Abstract

If traders are unwilling to display liquidity due to the information revealed by orders, then an opportunity to trade at a posted price can be welfare enhancing. We introduce a posted-price mechanism into a model of bargaining with incomplete information and strategic delay. Traders delay displaying liquidity even further when it is possible to trade at a posted price, but the net effect is to accelerate trade and increase welfare.
Traders do not like to reveal their trading intentions. A variety of market practices and market structures have been created to make it easier for traders to avoid disclosing their intentions. Examples include hidden or iceberg orders, dark pools, and workups and matching sessions.\(^1\) We model a situation in which a trader, in order to get better terms, prefers not to disclose how strongly he desires to trade. Consequently, the trader delays providing liquidity, producing delays in trade execution. In this environment, we include an opportunity to trade at a posted price. Liquidity in the posted-price mechanism is not revealed ex ante, so participating in the mechanism is an alternative to displaying liquidity. Our model is motivated by workups in the interdealer market for U.S. Treasury bonds, described by Duffie and Zhu (2017). In workups, trading in the limit order book is suspended momentarily and the price is frozen whenever there is a transaction, and traders can submit orders to transact at the frozen price. An obvious question is why traders who are willing to trade at the workup price did not do so beforehand, by submitting limit orders at that price. The answer to that question in our model is that posting orders reveals information leading to terms of trade that are worse than the posted price.

In the model, the opportunity to trade at a posted price arrives at a random date, and the posted price is also drawn randomly (rather than being a previous transaction price or a price from a lit market). Traders simultaneously and without coordination submit demands/supplies to the mechanism, and matching orders are executed at the posted price. Importantly, the mechanism in the model has not been fine tuned to maximize welfare gains. However, it does improve welfare. This is true despite the fact that the

\(^1\)Dark pools execute approximately 14% of total U.S. equity volume, and hidden orders on lit markets account for another 8% (Rosenblatt Securities, October, 2018). Duffie and Zhu (2017) describe workups in U.S. Treasury markets and matching sessions in markets for corporate bonds and credit default swaps. They cite Fleming and Nguyen (2018) that workups account for around half of total trading volume on the largest U.S. Treasuries trade platform, and they cite Collin-Dufresne, Junge and Trolle (2017) that matching sessions and workups account for 70% of trading volume on a particular swap execution facility.
mechanism exacerbates the original friction that we study—the trader becomes even more reluctant to post an order when he may later have an opportunity to trade at a posted price. Despite the original friction becoming worse, the net effect of the mechanism is to increase aggregate gains from trade.

An interesting example regarding the effects of displaying liquidity is the sanctioning by FINRA of Trillium Brokerage in 2010. Trillium was censured and fined, and multiple individuals were suspended from the securities industry, because Trillium’s traders entered orders to create “a false sense of buying or selling pressure,” inducing

other market participants to enter orders to execute against limit orders previously entered by the Trillium traders. Once their orders were filled, the Trillium traders would then immediately cancel orders that had only been designed to create the false appearance of market activity. . . . ‘Trillium’s trading conduct was designed to improperly bait unsuspecting market participants into executing trades at illegitimately high or low prices for the advantage of Trillium’s traders,’ said Thomas R. Gira, Executive Vice President, FINRA Market Regulation (FINRA News Release, September 13, 2010).

The traders at Trillium apparently believed that other market participants attempt to exploit the information in displayed liquidity. Likewise, in our model, displaying liquidity leads to exploitation—displaying liquidity early in our model signals a strong desire to trade, which is exploited by the trader’s counterparty.

Our model is a variation of the model of strategic delay in bargaining due to Admati and Perry (1987) and Cramton (1992), who build on the alternating-offers bargaining game.

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2The activity in which Trillium was alleged to have engaged is called spoofing. Another example is the trader Navinder Sarao, who was criminally charged by the U.S. Department of Justice for alleged spoofing that contributed to the flash crash in May 2010 (Miedema and Lynch, 2015). We thank Markus Baldauf for suggesting this example.
of Rubinstein (1982). We consider a buyer and seller who have different valuations for an asset. For simplicity, we assume the seller’s valuation is known, and the seller initially offers the asset at a price he chooses, without knowing the buyer’s valuation. The buyer can respond by accepting the offer or by delaying an arbitrary amount of time before making a bid. An early bid signals a ‘weak’ bidder, that is, a buyer with a high valuation who has a lot to lose by delaying acquisition of the asset. As Admati and Perry (1987) and Cramton (1992) show, the Rubinstein analysis implies that once the buyer has signaled his valuation, trade must take place at the midpoint of the buyer’s and seller’s valuations. Thus, greater delay, which signals a lower valuation, leads to a better price for the buyer. The buyer trades off the better price against the costs of delay. Eventually, trade takes place, but there is a welfare loss due to the delay. The possibility of trading at a posted price causes the buyer to delay even more before making a bid, but considering both trades made in the mechanism and trades made when the buyer bids, the net effect is to reduce the costs due to delay. This is a welfare gain.

