation are most similar at very long horizons since the state probabilities converge to their steady state values. Even so, long-term equity holdings under regime switching are generally below those applying to an IID investor who holds around 50% of the portfolio in stocks. This reflects the more fat-tailed distribution of stock returns under the three-state model.

Bond holdings are generally much smaller than those in stocks. At the six-month horizon, bond holdings are often at zero with exception of short periods lasting up to a year which see bond holdings rise to 35%. At longer horizons, bond holdings increase to 25% after the October 1987 stock market crash only to decline subsequently before rising steadily throughout the 1990s to a level around 40% at the end of the sample. Bond holdings under the IID model are generally much higher and close to 50% after 1987.

Real-time allocations to T-bills are strikingly different under regime switching and in the absence of predictability. In the absence of regime shifts, T-bills never enter into the optimal portfolio. In contrast, under regime switching T-bills act as a buffer, particularly at short horizons where their weight is often as high as 90% (e.g., in the aftermath of October 1987 or during the Kuwait invasion). At the longer horizons, the share allocated to T-bills rises from 20% in 1986 to a maximum of 60% in the early 1990s before declining to 20% at the end of the sample.

2.3. *State beliefs and horizon effects*

Assuming a time-invariant investment opportunity set and power utility, in a classic paper Samuelson (1969) showed that optimal asset holdings are identical across different investment horizons. In our setup, the relationship between the investment horizon and the optimal stock holdings is considerably more complicated since investment opportunities change both with the underlying state probabilities and with the investment horizon.

Fig. 3 shows the interaction between the perceived state probabilities and the investment horizon in determining optimal asset holdings. The optimal stock holdings can be either non-monotonic, rising or declining as a function of the investment horizon. The intuition for these findings is as follows. Since bull markets are more persistent and likely in steady-state than bear markets, at long horizons stocks and bonds offer a high risk premium at relatively low volatility. At shorter investment horizons the risk-return trade-off offered by stocks and bonds varies significantly with the underlying state.

Starting from the bear state which is unattractive for the risky assets, the allocation to stocks and bonds is initially at zero, while the allocation to T-bills forms 100% of the portfolio. As the investment horizon grows, the allocation to stocks increases to 40% while the allocation to bonds is a non-monotonic function of T, rising from zero to 50% at $T = 6$ months before declining to 40% at longer horizons. T-bills, in contrast, see their portfolio weight decline sharply from 100% to 20% as T grows from one to six months.

When the bull state is the initial regime, at the shortest investment horizon investors allocate 100% to stocks. As the investment horizon grows, this share declines to 40% while conversely the allocation to bonds rises from zero to 40% and that to T-bills grows from zero to 20%. The normal state sees non-monotonic patterns in the allocation to stocks and T-bills as a function of the horizon. The allocation to equities declines from 100% to 20% as T expands to six months, only to rise to 40% at longer horizons. The allocation to T-bills initially increases from zero to 45% before declining to a level near 20%. Bonds, in contrast, see their allocation rise steeply from zero to 40% as T is expanded.

2.4. *Effects of risk aversion*

So far we have fixed the coefficient of risk aversion at $\gamma = 5$. To study the effect on the optimal asset allocation of different values of the coefficient of relative risk aversion, we vary γ between 1 and 50. We show results for three different configurations of the initial state probabilities representing each of the three regimes. For comparison we also report optimal stock holdings under no predictability.

First consider the results in the left column of Fig. 4 which assume a short investment horizon of $T = 1$ month. As γ rises, the optimal stock holdings decline monotonically irrespective of which regime the economy starts from, although the level of stock holdings obviously differs across states. In the bull state, at levels of risk aversion $\gamma \leq 5$, the entire portfolio is allocated to stocks, but this number gradually declines to 20% as γ rises. In the bear state stock holdings decline more sharply as a function of γ.

At the longer 10-year horizon, the effect on stock holdings of raising γ appears to be even stronger and largely independent of the initial state. Thus, for a level of risk aversion $\gamma = 1$ (log-utility), a stock-only portfolio is chosen while for $\gamma = 5$, the weight on stocks is 40%. Demand for T-bills shows the reverse pattern, rising uniformly from zero to 90% as γ rises above 5. In the IID model, as γ rises the optimal allocation to stocks also declines uniformly while - in contrast with the three-state model - T-bills get a zero allocation.

At short investment horizons, consistent with Fig. 3 there is practically no demand for bonds. Interesting non-linearities arise in the relationship between γ and the optimal bond holdings when $T = 120$. Bond holdings are very small for $\gamma = 1$ and then increase rapidly to roughly 50% as γ goes from 2 to 4. As γ rises further, the optimal allocation to bonds declines to less than 10%. Aït-Sahalia and Brandt (2001) document a similar non-monotonic effect of the coefficient of risk aversion when predictability originates from a composite index of economic variables.

3. **Stock holdings under predictability from the dividend yield**

Studies such as Barberis (2000), Brandt (1999), Campbell and Viceira (1999), Kandel and Stambaugh (1996), and Lynch (2001) have considered optimal asset allocation under predictability from the dividend yield, so it is natural for us to extend our results to allow for predictability from this regressor. While other predictor variables could also be studied, Lynch (2001) finds that the yield generates the highest hedging demands for U.S. stocks.

Our analysis proceeds in two steps. First, we consider the optimal asset allocation when the dividend yield is the only predictor, basing the results on a VAR model similar to that used in earlier studies. Having ensured comparability with existing results, we next introduce regimes and investigate the results in the context of the regime switching model (3).

3.1. *Predictability from the dividend yield*

We use our more general model (3) to both consider a linear VAR and a three-state regime-switching VAR with the dividend yield added as a predictor variable. A three-state model extended to incorporate a lag to accommodate the strong evidence of a first-order autoregressive component in the yield continued to be supported by a model specification analysis. Table 4 presents full-sample parameter estimates for this model assuming either a linear (Panel A) or a three-state specification with an autoregressive component (Panel B), while Fig. 5 plots the smoothed state probabilities. In the linear model, autoregressive terms are insignificant for asset returns. As expected, the dividend yield is highly persistent. The coefficient of the dividend yield is marginally significant for stock returns but insignificant for bond returns.

Turning to the three-state model, Fig. 5 shows that the second regime covers most of the sample since 1983 while the third regime identifies a prolonged period from 1978-1983. Regime 1 is now slightly more persistent and expected to continue for two months before exiting mainly to state 2. As a consequence, the steady state probability of the first regime rises to 13%. Mean returns within this "bear" regime are -55% and -12% per annum for stocks and bonds, respectively.[7] Regimes 2 and 3 continue to be highly persistent with average durations exceeding 10 months and steady state

[7] These estimates were computed using the smoothed state probabilities as weights.

probabilities of 68% and 19%, respectively. Volatilities are low in the "normal" second state (12% and 9% per annum, for stocks and bonds, respectively), while mean excess returns are at 15.4% and 4.3% per annum. Finally, the third state is again a bull regime characterized by high mean excess returns of 18.6% and 10.4%, respectively.

The mean dividend yield also varies significantly across states, from a level of 2.3% in states 1 and 2 to a level of 5.7% in state 3, indicating that a large shift occurred in the mean dividend yield around 1983. Once regime-switching is accounted for, the dividend yield ceases to have predictive power over asset returns in two out of three states. In fact, the dividend yield forecasts asset returns only in the bull state, albeit with large coefficients in this regime. This is consistent with its marginal statistical significance in the linear VAR model.

3.2. *Asset holdings when the dividend yield is the only predictor*

Earlier papers on optimal stock holdings under predictability from the dividend yield all assume a single state. To compare our results to this literature and to disentangle the effect on asset holdings of predictability from the dividend yield and the presence of regimes, we first set $k = 1$, so that lagged values of the dividend yield is the only source of predictability and the model simplifies to a bivariate VAR:

$$\begin{pmatrix} \mathbf{R}_t \\ dy_t \end{pmatrix} = \begin{pmatrix} \boldsymbol{\mu}_R \\ \mu_{dy} \end{pmatrix} + \sum_{j=1}^{p} \begin{pmatrix} a_j^{r,r} & \mathbf{A}_j^{r,dy} \\ \mathbf{A}_j^{dy,r} & a_j^{dy,dy} \end{pmatrix} \begin{pmatrix} \mathbf{R}_{t-j} \\ dy_{t-j} \end{pmatrix} + \begin{pmatrix} \boldsymbol{\varepsilon}_{r,t} \\ \varepsilon_{dy,t} \end{pmatrix}. \tag{16}$$

We further constrain the model to match the assumptions in Barberis (2000) and Lynch (2001) by setting $p = 1$, and $a_j^{r,r} = 0$, $\mathbf{A}_j^{dy,r} = \mathbf{0}$.

Fig. 6 (left column) shows the real-time optimal asset allocation at three different horizons. The short sales constraint is now binding in many periods. At the longest 10-year horizon, it is almost always optimal to invest all money in stocks until 1997 except for a brief period in 1993 which sees a 20% allocation to bonds. Slow and persistent moves in the dividend yield are easy to detect in the short and medium term stock holdings. At short investment horizons, the model suggests going entirely out of stocks after 1993. In general bond weights are small and always below 20% prior to 1998. Consistent with results in Barberis (2000), the optimal allocation to risky assets appears to be an increasing function of the investment horizon while conversely T-bills are in higher demand for shorter horizons. Indeed, at the longest horizon the weight on T-bills is positive only after 1998, coinciding with the marked decline in the dividend yield around this time.