The welfare gain in our model can be understood in terms of the costs of signaling. If all trades are by posted orders, then patient buyer types separate themselves from impatient types by incurring the costs of delay. When we add the opportunity to trade at a posted price, two things happen. One is that the costs of signaling increase. It becomes more difficult for patient types to separate themselves from impatient types prior to the arrival of the posted price opportunity, so they incur greater costs of delay. However, trading at a posted price creates partial pooling—multiple buyer types are able to trade at the same price. The buyer types who are pooled avoid signaling costs. While these two forces operate in opposite directions, the net effect is for costs to fall and welfare to rise.3

The model we develop has implications beyond market microstructure. Consider, for

3We thank Ron Giammarino for suggesting that we emphasize this interpretation.
example, a company bargaining with a labor union. Suppose the company has an offer on
the table, and the union is delaying responding, perhaps engaged in a strike, in order to
signal its strength. An arbitrator could play the role of the posted price mechanism in our
paper, proposing a resolution which both parties can accept or reject. This can result in
pooling of types of the labor union, reducing signaling costs and hastening an agreement.4

1. Literature Review

Our conclusion about the welfare benefits of trading at posted prices is the opposite of
that reached by Antill and Duffie (2018), who address the same question in a very different
model. In their model, traders optimally use the posted-price mechanism (‘size discovery
sessions’), but overall welfare is reduced by the presence of the mechanism. The ‘normal’
market mechanism in Antill and Duffie’s model is a sequence of batch auctions. The friction
in this market is that each trader is concerned with price impacts and hence trades less
than he would in a Walrasian environment. The posted-price mechanism facilitates trade
by making it possible to avoid price impacts. However, traders respond to the posted-price
mechanism by reducing trades in the normal market so much that the net effect is actually
to lower welfare.

We model a different friction than that modeled by Antill and Duffie, and we obtain
a different result. The batch auctions in the Antill-Duffie model constitute the market
mechanism recommended by Budish, Cramton and Shim (2015). One interpretation of
the Antill-Duffie result is that posted-price mechanisms would not be welfare improving if
the Budish-Cramton-Shim market reforms were adopted. However, that leaves open the
question of whether such mechanisms improve welfare when the normal market mechanism
is an open limit order book. Because orders are submitted simultaneously and batched in

4We thank Hernan Ortiz-Molina for suggesting this application.
the Antill-Duffie model, the issues with displaying liquidity do not arise there. There are at least two issues with displaying liquidity. One is the ‘sniping’ risk addressed by Budish, Cramton, and Shim. The other is the information revealed by displayed liquidity, captured in the Trillium example and in our model. In our model, the posted-price mechanism is welfare enhancing, because it makes it possible to trade without revealing information ex ante.

Our conclusion about the welfare benefits of posted-price mechanisms is the same as that reached by Duffie and Zhu (2017), but we reach the conclusion for very different reasons. Duffie and Zhu assume the mechanism occurs only at date 0, so there is no issue of traders reducing their trades in the normal market (which is again a sequence of batch auctions) in anticipation of trading in the mechanism later. In contrast, our buyer does delay his bid even more when there is a possibility of trading at a posted price later. However, the social cost of this further delay is more than offset on average by the social benefits of trading at a posted price.

There are many papers on dark pools, but we do not know of any papers that model the dark pool as a device to avoid information leakage. Many model the lit market as a dealer market, in which investors cannot post orders (for example, Hendershott and Mendelson, 2000; Degryse et al., 2009). Usually, the traders in these models only want to trade one unit, which they do either through a market order or in the dark pool. Because they trade only once and do not post visible orders, there is no possibility of information leakage. Many papers on dark pools include noise traders (for example, Ye, 2012) or assume that trades in the dark pool are crossed against unmodeled traders, which makes welfare analysis problematic (for example, Bieklagk et al., 2019). Other papers assume traders are infinitesimal and hence are unconcerned with either price impact or information revelation (for example, Zhu, 2014).
One paper on dark pools that does make a welfare claim is Buti, Rindi and Werner (2017). They argue that dark pools reduce welfare, which is the opposite of our finding. Theirs is a four-date model in which a single trader arrives at each date. If a seller arrives at date 1 and posts an offer at the inside ask, then a seller who arrives at date 2 may choose to submit to the dark pool to avoid being behind the first seller in time/price priority (further price improvement beyond the inside ask is not possible in their model). Submitting to the dark pool avoids the tick size constraint—it creates a possibility of trading at the spread midpoint, which is between ticks and is therefore a price at which an order cannot be posted in the book. Unlike our paper, the Buti et al. paper is not focused on information leakage. In fact, the seller in the above example who went to the dark pool would prefer that it be lit, so any buyer who arrives at dates 3 or 4 would know that liquidity was available. The only friction in the limit order book in the Buti et al. model that trading in the dark could solve is the tick size constraint. In our model, there is an informational friction that trading at a posted price can and does mitigate.