3.3. *Regimes and dividend yield effects*

The right column in Fig. 6 shows that introducing states in the extended three-state model leads to significant changes in asset allocations. In stark contrast to the case with a single state, long-horizon ($T = 120$) stock holdings are now very low between 1986 and 1989. They are more similar - and very high - between 1990 and 1998 at which point they decline to a level near zero. At the short and medium investment horizons the two allocations are even more different. For example, at the three-year horizon, stock holdings are much higher in the 1980s under the single-state model, while the reverse holds in the mid-1990s. At the six-month horizon, the three-state model suggests an exit from U.K. stocks as late as in 1998 while, in contrast the single-state model suggests an exit as early as 1993. Despite the additional uncertainty associated with the introduction of regimes, their effect is not simply to reduce stock holdings at all points in time. Rather, regimes reduce stock holdings at some times (near bear states) while increasing them at others.

Bond and T-bill holdings are also very different in the two models. Under the single-state model bonds only play a substantial role after 1998. Bonds play a more important role under regime shifts rising to levels close to 60% for brief periods of time at the short (six-month) horizon. At the longest horizon, the weight on T-bills is often very large in the three-state model, not only towards the end of the sample but also in 1986-1989 and for a few months in 1994.

An important conclusion arising from these results is that inclusion of standard predictor variables does not subsume the importance of economic regimes. The intuition for these results is that both economic regimes and the dividend yield track mean reverting components in stock returns. However, whereas the dividend yield captures a slow-moving, mean reverting factor - as witnessed by the very high persistence of this variable - regimes capture mean reversion at a higher frequency. This means that the perceived state probability may have a stronger effect, the shorter the investment horizon, while the dividend yield variable has its strongest effect at the longest horizons.

4. **Portfolio rebalancing**

So far we have ignored the possibility of rebalancing. This may be realistic for an investor who faces high transaction costs and only gets to adjust portfolio weights infrequently. It also may be plausible for investors such as pension funds whose strategic asset allocation is constrained by trustees to lie within relatively narrow bands which precludes the pursuit of market timing opportunities. For other investors this assumption is likely to be unrealistic so this section studies the effects of rebalancing.

Suppose that investors can adjust portfolio weights every $\varphi = \frac{T}{B}$ months at B equally spaced points t, $t+\frac{T}{B}$, $t+2\frac{T}{B}$, ..., $t+(B-1)\frac{T}{B}$ and let $\boldsymbol{\omega}_b$ $(b = 0, 1, ..., B-1)$ be the portfolio weights on the risky assets at these rebalancing times. When $B = 1$, $\varphi = T$ and the investor simply implements a buy-and-hold strategy. Under power utility the derived utility of wealth conveniently simplifies to

$$J(W_b, \mathbf{R}_b, \boldsymbol{\theta}_b, \boldsymbol{\pi}_b, t_b) = \frac{W_b^{1-\gamma}}{1-\gamma} Q(\mathbf{R}_b, \boldsymbol{\theta}_b, \boldsymbol{\pi}_b, t_b). \tag{17}$$

Investors are assumed to update their beliefs about the underlying state at each point in time using the formula (c.f. Hamilton, 1989):

$$\boldsymbol{\pi}_{t_b+1}(\hat{\boldsymbol{\theta}}_t) = \frac{\boldsymbol{\pi}_{t_b}(\hat{\boldsymbol{\theta}}_t)\hat{\mathbf{P}}_t^{\varphi(b+1)} \odot \boldsymbol{\eta}(\mathbf{R}_{t_b+1}; \hat{\boldsymbol{\theta}}_t)}{(\boldsymbol{\pi}_{t_b}(\hat{\boldsymbol{\theta}}_t)\hat{\mathbf{P}}_t^{\varphi(b+1)} \odot \boldsymbol{\eta}(\mathbf{R}_{t_b+1}; \hat{\boldsymbol{\theta}}_t))'\boldsymbol{\iota}_k}, \tag{18}$$

where $\boldsymbol{\eta}(\mathbf{R}_{t_b+1}; \hat{\boldsymbol{\theta}}_t)$ is the predictive density of the stock return at time $t_b + 1$ and $\hat{\mathbf{P}}_t^{\varphi(b+1)}$ is the updated $\varphi(b+1)$-step ahead transition probability matrix, $\hat{\mathbf{P}}_t^{\varphi(b+1)} \equiv \prod_{i=1}^{\varphi(b+1)} \hat{\mathbf{P}}_t$.[8] Portfolio weights at the rebalancing points, $\boldsymbol{\omega}_b$, are chosen to maximize

$$E_{t_b}\left[\left\{(1-\boldsymbol{\omega}_b'\boldsymbol{\iota}_2)\exp\left(\varphi r^f\right) + \boldsymbol{\omega}_b'\exp(\mathbf{R}_{b+1}(s_b) + \varphi r^f \boldsymbol{\iota}_2)\right\}^{1-\gamma} Q(\boldsymbol{\pi}_{b+1}^j, t_{b+1})\right]. \tag{19}$$

We solve this equation numerically by discretising the domain of each of the state variables on G equi-distant points and using backward induction methods. The multiple integral defining the conditional expectation is again calculated by Monte Carlo methods. For each $\pi_b = \pi_b^j$, $j = 1, 2, ...,$ G^{k-1} on the grid we draw in calendar time N samples of $\varphi-$period excess returns $\{R_{b+1,n}(S_b)\}_{n=1}^N$ from the regime switching model (3) where $R_{b+1,n}(S_b) \equiv \sum_{i=1}^{\varphi} R_{t_b+i,n}(S_b)$. The expectation (19) is then approximated as

$$N^{-1}\sum_{n=1}^{N}\left[\left\{(1-\boldsymbol{\omega}_b'\boldsymbol{\iota}_2)\exp\left(\varphi r^f\right) + \boldsymbol{\omega}_b'\exp(\mathbf{R}_{b+1}(s_b) + \varphi r^f \boldsymbol{\iota}_2)\right\}^{1-\gamma} Q(\boldsymbol{\pi}_{b+1}^{(j,n)}, t_{b+1})\right], \tag{20}$$

[8] The symbol $\odot$ denotes element by element multiplication.

where $\pi_{b+1}^{(j,n)}$ denotes the element π_{b+1}^{j} on the grid used to discretise the state space.

Table 5 reports the outcome of this exercise. Again we study optimal asset holdings under three scenarios for the state probabilities. For each value of $\hat{\boldsymbol{\pi}}$ we consider a range of rebalancing frequencies, $\varphi = 6$, 12, 24, and T months and report optimal weights for investment horizons $T =$ 1, 6, 12, 24, 60, and 120 months.

Several interesting results emerge. Our earlier finding that equity demand (as a function of T) is downward sloping in the bull state and upward sloping in the bear state continues to be supported under rebalancing. However, investment schedules become flatter, the lower is φ.[9]

At all investment horizons the possibility of rebalancing makes an investor use information about the current state relatively aggressively. In bull states, the weight assigned to stocks is uniformly increasing as φ declines. Indeed, while under buy-and hold the long-horizon optimal weight (40%) is roughly two-thirds of its level when predictability is ignored, when $\varphi = 6$ or $\varphi = 12$ the optimal stock holding is close to the IID level. The allocation to long-term bonds appears to be less sensitive to the rebalancing frequency with the weights generally close to their level in the IID model. Market timing in the bull state is therefore largely implemented by increasing the position in equities and reducing the position in T-bills. In the normal state, similar - though weaker - patterns emerge: the more frequent the rebalancing, the higher the exposure to stocks and (in general) lower the exposure to bonds.

In bear states ω_t^s is a uniformly increasing function of φ, i.e. as the time between rebalancing points increases, the allocation to equities increasing. Indeed, while under buy-and-hold the long-horizon optimal weight (42%) is still substantial, under six-month rebalancing, the optimal weight is only 30%. Long-run buy-and-hold investors anticipate the end of the bear state and know that this state will occur infrequently in the long run. They are therefore willing to hold large investments in stocks even in the bear market. The possibility of rebalancing makes it optimal even for long-run investors to drastically reduce the commitment to stocks in a bear state. For similar reasons, the allocation to bonds increases from 40% to 48% as rebalancing becomes more frequent.

Under high uncertainty about the current state - a situation approximated by initializing the current state probabilities at their steady-state values - stock holdings are at intermediate levels

[9] Of course, for $\varphi \geq T$, by construction the buy-and-hold and rebalancing results coincide. This explains why some entries in the table are unavailable.

and not very sensitive to the investment horizon. Since investors are unsure of the current state of the financial markets, they are unwilling to take extreme positions that condition on very good (bull state) or very poor (bear state) prospects for stock and bond markets.

We conclude from this analysis that the possibility of frequent rebalancing makes an investor use the information in state probabilities more aggressively. In the bear state, the lower is φ, the more drastic the reduction in the optimal allocation to stocks and the increase in the bond weight. Conversely, at times when markets are perceived to be in the bull or normal states, an investor increases (decreases) the weight on stocks (bonds) by more, the higher the rebalancing frequency.

5. Robustness and economic significance of regime shifts

5.1. *Welfare costs of ignoring regimes*

An economic assessment of the costs from ignoring regimes in asset returns is best conducted using a metric based on expected utility. In our setting this is equivalent to maximizing utility subject to constraining investors to choose at time t an optimal allocation $\boldsymbol{\omega}_t^{IID}$ under the assumption that asset returns simply follow a normal distribution with mean vector $\hat{\boldsymbol{\mu}}_t$, and covariance $\hat{\Sigma}_t$. In this case the portfolio choice is independent of the investment horizon and the value function for the constrained investor is

$$\begin{aligned} J_t^{IID} &\equiv \frac{1}{1-\gamma}\sum_{b=0}^{B}\beta^b E_t\left[W_b^{1-\gamma}\right] \\ W_b &= W_{b-1}\left[(1-\boldsymbol{\iota}_2'\boldsymbol{\omega}_t^{IID})\exp\left(\varphi r^f\right)+\boldsymbol{\omega}_b'\exp\left(\mathbf{R}_{b+1}+\varphi r^f\boldsymbol{\iota}_2\right)\right]. \end{aligned} \tag{21}$$

The assumption of independent and identically distributed returns is a constrained version of the model with regime switching and predictor variables, so

$$J_t^{IID} \leq J(W_t, \mathbf{R}_t, \mathbf{z}_t, \boldsymbol{\theta}_t, \pi_t, t). \tag{22}$$

The increase in initial wealth, η_t^{IID}, an investor would require to derive the same level of expected utility from the constrained and unconstrained asset allocation problems solves the following equation:

$$(1+\eta_t^{IID})^{1-\gamma}\sum_{b=0}^{B}\beta^b E_t\left[(W_b)^{1-\gamma}\right] = Q(\mathbf{R}_b, \mathbf{z}_b, \boldsymbol{\theta}_b, \pi_b, t_b), \tag{23}$$

with solution

$$\eta_t^{IID} = \left\{ \frac{Q(\mathbf{R}_b, \mathbf{z}_b, \boldsymbol{\theta}_b, \boldsymbol{\pi}_b, t_b)}{\sum_{b=0}^{B} \beta^b E_t \left[(W_b)^{1-\gamma} \right]} \right\}^{\frac{1}{1-\gamma}} - 1. \tag{24}$$

This expression is relatively easy to calculate since $Q(\mathbf{R}_b, \mathbf{z}_b, \boldsymbol{\theta}_b, \boldsymbol{\pi}_b, t_b)$ is a by-product of the numerical solution to the investor's portfolio choice. $\sum_{b=0}^{B} \beta^b E_t \left[(W_b)^{1-\gamma} \right]$ can be computed through Monte Carlo methods.

Table 6 reports implied welfare costs starting from the steady-state probabilities so as to avoid conditioning on different initial points for the IID and three-state model. Again we consider rebalancing at different frequencies, φ, and investment horizons, T. To make results comparable across alternative investment horizons, we report the annualized, riskless percentage return required by an investor to ignore the evidence of regime shifts, i.e. $100 \left[(1 + \eta_t^{IID})^{\frac{12}{T}} - 1 \right]$.

The cost of ignoring regimes increases uniformly as a function of the investment horizon, T, but decreases in the rebalancing frequency, φ. Welfare costs increase in the investment horizon because the present value of a suboptimal asset allocation is higher, the longer this position is locked in. Conversely, the shorter the rebalancing frequency, φ, the more valuable it is for investors to use their information about the underlying state, a possibility that does not arise in the absence of regime shifts. For example, investors can react more aggressively in reducing stock holdings in the short-lived bear state provided that rebalancing is possible.

For the buy-and-hold investor the utility cost rises from 0.05% at the 1-month horizon to about 0.20% per annum at the two-year horizon and 1.5% per annum at the 10-year horizon. Rebalancing leads to much higher utility costs, around two or three percent per annum for an investor with a two year horizon and around 10% per annum for an investor with an investment horizon exceeding five years.

These costs are sufficiently large to be economically relevant for investors with a long horizon such as a pension fund. While we have ignored transaction costs, these can reasonably be assumed to be lower than the estimated potential gains, partly because a shift between stocks, bonds and T-bills can be inexpensively implemented through positions in futures contracts.

5.2. *Parameter estimation uncertainty*

While our analysis accounts for learning in the sense that the investor is assumed to optimally update his state beliefs, our analysis so far has ignored parameter estimation uncertainty. However, many of the parameter estimates $\widehat{\boldsymbol{\theta}}$ reported in Tables 2 and 4 are not very precisely estimated. For instance, some of the state-dependent mean parameter estimates have relatively large standard errors. We know from the literature on mean-variance portfolio choice that optimal weights tend to be very sensitive to mean excess returns (see Best and Grauer, 1990). This suggests that parameter estimation uncertainty could significantly affect the optimal asset allocation. To investigate this question, we adopted the following procedure. We know that asymptotically (e.g., Krolzig, 1997),

$$\sqrt{T}\left(\widehat{\boldsymbol{\theta}}-\boldsymbol{\theta}\right)\overset{A}{\sim}N(\mathbf{0},V_{\theta}). \tag{25}$$

We utilize this result in the following algorithm:

1. For a particular trial, j, draw a vector $\widehat{\widehat{\boldsymbol{\theta}}}^{j}$ from the distribution $N(\widehat{\boldsymbol{\theta}},T^{-1}\hat{V}_{\theta})$, where $\hat{V}_{\theta}$ is the estimated covariance matrix of $\widehat{\boldsymbol{\theta}}$.

2. Conditional on $\widehat{\widehat{\boldsymbol{\theta}}}^{j}$, solve (14) to obtain a new vector of portfolio weights $\widehat{\widehat{\boldsymbol{\omega}}}^{j}$.

3. Repeat steps 1-2 many times, $j=1,2,...,J$, independent of previous trials.

4. Form $100(1-\alpha)\%$ confidence intervals for the optimal asset allocation $\hat{\omega}_t$ from the simulated distribution for $\widehat{\widehat{\boldsymbol{\omega}}}^{q}$, $j=1,2,...,J$. For example, the fifth quantile, $\widehat{\widehat{\boldsymbol{\omega}}}_{0.05}$, and the 95th quantile, $\widehat{\widehat{\boldsymbol{\omega}}}_{0.95}$, form the lower and upper bounds of a 90% confidence interval for the optimal weights.

To ensure the calculations are feasible, we report confidence bands only for the buy-and-hold allocations, assuming $J=1,500$ simulations. The outcome of our analysis is shown in Table 7 which for stocks, bonds and T-bills presents the mean allocation along with 90% confidence bands and the number of simulations above the IID allocation. Two conclusions emerge from this analysis. First, parameter estimation uncertainty is clearly important as the 90% bands are very wide. This is not a finding particular to our paper as emphasized by Britten–Jones (1999) with reference to mean-variance portfolio problems. Even so, at horizons longer than six months and independent of the initial state probabilities, close to 95% of the simulations yield a *lower* allocation to stocks under the three-state model than under the model that ignores regimes. Despite the wide confidence intervals, our optimal asset allocation results are thus different from those obtained in the absence of regime shifts in a statistical sense. Similarly, allocations to T-bills tend to be significantly larger under the regime switching model than under the IID model.

Parameter estimation uncertainty will also affect the magnitude of the expected utility costs due to ignoring the presence of economic regimes. It is clearly undesirable that our conclusion for the importance of such welfare losses is driven by a particular choice of parameter values that ignores their standard errors. We therefore apply a similar simulation approach to obtain 90% confidence bands for the compensatory riskless rate of return when state probabilities are initialized at their steady-state values. Fig. 7 shows the results. Three points emerge. First, due to Jensen's inequality, the average welfare cost computed across different parameter values is typically higher than the value reported in table 6 which conditions on a particular parameter estimate. In particular, at short investment horizons welfare costs may be somewhat larger than what our final parameter estimates led us to believe, and generally exceed 1% per annum. In this sense, the values in Table 6 may underestimate the expected utility costs. Second, confidence bands are rather wide. Third, even focussing on the lower 90% band, welfare losses remain economically important. For instance, the annualized loss at a 10-year horizon is around 0.50% per annum. Conversely, the upper 90% band supports compensatory rates of return of 4-5% per annum.

5.3. *Regime shifts and no arbitrage*

So far we have not explored the potential equilibrium implications of time-variations in investors' asset allocation. Instead we considered the optimal asset allocation of a small investor whose actions do not affect equilibrium rates of return. However, it is important to verify that our regime switching model is not inconsistent with equilibrium. Whitelaw (2001) constructs an equilibrium model where consumption growth follows a two-state process so that the investors' intertemporal marginal rate of substitution also follows a regime process. This setup is consistent with the process for returns assumed in our paper and suggests that regime-switching in asset returns does not violate arbitrage conditions in financial markets.

Suppose that investors have constant relative risk aversion and that asset returns are determined from the no-arbitrage relation

$$E\left[m_{t+1}r_{t+1}|\Im_t\right] = 1, \tag{26}$$

where m_{t+1} is the pricing kernel which is commonly restricted to be $m_{t+1} \equiv \beta\left(\frac{c_{t+1}}{c_t}\right)^{-\gamma}$, where c_t is real per-capita consumption. Excess returns on risky assets (over and above the risk-free rate, r_t^f) $R_{t+1} = r_{t+1} - r_t^f$ are then given by

$$E[R_{t+1}|\Im_t] = -\frac{Cov(m_{t+1}, R_{t+1}|\Im_t)}{E[m_{t+1}|\Im_t]}. \tag{27}$$

Suppose that consumption growth follows a regime switching process

$$g_{t+1} = \ln(c_{t+1}/c_t) \sim N(\mu_{gS_{t+1}}, \sigma^2_{gS_{t+1}}). \tag{28}$$

Then m_{t+1} is log-normally distributed with state-specific mean and variance:

$$\begin{aligned} E[m_{t+1}|S_{t+1}] &= \beta \exp(-\gamma\mu_{gS_{t+1}} + \frac{\gamma^2}{2}\sigma^2_{gS_{t+1}}), \\ Var[m_{t+1}|S_{t+1}] &= \beta^2 \exp(-2\gamma\mu_{gS_{t+1}} + \gamma^2\sigma^2_{gS_{t+1}})(\exp(\gamma^2\sigma^2_{gS_{t+1}}) - 1). \end{aligned} \tag{29}$$

Using this result and denoting $\pi_{S_{t+1}|t} = E[S_{t+1}|\Im_t]$, we have

$$E[R_{t+1}|\Im_t] = -\frac{\beta\sum_{S_{t+1}}\pi_{S_{t+1}|t}Cov\left[\left(\frac{c_{t+1}}{c_t}\right)^{-\gamma}, R_{t+1}|S_{t+1}\right]}{\beta\sum_{S_{t+1}}\pi_{S_{t+1}|t}\exp\left(-\gamma\mu_{gS_{t+1}} + \frac{\gamma^2}{2}\sigma^2_{gS_{t+1}}\right)}. \tag{30}$$

This simple model thus implies that excess returns on risky assets follow a regime-switching process driven by the states in the underlying pricing kernel that reflect time-varying expected consumption growth and time-varying conditional covariances between asset returns and consumption growth. A full equilibrium model that could quantify such effects and consider the cross-equation restrictions on different assets implied by (30) is, of course, well beyond the present paper, but it is worth establishing that our model need not violate arbitrage conditions.