2. Model and Overview

We model the trade of a single unit of an asset, held initially by an agent called the seller. The asset may possibly be traded to an agent called the buyer. It is common knowledge that the seller’s value for the asset is 0. The buyer’s value for the asset is denoted by \( b \) and is uniformly distributed on \([0, 2]\). If a trade eventually occurs at price \( p \) at date \( t \), then the present value of the seller’s gain from trade is \( e^{-rt}p \), and the present value of the buyer’s gain is \( e^{-rt}(b-p) \), for a constant \( r \). The discount rate \( r \) embodies not just the interest rate but also other factors creating an urgency to trade, including the risk that the market may move and valuations may change before trade occurs. We do not model that risk, except for considering aversion to it as partially underlying the parameter \( r \). When trade occurs, the gain from trade is the difference between the buyer’s and seller’s valuations, which is
simply $b$. The maximum possible expected gain from trade is the mean of $b$, which is 1. As we show below, trade always occurs in our model. However, there is an inefficiency due to delay.

At date 0, the seller offers the asset at a price $p$ that he chooses. The buyer chooses whether to accept this offer. If he declines it, then he has the opportunity to submit a bid at a later date—a date that he chooses. The assumption that the seller does not make a new offer before the buyer bids is motivated by the idea that the seller should not want to compete against himself.

At an exponentially distributed time $\tau$, if the buyer and seller have not already traded by $\tau$, then they are confronted with an opportunity to trade at a random price $q$ drawn from the uniform distribution on $[0, 1]$. The price $q$ is observed by both the buyer and seller, and then they choose simultaneously whether to participate. If both choose to participate, then trade occurs at the price $q$. The buyer and seller learn nothing if they do not participate, and learn whether their order executed if they do participate. If the seller chooses to participate, but the buyer does not, then the seller learns that the buyer did not, which is informative to the seller about the buyer’s valuation. After such an event, the game continues as before from this new information state. For simplicity, we assume the posted-price mechanism occurs only once. Let $\lambda$ denote the parameter of the exponential time, so the probability of arrival in an instant $dt$ is $\lambda dt$. If either the buyer or seller chooses not to participate in the mechanism, then both parties resume waiting until the buyer makes a bid. Thus, the buyer is still ‘on the clock’ in our model.

In equilibrium, trade always occurs. If the buyer has a sufficiently high valuation, then he will accept the seller’s initial offer. If he does not, then there are three possibilities: (i) the buyer makes a bid before the posted-price mechanism arrives, and the bid is accepted by the seller, (ii) the buyer does not make a bid before the mechanism arrives, and trade
occurs in the mechanism, (iii) the buyer does not make a bid before the mechanism arrives, trade does not occur in the mechanism, and after an additional delay the buyer makes a bid that is accepted by the seller.

The general form of the equilibrium is that at each date $t$ there is some buyer type $b = \xi(t)$ who would bid at that date. Thus, the seller can infer the buyer’s valuation from the timing of the bid. Of course, the function $\xi(\cdot)$ must be incentive compatible—no type of buyer can benefit from bidding at the time the seller expects some other type of buyer to be bidding. For a given price $p$, the dollar loss due to discounting is higher for higher buyer types, so higher types are less patient. Thus, $\xi(\cdot)$ will turn out to be a decreasing function.