6. **Conclusion**

This paper found clear empirical evidence of regimes with very different risk and return characteristics for U.K. stocks and bonds. Our results suggest that a three-state specification with a transitory high-volatility regime with negative mean returns, a highly persistent, 'normal' state with mean returns and volatility levels close to historical averages and a persistent high-return 'bull' state capture important features of U.K. stock and bond returns. Predictability from the dividend yield by no means subsumes the effects of predictability due to the presence of underlying states in asset returns. However, the relative effects of the two factors very much depends on the assumed investment horizon. The shorter the investment horizon, the more important the perceived state probabilities are relative to variations in the dividend yield.

One way to summarize the economic difference between our three-state model and a model that assumes no predictability is by comparing the optimal portfolio holdings associated with these models at the end of our sample. At this point, for an investor with a six month horizon, the optimal stock holdings would have been close to twice as high under the single-state model compared to the three-state model (25%) that accounts for the possibility of bull and bear markets. The weight on bonds would also have been quite different, 50% for the single-state model and 30% for the three-state model. These differences translate into very different investments in T-bills. An investor that ignores predictability should not hold T-bills, while under regime shifts the investor would hold almost 50% in T-bills to hedge against the possibility of shifts in the underlying state of the financial markets. Unsurprisingly in view of such differences in optimal asset allocations, the expected utility costs arising from ignoring regimes can be quite significant.

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Table 1

Specification Tests

Model	Number of parameters	Equity Excess Returns				Bond Excess Returns			
		Jarque-Bera test	LR_2	LR_3	LR_6	Jarque-Bera test	LR_2	LR_3	LR_6
MSIA(1,0)	5	470.96 (0.000)	0.000 (0.999)	3.380 (0.337)	12.476 (0.050)	133.76 (0.000)	0.000 (0.999)	4.082 (0.253)	6.966 (0.324)
MSIH(2,0)	12	4.159 (0.125)	0.028 (0.986)	2.372 (0.499)	8.064 (0.233)	2.580 (0.275)	0.064 (0.969)	3.910 (0.271)	6.320 (0.388)
MSIH(3,0)	21	1.852 (0.396)	0.036 (0.982)	2.200 (0.532)	7.308 (0.293)	1.587 (0.452)	0.022 (0.989)	4.666 (0.198)	7.240 (0.299)
MSIH(4,0)	32	0.345 (0.842)	0.000 (0.999)	2.638 (0.451)	7.450 (0.281)	0.562 (0.755)	0.010 (0.995)	3.332 (0.345)	6.868 (0.333)

Notes: This table reports specification tests for the multivariate regime-switching models

$$\boldsymbol{R}_t = \boldsymbol{\mu}_{S_t} + \boldsymbol{\varepsilon}_t$$

where $\boldsymbol{\mu}_{s_t}$ is the intercept vector in state s_t, and $\boldsymbol{\varepsilon}_t = [\varepsilon_{1t}\ \varepsilon_{2t}]' \sim \boldsymbol{N}(\mathbf{0}, \Sigma_{S_t})$. The unobserved state variable S_t is governed by a first-order Markov chain that can assume *k* distinct values. The two monthly return series comprise excess returns on the FTSE All Share index and FTA portfolio of 15-year government bonds (including coupon payments). Under the null of correct specification of the model, the probability integral transform of the one-step-ahead standardized forecast errors should follow an IID uniform distribution over the interval (0,1). A further Gaussian transform is applied to perform likelihood ratio tests of the null that (under correct specification) the transformed z-scores are IIN(0,1). p-values are reported in brackets below the test statistics. The sample period is 1976:01 – 2000:12. MSIAH(k,p) stands for Markov Switching Intercept Autoregressive Heteroskedasticity model with k states and p autoregressive lags.

Table 2

Parameter Estimates for the Three-State Model

	Panel A – Single State Model		
	Stocks		Bonds
1. Mean excess return	0.577 (0.289)		0.296 (0.185)
2. Correlations/Volatilities			
Stock Returns	4.995***		
Bond Returns	0.414**		3.206**
	Panel B – Two State Model		
	Stocks		Bonds
1. Mean excess return			
Bear State	-11.060 (2.716)		-1.975 (1.646)
Normal State	0.608 (0.364)		0.115 (0.191)
Bull State	1.708 (0.712)		0.833 (0.571)
3. Correlations/Volatilities			
Bear state:			
Stock Returns	8.181**		
Bond Returns	-0.456		3.936**
Normal state:			
Stock Returns	3.748***		
Bond Returns	0.375***		1.995***
Bull state:			
Stock Returns	4.891***		
Bond Returns	0.553***		4.409***
3. Transition probabilities	Bear State	Normal State	Bull State
Bear State	0.237 (0.175)	0.027 (0.086)	0.735
Normal State	0.046 (0.026)	0.914 (0.037)	0.041
Bull State	0.001 (0.031)	0.145 (0.072)	0.855

Notes: This table shows parameter estimates for the three-state regime switching model

$$\boldsymbol{R}_t = \boldsymbol{\mu}_{S_t} + \boldsymbol{\varepsilon}_t$$

where $\mathbf{R}_t$ is a 2×1 vector collecting excess returns on the FTSE All Share index and the FTA portfolio of 15-year government bonds, $\boldsymbol{\mu}_{S_t}$ is the intercept vector in state s_t, and $\boldsymbol{\varepsilon}_t = [\varepsilon_{1t}\ \varepsilon_{2t}]' \sim N(\mathbf{0}, \Sigma_{S_t})$. The unobservable state S_t is governed by a first-order Markov chain that can assume three values. For comparison, the first panel reports estimates for the single-state case, k = 1. For mean coefficients and transition probabilities, standard errors are reported in parenthesis. The sample period is 1976:01 – 2000:12. * denotes 10% significance, ** significance at 5%, *** significance at 1%.

Table 3

Estimates of Three-State Model with ARCH Effects

	Stocks	Bonds
1. Mean return		
Bear State	-11.111 (5.963)	-2.310 (2.393)
Normal State	0.709 (0.472)	0.166 (0.452)
Bull State	1.355 (0.750)	0.678 (0.488)
2. ARCH model		
Bear State:		
Constants (K_1)	66.906**	-14.629
	-14.629	15.545
ARCH coefficients (Δ_1)	0.329	0.861
	-0.041	-0.730
Normal State:		
Constants (K_2)	13.786***	2.625**
	2.625**	3.823**
ARCH coefficients (Δ_2)	-0.170	0.002
	-0.098	0.253
Bull State:		
Constants (K_3)	23.310***	13.334**
	13.334**	18.848***
ARCH coefficients (Δ_3)	0.311**	-0.171
	-0.106	0.088

3. Transition probabilities	Bear State	Normal State:	Bull State
Bear State	0.094 (0.229)	0.000 (0.217)	0.906
Normal State	0.040 (0.024)	0.923 (0.034)	0.037
Bull State	0.000 (0.083)	0.131 (0.151)	0.869

Notes: This table shows estimation results for the regime switching model

$$\boldsymbol{R}_t = \boldsymbol{\mu}_{S_t} + \boldsymbol{\varepsilon}_t$$

where $\mathbf{R}_t$ is a 2×1 vector collecting excess returns on the FTSE All Share index the FTA portfolio of 15-year government bonds, $\boldsymbol{\mu}_{S_t}$ is the intercept vector in state s_t, and $\boldsymbol{\varepsilon}_t = [\varepsilon_{1t}\ \varepsilon_{2t}]' \sim \boldsymbol{N}(\mathbf{0}, \Sigma_{S_t})$. Within each regime, $\boldsymbol{\Sigma}_{S_t}$ follows a multivariate ARCH model

$$\Sigma_{S_t} = \boldsymbol{K}_{s_t} + \Delta_{s_t} \boldsymbol{\varepsilon}_{t-1} \boldsymbol{\varepsilon}'_{t-1} \Delta'_{s_t}$$

where K_{St} is symmetric and positive definite.The unobservable state S_t is governed by a first-order Markov chain that can assume three values. For mean coefficients and transition probabilities, standard errors are reported in parenthesis. The sample period is 1976:01 – 2000:12. The sample period is 1976:01 – 2000:12. * denotes 10% significance, ** significance at 5%, *** significance at 1%.

Table 4

Estimates of a Three-State Model for Stock, Bond Returns and the Dividend Yield

	Panel A – Single State Model		
	Stocks	Bonds	Dividend Yield
1. Mean return	-1.585 (1.136)	-0.390 (0.735)	0.081 (0.057)
2. VAR(1) Matrix			
Stock returns	0.019 (0.063)	0.168 (0.098)	0.473 (0.245)
Bond returns	0.022 (0.041)	0.069 (0.063)	0.148 (0.159)
Dividend Yield	-0.002 (0.003)	-0.009 (0.005)	0.981 (0.012)
3. Correlations/Volatilities			
Stock Returns	4.937**		
Bond Returns	0.405	3.193**	
Dividend Yield	-0.939**	-0.439*	0.248*
	Panel B – Three State Model		
	Stocks	Bonds	Dividend Yield
1. Intercept Coefficients			
Bear State	-9.202 (11.584)	4.205 (1.404)	0.027 (0.550)
Normal State	0.499 (1.562)	-1.445 (0.650)	0.034 (0.065)
Bull State	-28.276 (6.943)	-11.593 (5.940)	2.035 (0.418)
2. VAR(1) Matrices			
Bear State:			
Stock Returns	0.083 (0.145)	0.715 (0.407)	1.068 (0.465)
Bond Returns	-0.016 (0.055)	0.449 (0.171)	-1.076 (0.281)
Dividend Yield	-0.008 (0.007)	-0.047 (0.021)	1.037 (0.023)
Normal State:			
Stock Returns	-0.136 (0.074)	0.146 (0.085)	0.207 (0.690)
Bond Returns	-0.038 (0.057)	0.105 (0.067)	0.449 (0.335)
Dividend Yield	0.006 (0.003)	-0.006 (0.003)	0.982 (0.034)
Bull State:			
Stock Returns	0.088 (0.111)	0.161 (0.151)	5.105 (1.218)
Bond Returns	0.220 (0.111)	-0.133 (0.130)	2.127 (1.038)
Dividend Yield	-0.006 (0.006)	-0.010 (0.009)	0.639 (0.074)
3. Correlations/Volatilities			
Bear State:			
Stock Returns	6.761***		
Bond Returns	-0.127	2.230	
Dividend Yield	-0.940	0.157	0.328*
Normal State:			
Stock Returns	3.535**		
Bond Returns	0.491	2.593*	
Dividend Yield	-0.957*	-0.515	0.148
Bull State:			
Stock Returns	4.443***		
Bond Returns	0.507***	4.135***	
Dividend Yield	-0.986*	-0.462	0.253*
4. Transition probabilities	Bear State	Normal State:	Bull State
Bear State	0.463 (0.101)	0.450 (0.144)	0.087
Normal State	0.086 (0.055)	0.909 (0.039)	0.005
Bull State	0.046 (0.032)	0.016 (0.023)	0.938

Notes: This table shows estimation results for the regime switching model

$$\boldsymbol{y}_t = \boldsymbol{\mu}_{S_t} + \boldsymbol{A}^*_{S_t}\boldsymbol{y}_{t-1} + \boldsymbol{\varepsilon}_t$$

where $\mathbf{y}_t$ is a 3×1 vector collecting excess returns on the FTSE All Share index, excess returns on a FTA portfolio of 15-year government bonds, and the dividend yield, $\boldsymbol{\mu}_{S_t}$ is the intercept vector in state s_t, $\boldsymbol{A}^*_{S_t}$ is the matrix of first-order autoregressive coefficients in state s_t and $\boldsymbol{\varepsilon}_t = [\varepsilon_{1t}\ \varepsilon_{2t}\ \varepsilon_{3t}]' \sim N(0, \Omega_{s_t})$. The unobservable state S_t is governed by a first-order Markov chain that can assume one of three values. The sample period is 1976:01 – 2000:12. * denotes 10% significance, ** significance at 5%, *** significance at 1%.

Table 5

Optimal Asset Allocation – Effect of Rebalancing

Rebalancing Frequency φ	Panel A: Allocation to Equities						Panel B: Allocation to Bonds					
	T=1	**T=6**	**T=12**	**T=24**	**T=60**	**T=120**	**T=1**	**T=6**	**T=12**	**T=24**	**T=60**	**T=120**
							Bear regime					
φ = T (buy-and-hold)	0.00	0.28	0.33	0.40	0.41	0.42	0.00	0.51	0.46	0.41	0.40	0.40
φ = 24 months	NA	NA	NA	NA	0.40	0.40	NA	NA	NA	NA	0.43	0.43
φ = 12 months	NA	NA	NA	0.33	0.33	0.33	NA	NA	NA	0.46	0.46	0.46
φ = 6 months	NA	NA	0.30	0.30	0.30	0.30	NA	NA	0.48	0.48	0.48	0.48
IID (no predictability)	0.59	0.59	0.59	0.59	0.59	0.59	0.41	0.41	0.41	0.41	0.41	0.41
							Normal regime					
φ = T (buy-and-hold)	1.00	0.20	0.31	0.36	0.40	0.40	0.00	0.34	0.38	0.38	0.40	0.40
φ = 24 months	NA	NA	NA	NA	0.41	0.41	NA	NA	NA	NA	0.38	0.38
φ = 12 months	NA	NA	NA	0.42	0.42	0.42	NA	NA	NA	0.37	0.37	0.37
φ = 6 months	NA	NA	0.43	0.43	0.43	0.43	NA	NA	0.35	0.35	0.35	0.35
IID (no predictability)	0.59	0.59	0.59	0.59	0.59	0.59	0.41	0.41	0.41	0.41	0.41	0.41
							Bull regime					
φ = T (buy-and-hold)	1.00	0.81	0.63	0.52	0.47	0.44	0.00	0.19	0.36	0.39	0.39	0.38
φ = 24 months	NA	NA	NA	NA	0.49	0.49	NA	NA	NA	NA	0.37	0.37
φ = 12 months	NA	NA	NA	0.59	0.59	0.59	NA	NA	NA	0.40	0.40	0.40
φ = 6 months	NA	NA	0.66	0.66	0.66	0.66	NA	NA	0.30	0.30	0.30	0.30
IID (no predictability)	0.59	0.59	0.59	0.59	0.59	0.59	0.41	0.41	0.41	0.41	0.41	0.41
							Steady-State Probabilities					
φ = T (buy-and-hold)	0.59	0.45	0.44	0.44	0.43	0.42	0.41	0.44	0.40	0.39	0.39	0.38
φ = 24 months	NA	NA	NA	NA	0.42	0.42	NA	NA	NA	NA	0.41	0.41
φ = 12 months	NA	NA	NA	0.44	0.44	0.44	NA	NA	NA	0.40	0.42	0.42
φ = 6 months	NA	NA	0.44	0.44	0.44	0.44	NA	NA	0.42	0.42	0.42	0.42
IID (no predictability)	0.59	0.59	0.59	0.59	0.59	0.59	0.41	0.41	0.41	0.41	0.41	0.41

Notes: This table reports the optimal weight invested in equities and long-term bonds as a function of the rebalancing frequency φ for an investor with power utility and a constant relative risk aversion coefficient of 5. Excess returns on the FTSE-All Share index and long-term bonds are assumed to be generated by a three-state model.

Table 6

Expected Utility Costs

Rebalancing Frequency φ	Investment Horizon T (months)					
	T=1	T=6	T=12	T=24	T=60	T=120
φ = T (buy-and-hold)	0.050	0.072	0.079	0.167	0.482	1.502
φ = 24 months	0.050	0.072	0.079	0.167	4.386	9.703
φ = 12 months	0.050	0.072	0.079	2.437	10.875	9.984
φ = 6 months	0.050	0.078	0.096	2.681	11.007	10.346

Notes: This table reports the expected utility cost due to being constrained to take portfolio decisions on the basis of a model that ignores regimes for an investor with power utility and a coefficient of relative risk aversion $\gamma = 5$. Excess returns on the FTSE-All Share index and long-term bonds are assumed to be generated by a three-state regime-switching model. Expected utility costs are measured by the annualized (riskless) percentage return required by an investor to ignore regimes. Initial state probabilities are set at their steady-state values.

Table 7

Effect of Parameter Estimation Uncertainty

		Investment Horizon T						
		T=1	T=6	T=24	T=48	T=72	T=96	T=120
Panel A: Allocation to Equities								
Bear state	Upper 90% band	0.00	0.50	0.55	0.57	0.58	0.58	0.58
	Mean	0.00	0.22	0.27	0.29	0.29	0.29	0.29
	Lower 90% band	0.00	0.00	0.00	0.00	0.00	0.00	0.00
	% above IID	0.00	0.00	2.33	3.83	4.17	4.33	4.50
Normal state	Upper 90% band	1.00	0.51	0.55	0.57	0.58	0.58	0.59
	Mean	0.71	0.23	0.28	0.29	0.29	0.30	0.30
	Lower 90% band	0.36	0.00	0.00	0.00	0.00	0.00	0.00
	% above IID	62.50	0.00	2.17	4.17	4.33	4.00	5.00
Bull state	Upper 90% band	1.00	0.77	0.62	0.62	0.61	0.60	0.60
	Mean	0.79	0.46	0.35	0.33	0.32	0.31	0.31
	Lower 90% band	0.46	0.11	0.04	0.02	0.00	0.00	0.00
	% above IID	68.00	24.67	9.00	7.00	6.50	1.17	5.83
Steady-state	Upper 90% band	0.96	0.58	0.59	0.60	0.59	0.59	0.60
	Mean	0.55	0.32	0.31	0.31	0.30	0.30	0.30
	Lower 90% band	0.12	0.00	0.00	0.00	0.00	0.00	0.00
	% above IID	40.33	5.00	5.00	6.00	6.17	6.17	5.50
Panel B: Allocation to Bonds								
Bear state	Upper 90% band	0.20	0.79	0.79	0.78	0.78	0.78	0.78
	Mean	0.07	0.41	0.39	0.38	0.38	0.38	0.38
	Lower 90% band	0.00	0.02	0.00	0.00	0.00	0.00	0.00
	% above IID	1.17	38.00	37.00	34.67	34.83	34.67	34.67
Normal state	Upper 90% band	0.50	0.68	0.75	0.76	0.76	0.76	0.78
	Mean	0.17	0.32	0.36	0.36	0.37	0.37	0.37
	Lower 90% band	0.00	0.00	0.00	0.00	0.00	0.00	0.00
	% above IID	10.00	27.83	33.67	33.17	33.17	34.00	33.50
Bull state	Upper 90% band	0.50	0.76	0.78	0.77	0.78	0.78	0.78
	Mean	0.20	0.37	0.38	0.38	0.37	0.37	0.37
	Lower 90% band	0.00	0.00	0.00	0.00	0.00	0.00	0.00
	% above IID	14.67	33.50	35.17	35.17	33.17	33.33	33.33
Steady-state	Upper 90% band	0.76	0.80	0.80	0.78	0.78	0.78	0.78
	Mean	0.33	0.39	0.38	0.37	0.37	0.35	0.35
	Lower 90% band	0.00	0.00	0.00	0.00	0.00	0.00	0.00
	% above IID	29.17	36.83	35.50	33.67	34.00	35.50	34.00