In the formal game, there are alternating offers—when the buyer makes a bid, the seller can reject it and then later make another offer, and so on. There is a minimum amount of time $\Delta$ that must pass between offers. However, we are interested in the limit as $\Delta \to 0$. The subgame following the posted-price mechanism with no trade is the same as a subgame in Cramton (1992), because we assume the mechanism can only occur once. This game is the Rubinstein (1982) alternating-offers bargaining game with one-sided incomplete information and with endogenous delay beyond $\Delta$. With a minimum delay of $\Delta > 0$ between orders, if a buyer bids and reveals his value $b$, then his equilibrium bid is $B$ such that the seller is indifferent between (i) accepting the bid and (ii) rejecting the bid and submitting an ask $A$ after the minimum delay of $\Delta$. Thus, $B = e^{-r\Delta}A$. Likewise, if the seller makes an offer, it is such that the buyer is indifferent between accepting it and rejecting it and making a new bid after the minimum delay, which implies $b - A = e^{-r\Delta}(b - B)$. The solution to this pair of equations is

$$A = \frac{b}{1 + e^{-r\Delta}}, \quad B = \frac{e^{-r\Delta}b}{1 + e^{-r\Delta}}.$$  (1)
In the limit as $\Delta \to 0$, we have $A = B = b/2$. If a buyer of type $b$ submits a bid at a time $t$ when a different buyer type $b'$ should have bid, then the bid is $b'/2$, and the resulting profit of the buyer is $e^{-rt}(b - b'/2)$. The reason the bid has to be $b'/2$ is that the seller will reject any lower bid, thinking the buyer is of type $b'$ and then quote the ask of $b'/2$ as just described. The seller continues this behavior indefinitely. Hence, $b'/2$ is the best price the buyer can get after bidding when $b'$ should have bid. This is the sense in which moving early—when $b = \xi(t)$ is higher—leads to worse terms of trade for the buyer. The seller's inference about the buyer's valuation when the buyer displays liquidity by making a bid results in an incentive to delay bidding.

Optimal behavior is easily determined in the posted-price mechanism, given that the Cramton equilibrium is expected to be followed if there is no trade in the mechanism. We take into consideration that the information state can change as a result of the mechanism: If the seller chooses to participate, but the buyer does not, then the seller learns that the buyer did not participate and may therefore revise his estimate of the buyer's type downwards.

We derive the equilibrium strategies before the mechanism arrives. As in the subgame following the mechanism, if a bidder with valuation $b$ bids prior to the arrival of the mechanism, then the buyer bids $b/2$ in the limit as $\Delta \to 0$. The reasoning above leading to the pair of equations (1) has to be modified only by including the possibility of trading in the mechanism if it arrives in the time period of length $\Delta$. For example, the ask price is such that the buyer is indifferent between accepting it on the one hand, and, on the other, rejecting it and either trading in the mechanism if it arrives in the time period of length $\Delta$ or making a bid after $\Delta$ has passed. These considerations produce equations that differ from (1) only by including the value of possibly trading in the mechanism. Since the probability of trading in the mechanism during a time period of length $\Delta$ goes to zero as
\( \Delta \) goes to zero, we again obtain \( A = B = b/2 \) in the limit.

The fact that the buyer bids \( b/2 \) implies that the buyer will accept the seller’s initial offer \( p \) (or a posted price \( q \)) if and only if \( b \geq 2p \) (or \( b \geq 2q \)). The reason is straightforward. Let \( x \) denote the marginal buyer value, so the buyer accepts the offer if and only if \( b \geq x \) and is indifferent if \( b = x \). Following a rejected offer, we enter a signaling game, in which the buyer signals by the time of his bid. It is always true in a separating equilibrium of a signaling game that the ‘worst’ type (here, the most impatient type of buyer) realizes his full-information value. The action that produces the full-information value must be optimal, because no worse inference can be made than that the buyer is the worst type. So, the highest valuation buyer must bid immediately following the mechanism, that is, the buyer of type \( x \) bids \( x/2 \) immediately. Because he is indifferent about accepting the offer on one hand and rejecting it and bidding \( x/2 \) immediately, it must be that the offer is \( x/2 \), which means that the marginal type \( x \) is twice the offer.

While bid prices are not affected by the posted-price mechanism, the timing of bids is very much affected. The bidder delays his bid even more because of the possibility of trading in the mechanism. Because each bidder type delays longer, bidding at any particular time signals an even higher valuation than when the mechanism does not exist. Thus, the friction that bidding signals a high valuation and consequently leads to an unfavorable price is worsened by the presence of the mechanism.

One feature of our model that deserves a bit more explanation is the assumption that the price in the mechanism is drawn from \([0, 1]\) even though the buyer’s value is distributed on \([0, 2]\). We make this assumption because of the fact that the buyer bids \( b/2 \). Even if the seller quoted an unreasonable price at date 0 which the buyer rejects with probability one, the buyer could still bid 1 immediately afterwards, which the seller would accept. Thus, the buyer would never to agree to pay a price higher than 1 in this model. We could have
assumed prices in the mechanism are drawn from $[0, 2]$, but then the mechanism would be irrelevant half of the time. In fact, we will see that the seller’s initial offer is $3/4$ or smaller, and trade never occurs in this model at prices above $3/4$. We discuss making our mechanism a bit smarter, drawing prices from a subset of $[0, 1]$, in Section 6.

3. Equilibrium with Only Posted Orders

If there is no possibility of trading at a posted price ($\lambda = 0$), then the solution to our model is given by Cramton (1992). Suppose that if the seller’s initial offer is rejected then he believes that the buyer’s value is uniformly distributed on $[0, x]$ for some $x \leq 2$. Then, if the buyer’s value is $b \leq x$, he bids at the date $t$ satisfying

$$e^{-rt} = \frac{b}{x}. \tag{2}$$

Note that the equilibrium discount factor (2) is independent of the discount rate $r$. Each buyer type has to incur a certain cost in order to separate from higher buyer types. If the discount rate $r$ is lower, then the buyer has to wait a longer time $t$ in order to incur the cost, exactly offsetting the lower $r$. Both the buyer and the seller realize a gain of $b/2$ when trade occurs, so the discounted gain from trade for each trader is $b^2/(2x)$. The expected discounted gain of each trader conditional on $b \leq x$ is

$$\int_0^x \frac{b^2}{2x} \cdot \frac{1}{x} \, db = \frac{x}{6}. \tag{3}$$

As discussed above, the buyer will accept the seller’s initial offer if $b \geq 2p$, so we can take $x = 2p$ in the above. The probability of the offer being accepted is $(2 - 2p)/2 = 1 - p$. Thus, using (3) with $x = 2p$, the seller’s expected gain from an offer at $p$ is

$$(1 - p) \cdot p + p \cdot \frac{2p}{6} = p - \frac{2}{3}p^2.$$

The optimal price is $p = 3/4$, producing an expected gain of $3/8$. The buyer accepts the
seller’s initial offer, gaining \( b - 3/4 \), if \( b \geq 3/2 \) and otherwise gains \( b^2/(2x) = b^2/(4p) = b^2/3 \). Thus, the buyer’s expected gain is

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\frac{1}{2} \int_0^{3/2} \frac{b^2}{3} \, db + \frac{1}{2} \int_{3/2}^2 \left( b - \frac{3}{4} \right) \, db = \frac{7}{16}.
\]

The total expected discounted gain from trade for the buyer and seller is \( 3/8 + 7/16 = 13/16 \).

The efficient outcome in this model is for trade to occur at date 0, producing a gain of \( b \). So, the maximum possible expected discounted gain from trade is \( E[b] = 1 \). It follows that there is a welfare loss of \( 3/16 \) due to delays in trading when there is no opportunity for trading at a posted price. We will see in Section 5 that the possibility of trading at a posted price reduces this welfare loss.

Figure 1 shows how the seller’s inference about the upper bound on the buyer’s valuation evolves over time when there is no possibility of trading at a posted price. At date 0, if the seller’s initial offer is rejected, the seller infers that \( b \leq 2p = 3/2 \). From (2) with \( x = 2p = 3/2 \), we see that at each time \( t \), the buyer should have bid prior to \( t \) if \( b > 3e^{-rt}/2 \). Thus, if the buyer does not accept the seller’s initial offer and has not bid prior to \( t \), then the seller can infer that \( b \leq 3e^{-rt}/2 \).
Figure 1: Seller’s Inference with Only Posted Orders
The upper boundary of the shaded area specifies the type of buyer who should bid at each date \( t \), when trading is only by posted orders. At each date \( t \), if the buyer has not bid by \( t \), the seller infers that the buyer’s value is uniformly distributed over the shaded vertical slice at \( t \). In this figure, \( r = 1 \).
4. Equilibrium with Posted Prices

We describe the equilibrium of our model here, deferring proofs to the appendix. We start from the end of the game and work backwards to describe equilibrium strategies. For our main results, we invoke a parametric assumption that the arrival rate of the mechanism is not too high. Specifically, assume $\lambda < 8r$. We discuss higher arrival rates in Section 6.

4.1. Trading After the Posted-Price Mechanism

Suppose the posted price mechanism arrives and there is no trade in it. The subgame following this event is the same as in Cramton. Letting $x$ denote the seller’s perceived upper bound of the support of $b$ after the mechanism, the buyer’s bidding rule for $b \leq x$ is given in (2). The buyer’s expected value for $b \leq x$ is $b^2 / (2x)$, and the seller’s expected value is given in (3). If, out of equilibrium, $b > x$, then the optimal action for the buyer is to bid $x/2$ immediately, producing a value of $b - x/2$.

4.2. Equilibrium Behavior in the Posted-Price Mechanism

We use the values attained in the subgame following the mechanism to derive optimal behavior in the mechanism.