Notes: This table reports 90% confidence bands for a buy-and-hold investor's optimal portfolio weights at different investment horizons, T, assuming a constant relative risk aversion coefficient of 5. Under regime switching, portfolio returns are assumed to be generated by the model

$$\boldsymbol{R}_t = \boldsymbol{\mu}_{S_t} + \boldsymbol{\varepsilon}_t$$

where $\mathbf{R}_t$ is a 2×1 vector collecting excess returns on the FTSE All Share index and the FTA portfolio of 15-year government bonds, $\boldsymbol{\mu}_{S_t}$ is the intercept vector in state s_t, and $\boldsymbol{\varepsilon}_t = [\varepsilon_{1t}\ \varepsilon_{2t}]' \sim N(\mathbf{0}, \Sigma_{S_t})$. In the IID case, $k = 1$.

Table 7 - continued

		Investment Horizon T						
		T=1	**T=6**	**T=24**	**T=48**	**T=72**	**T=96**	**T=120**
		Panel C: Allocation to T-bills						
Bear state	Upper 90% band	1.00	0.79	0.74	0.72	0.72	0.72	0.72
	Mean	0.93	0.37	0.34	0.33	0.33	0.33	0.33
	Lower 90% band	0.77	0.00	0.00	0.00	0.00	0.00	0.00
	% above IID	99.00	87.67	88.17	86.83	87.67	90.00	86.83
Normal state	Upper 90% band	0.38	0.96	0.78	0.76	0.74	0.74	0.74
	Mean	0.12	0.45	0.36	0.35	0.34	0.33	0.33
	Lower 90% band	0.00	0.02	0.00	0.00	0.00	0.00	0.00
	% above IID	47.33	95.67	91.17	91.00	89.83	89.67	88.33
Bull state	Upper 90% band	0.00	0.48	0.63	0.67	0.69	0.70	0.71
	Mean	0.01	0.17	0.27	0.29	0.31	0.32	0.32
	Lower 90% band	0.00	0.00	0.00	0.00	0.00	0.00	0.00
	% above IID	1.33	66.83	81.83	83.17	86.33	85.17	87.00
Steady-state	Upper 90% band	0.36	0.68	0.70	0.72	0.72	0.72	0.72
	Mean	0.12	0.29	0.31	0.32	0.33	0.34	0.34
	Lower 90% band	0.00	0.00	0.00	0.00	0.00	0.00	0.00
	% above IID	55.83	82.00	83.33	85.17	87.17	87.00	87.33

Notes: see panels A – B.

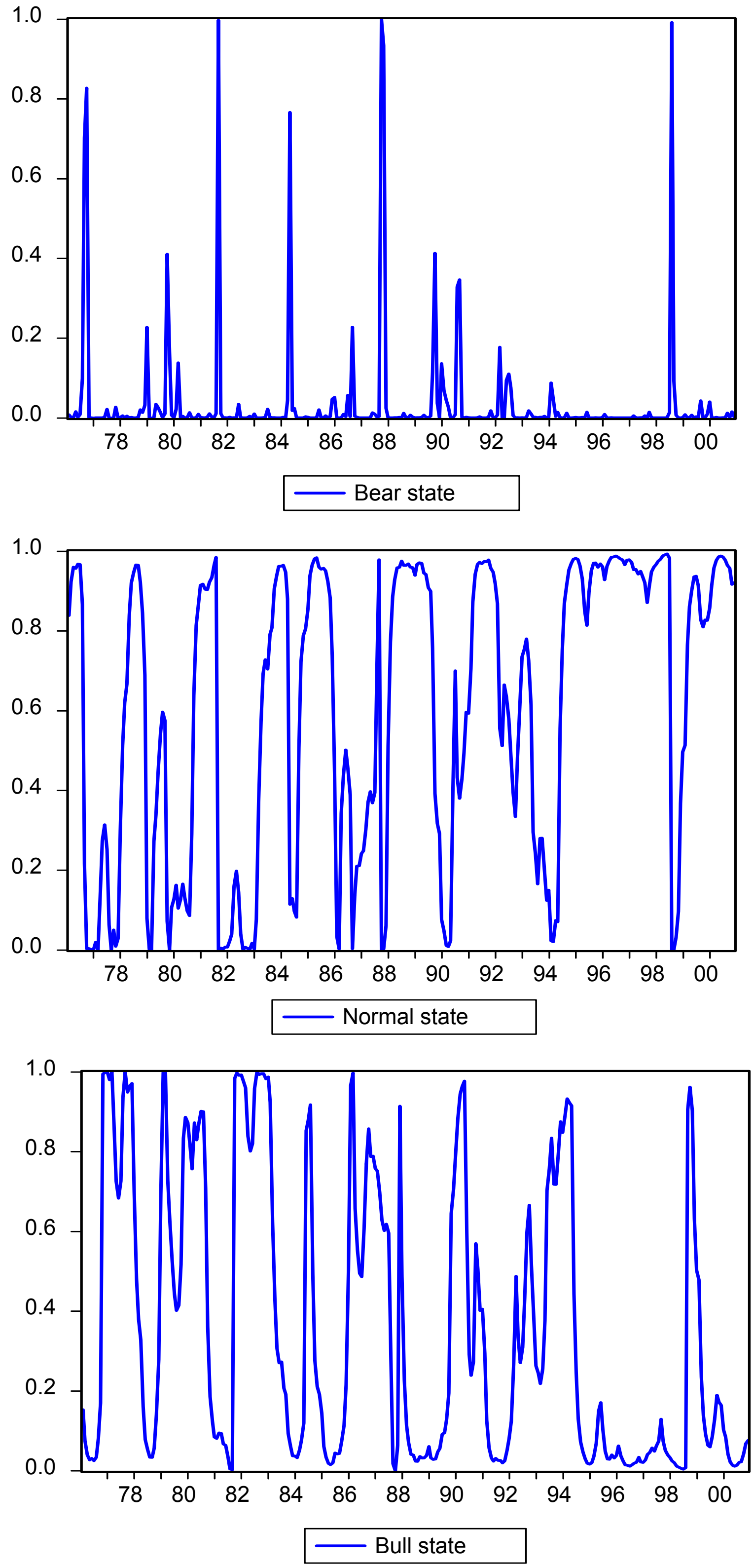

Fig. 1 - Smoothed State Probabilities: Three-State Model for Stock and Bond Excess Returns