Proposition 1. Suppose the mechanism arrives and the seller believes the buyer’s value is uniformly distributed on $[0,x]$. Let $q$ denote the price in the mechanism. The following are true for all buyer valuations $b \in [0, 2]$, including $b > x$.

(a) If $q < x/2$, then it is optimal for the buyer to participate in the mechanism if and only if $b \geq 2q$.
(b) If $q > x/2$, then it is never optimal for the buyer to participate in the mechanism.

If $q > x/4$ and to not participate when $q \leq x/4$. 

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With the notation of the proposition, if the buyer had bid an instant before the mechanism arrived, then the seller would have insisted on a price of $x/2$. However, the seller will accept $x/4$ in the mechanism. The reason is that a bid just before the mechanism would reveal information about the buyer’s type, namely that the buyer is the most impatient type to have not yet bid. This information is disadvantageous to the buyer. The mechanism creates pooling of buyer types. Given a price $q \in [x/4, x/2]$, all buyer types $b$ between $2q$ and $x$ get execution in the mechanism at $q$. These buyer types avoid additional costs of signaling that would have been incurred in the absence of the mechanism.

If the mechanism arrives and $q < x/4$ or $q > x/2$, then both the buyer and seller know ex ante that there will be no trade. The seller learns nothing about the buyer’s type, either because the seller does not participate ($q < x/4$) or because the seller knows ex ante that the buyer will not participate ($q > x/2$). In either case, the signaling game described in the previous subsection commences with the upper bound $x$ on the buyer’s valuation being the same as the upper bound prior to the mechanism. However, if $q > x/4$ and $b < 2q < x$, then the seller participates and the buyer does not, and the seller learns that the buyer’s type is below $2q$. In this case, the signaling game described in the previous subsection commences with $x = 2q$.

Figure 2 illustrates a possible path of the game. Prior to the arrival of the mechanism, the seller’s inference about the buyer’s type is based on the equilibrium bidding function $\xi$ that is given below in (11). The figure illustrates the case $q > x/4$ and $b < 2q < x$. The seller gains information when there is no trade in the mechanism. The additional delay after the mechanism is determined by the Cramton bidding rule (2).
In this example, the date \( \tau \) at which the posted-price mechanism arrives is such that the seller has inferred by that date that \( b \leq 1 \) (buyers with values \( b > 1 \) should have bid prior to \( \tau \)). At this time, all prices \( q > 0.25 \) are acceptable to the seller, even though the seller would have required a price of 0.5 if the buyer had bid just before \( t \). The realized posted price in this example is \( q = 0.3 \). All buyers with values \( b > 0.6 \) will accept the price. Thus, all buyers with valuations \( 0.6 < b < 1 \) get pooled at the price \( q = 0.3 \). In this particular example, \( b = 0.4 \), so the buyer does not participate in the mechanism and instead later bids 0.20 after an additional delay defined by (2), with \( x = 2q = 0.6 \). In this figure, \( r = 1 \).
Based on Proposition 1 and the values in the subgame following the mechanism, we can calculate the expected gains from trade for the buyer and seller at the time of the mechanism, discounted to the time of the mechanism, by integrating over $q$. This produces the following.

**Proposition 2.** Suppose the mechanism arrives and the seller believes the buyer’s value is uniformly distributed on $[0, x]$. Given equilibrium behavior, the seller’s value at that time, unconditional on the price $q$ of the mechanism, is

$$\frac{x}{6} + \frac{x^2}{288}.$$  \hfill (4)

The buyer’s value unconditional on $q$ is $\delta(b, x)$, where, for $b, x \in (0, 2)$,

$$\delta(b, x) = \begin{cases} 
\frac{b^2}{2} \left( \frac{1}{x} + \frac{2 \log(2) - 1}{4} \right) & \text{if } b < x/2, \\
\frac{b^2}{2} \left[ \frac{1}{x} + \frac{1}{2} \log \left( \frac{x}{b} \right) \right] - \frac{bx}{2} + \frac{x^2}{32} & \text{if } x/2 < b < x, \\
b - \frac{x}{2} + \frac{x^2}{32} & \text{if } x < b.
\end{cases}$$  \hfill (5)

Figure 3 plots $\delta(b, x)$ as a function of $x$ for a fixed value of $b$. The dotted line shown in Figure 3 is a lower bound on the buyer’s value that can be achieved by bidding $x/2$ immediately after the mechanism. The difference between the buyer’s value $\delta(b, x)$ and this lower bound is due to three things: (i) the possibility of trading in the mechanism, (ii) the possibility that there will be no trade in the mechanism but the seller’s perceived maximum value will fall as a result of the lack of trade, and (iii) the possibility that there will be no trade in the mechanism and the buyer will delay bidding after the mechanism to get a better price than $x/2$. 