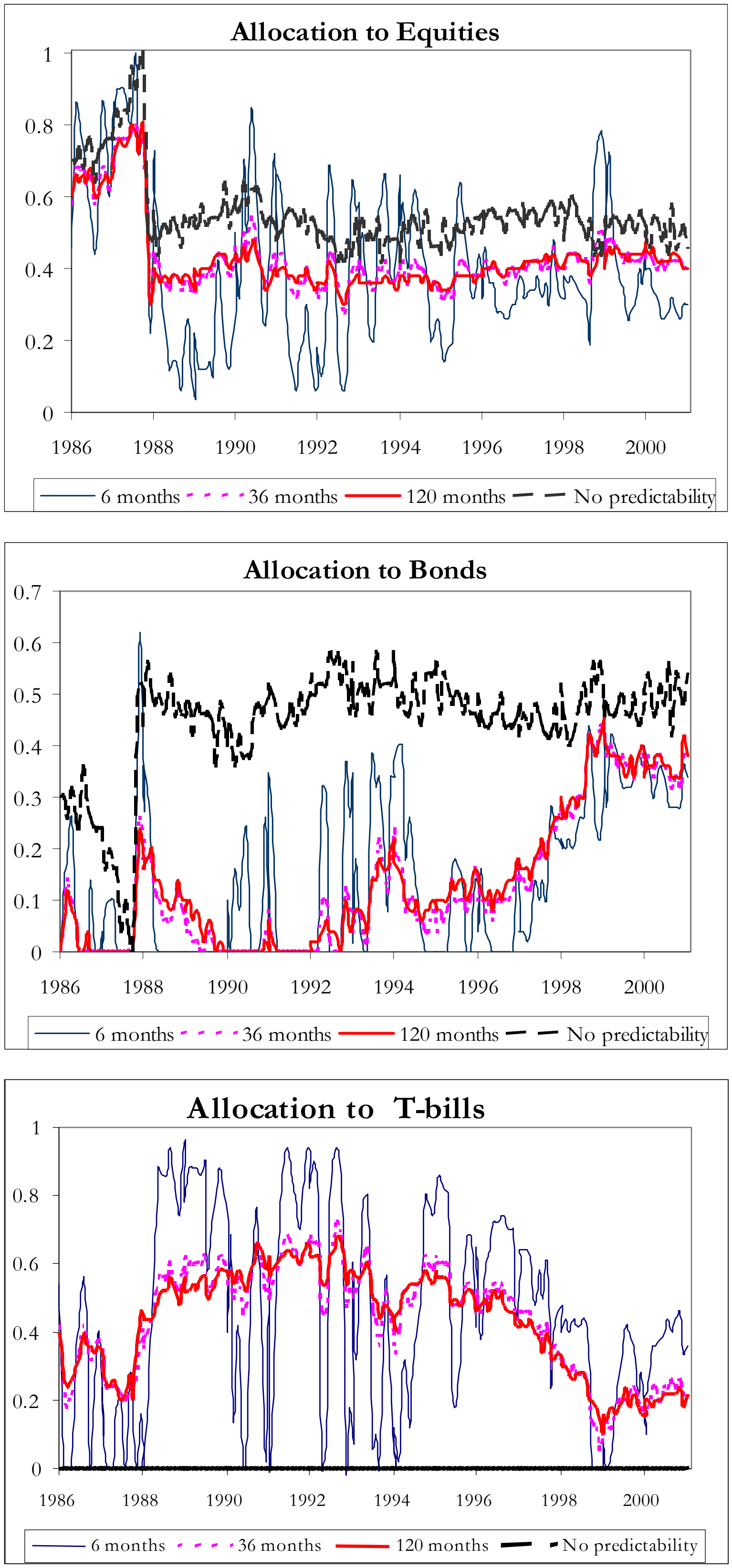

Fig. 2 - Real-Time Optimal Asset Allocations

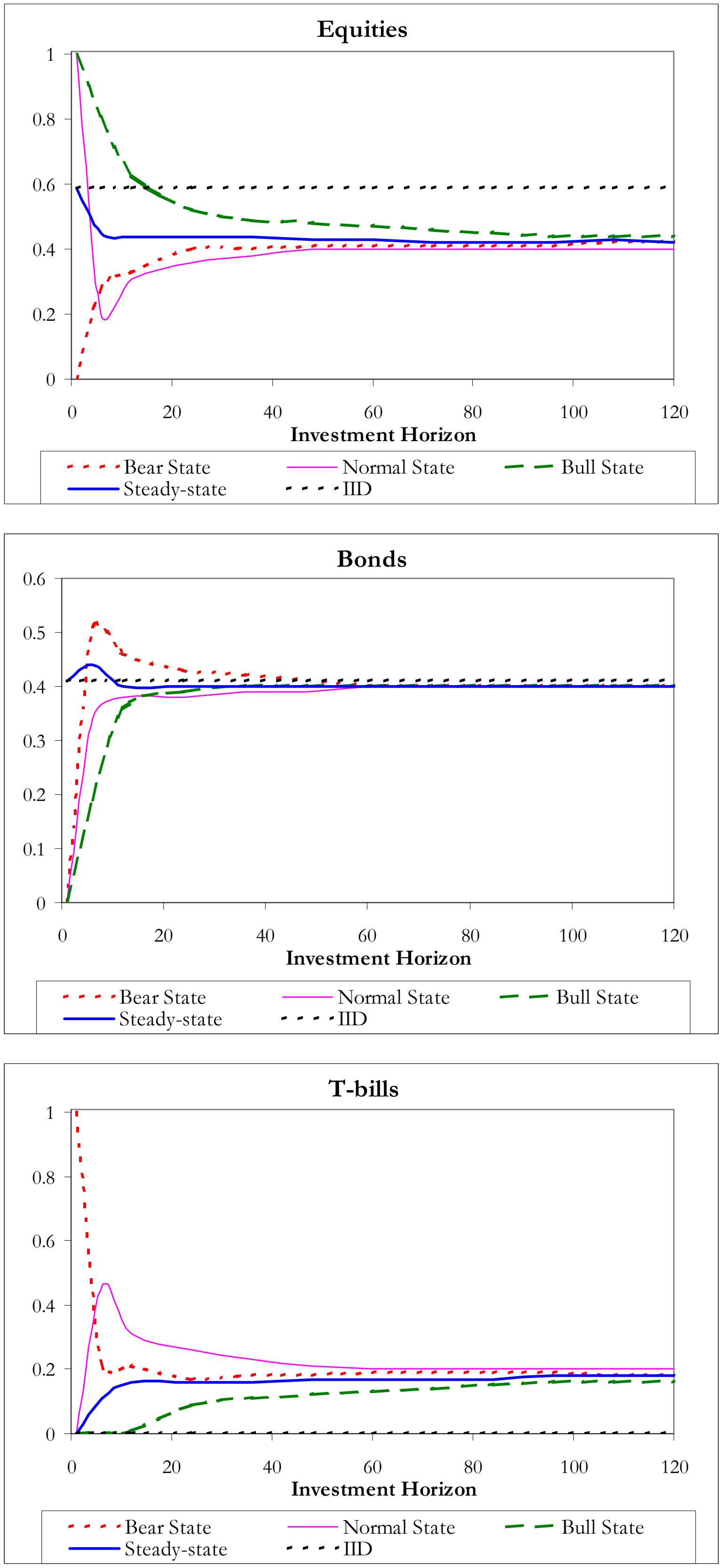

Fig. 3 - Optimal Asset Allocation as a Function of the Investment Horizon

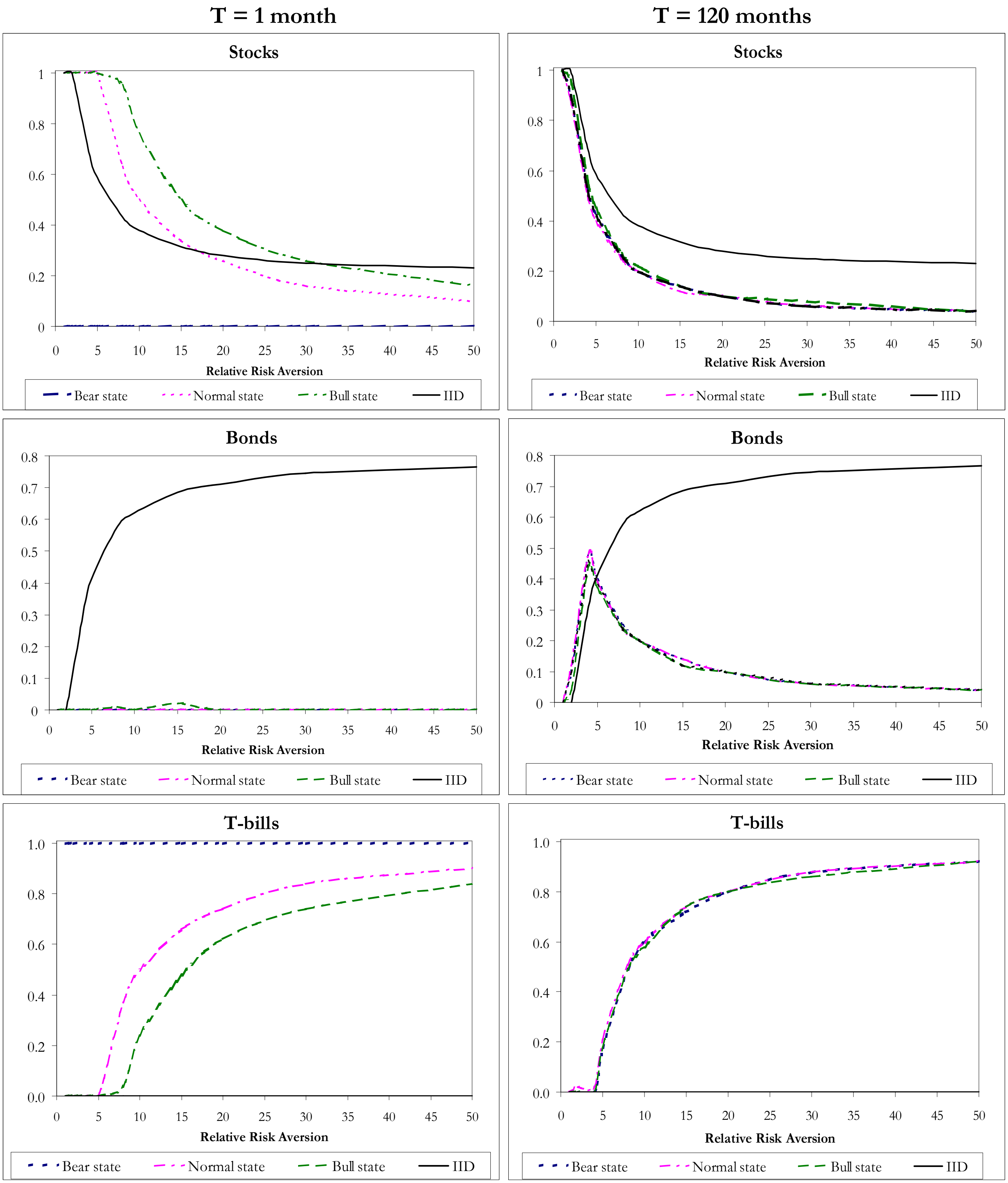

Fig. 4 - Effect of Relative Risk Aversion on Optimal Asset Allocation