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Figure 3: Buyer’s Value in the Posted-Price Mechanism
In this figure, $r = 1$, $\lambda = 6$, and $b = 1$. This is the function $x \mapsto \delta(b, x)$, which is the expected value for the buyer when the posted-price mechanism arrives, unconditional on the price $q$ in the mechanism, given that $x$ is perceived by the seller to be the maximum possible buyer valuation. In equilibrium, we always have $x \geq b$, so the area to the right of $x = 1$ is what occurs on the equilibrium path. The dotted line is the plot of $b - x/2 = 1 - x/2$.

4.3. Incentive Compatibility and the Equilibrium Bidding Rule

When the seller makes an initial offer at price $p$, all buyer types $b \leq 2p$ reject it, and plan to bid $b/2$ at some date $t$ if the posted price mechanism does not arrive before $t$. Let $t = \theta(b \mid p)$ denote the date at which type $b$ plans to bid, for $b \leq 2p$. Let $\xi(\cdot \mid p)$ denote the inverse of $\theta(\cdot \mid p)$, so $b = \xi(t \mid p)$ is the type of buyer who bids at date $t$. For $p \in [0, 1]$, $b \in [0, 2]$, and $b' \in [0, 2p]$, define

$$L(b, b' \mid p) = \left(b - \frac{b'}{2}\right) e^{-(r+\lambda)\theta(b' \mid p)} + \lambda \int_{0}^{\theta(b' \mid p)} e^{-(r+\lambda)t} \delta(b, \xi(t \mid p)) \, dt .$$

(6)

This is the value of the game to the buyer at date 0 if he rejects the seller’s initial offer at price $p$, his valuation is $b$, and he adopts the strategy of type $b'$. The first term in the function $L$ is the gain from a bid of $b'/2$ at the time $\theta(b' \mid p)$ multiplied by the probability
that $\theta(b'|p) < \tau$, that is, the buyer bids before the mechanism arrives. The second term is the value of the game to the buyer when the mechanism arrives, integrated over the arrival time density of the mechanism up to the time $\theta(b'|p)$, and discounted back to date 0.

An incentive compatibility condition, following the seller’s initial offer at price $p$, is, for $b \leq 2p$,

$$L(b, b|p) = \max_{b' \leq 2p} L(b, b'|p).$$

The maximization is over $b' \leq 2p$, because higher buyer types accept the seller’s initial offer. This condition says that any buyer type who in equilibrium rejects the seller’s initial offer would not prefer to mimic any other buyer type that rejects the seller’s initial offer. The first-order condition for (7) is

$$\frac{\partial L(b, b'|p)}{\partial b'} \bigg|_{b'=b} = 0.$$ 

This is a differential equation in $\theta$. By solving the differential equation, we arrive at the following equilibrium $\theta$ and $\xi$.

Define

$$c = \frac{\lambda}{16r} < \frac{1}{2}. \tag{8}$$

For any $x$, define

$$K(x) = \frac{x}{1 - cx}. \tag{9}$$

Our parametric assumption $\lambda < 8r$ implies $1 - cx > 0$ for all $x \leq 2$. The following defines the time $t = \theta(b|p)$ at which the buyer of type $b \leq 2p$ bids:

$$e^{-r\theta(b|p)} = \frac{K(b)}{K(2p)}. \tag{10}$$

Notice that (10) is of the same form as (2), but with $b$ and $x$ replaced by $K(b)$ and $K(x)$
respectively. The inverse of $\theta$ is

$$
\xi(t|p) = \frac{1}{c + e^{rt}/K(2p)}.
$$

The differential equation we solved is a necessary condition for equilibrium. In the proof of the following, we establish sufficiency, that is, we verify second-order conditions.

**Proposition 3.** For any $p \in [0, 1]$,

$$(\forall b > 2p) \quad b - p \geq \max_{b' \leq 2p} L(b, b'|p),$$

$$(\forall b \leq 2p) \quad L(b, b|p) = \max_{b' \leq 2p} L(b, b'|p) \geq b - p,$$

when $L$, $\theta$ and $\xi$ are defined by (7), (10), and (11).