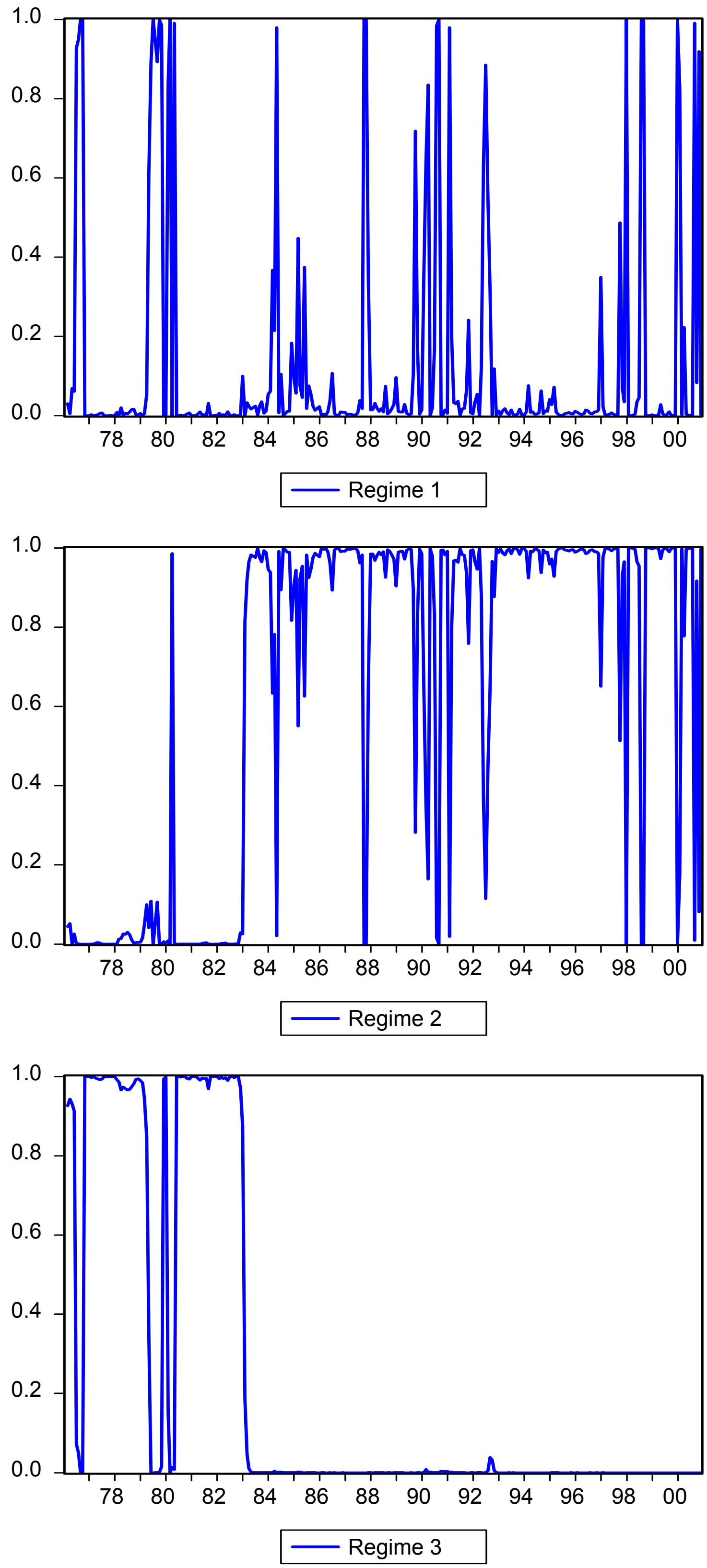

Fig. 5 - Smoothed State Probabilities for extended three-State Model

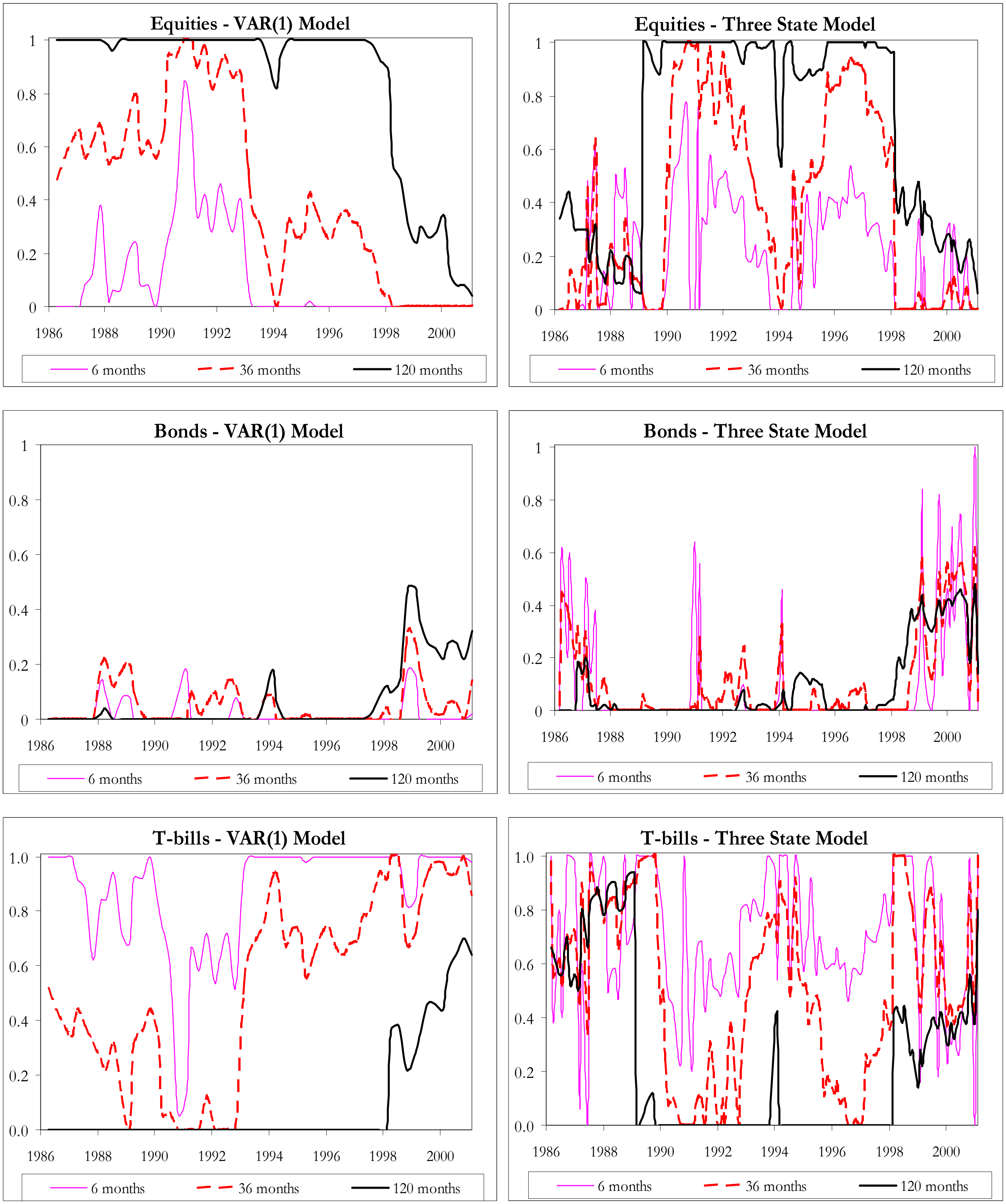

Fig. 6 - Real-Time Asset Allocation under Predictability from the Dividend Yield

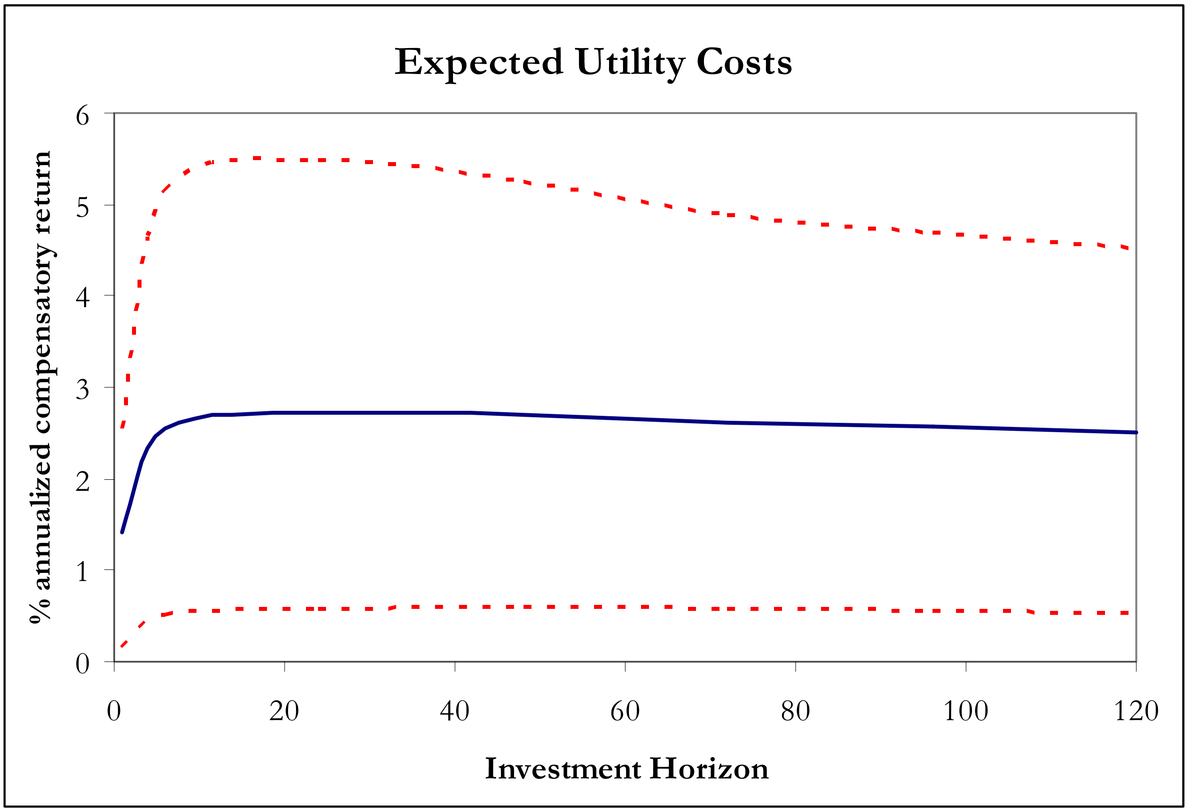

Fig. 7 - Effect of Parameter Estimation Uncertainty on Expected Utility Costs