The function $L$ and the incentive compatibility conditions are illustrated in Figures 4 and 5. Figure 6 shows the equilibrium bidding function $\theta(\cdot | p)$, where $p$ is the seller’s equilibrium initial offer price (which we derive below). Figure 6 shows that delay is longer due to the presence of the mechanism, and it shows that delay is longer when the arrival intensity of the mechanism is higher.
Figure 4: Incentive Compatibility

In this figure, \( r = 1, \lambda = 6, b = 1, \) and the seller’s initial offer is \( p = 0.75. \) The plot is of the function \( b' \mapsto L(b, b' \mid p), \) which is the value of the game to the buyer at date 0 if he rejects the seller’s initial offer at price \( p, \) his valuation is \( b, \) and he adopts the equilibrium strategy of type \( b'. \) The maximum of the function occurs at \( b' = b = 1. \) The left panel shows the function over the range \([0, 2p]\). The right panel zooms in on a portion of the \( x \) axis near \( b' = 1 \) to better show that the maximum occurs there.
Figure 5: Buyer’s Decision at Date 0
In this figure, \( r = 1 \) and \( \lambda = 6 \), and the seller’s initial offer is \( p = 0.75 \). The dashed line plots \( b - p \), which is the value of accepting the initial offer. The solid curve is the value \( L(b, b|p) \). Buyer types \( b < 2p \) should reject the seller’s initial offer.
Figure 6: Equilibrium Delay
This is the equilibrium delay function $t = \theta(b \mid p)$ with $r = 1$, where $p$ is the seller’s equilibrium initial offer price. The figure shows that the bidder delays longer when there may be an opportunity to trade at a posted price. The plot is truncated at $b = 2p$, because bidders with values above $2p$ accept the seller’s initial offer at $p$ and hence trade at date 0.
4.4. Initial Offer Price

Knowing the buyer’s behavior, we can compute expected gains from trade for the seller and use that to determine the price at which the seller offers the asset at date 0. As with the buyer’s gains, we compute the seller’s expected discounted gain from trade by integrating over an arrival time density. In this case, we use the arrival time of $\tau \wedge \theta(b|p)$, where $b$ is viewed as uniformly distributed on $[0, 2p]$. The value depends on the perceived upper bound of the distribution of $b$.

**Proposition 4.** If the seller offers the asset at any price $p < 1$, then the seller’s expected discounted gain from trade is $(1 - p)p + pJ(2p)$ where, for $x \in [0, 2]$, we define

$$J(x) = \frac{\lambda + 3r}{6r x K(x)^{1+\lambda/r}} \int_0^x K(u)^{2+\lambda/r} \, du - \frac{\lambda}{72r x K(x)^{1+\lambda/r}} \int_0^{x^2} K(\sqrt{u})^{2+\lambda/r} \, du. \quad (13)$$

If the seller offers the asset at any price $p \geq 1$, then the seller’s expected discounted gain from trade is $J(2)$. The seller’s equilibrium initial offer is at the price $p < 1$ that solves

$$\max_{p \leq 1} (1 - p)p + pJ(2p). \quad (14)$$

The possibility of trading at a posted price is attractive to the buyer and makes him less likely to accept the seller’s initial offer. Consequently, the seller cuts the price when there is a possibility of trading later at a posted price. This is illustrated in Figure 7.
The seller’s initial offer price is $p = 0.75$ when there is no possibility of trading at a posted price ($\lambda = 0$). It is smaller when there is a possibility of trading at a posted price, and it is a decreasing function of the arrival intensity $\lambda$ of the posted-price mechanism. In this figure, $r = 1$. 

Figure 7: Initial Offer Price
5. Welfare Gains from Posted Prices

The maximum possible gain from trade in this model is 1. This could be achieved if the buyer could make a take-it-or-leave-it offer. In the alternating offers framework, take-it-or-leave-it offers are ruled out by subgame perfection. Efficiency could also be achieved if the buyer’s value were common knowledge. In that case, in the alternating offers game (as the time interval between offers shrinks to zero), the outcome is for the seller to offer the asset at \( b/2 \), which is accepted by the buyer.

In this model, with only posted orders, the total gain from trade is \( 13/16 \), as discussed in Section 4. Hence, the welfare loss due to private information and delayed trading is \( 3/16 \). Figure 8 shows the fraction of this welfare loss that is eliminated by the possibility of trading at a posted price. When the arrival intensity of the mechanism is high, as much as 5% of the welfare loss is eradicated. This is a relatively small number, but the mechanism has not been optimized. It only occurs once, and the price is randomly drawn from \([0, 1]\).

The welfare gains are due to trade being accelerated. The acceleration is not uniform across buyer types. Figure 9 shows how the expected discount factor \( e^{-rt} \) in the model with posted prices compares to the expected discount factor in the model without posted prices. The average across \( b \) of this difference in expected discount factors is the expected welfare gain. More trade occurs at date 0 with posted prices, because the seller’s equilibrium offer price is lower, so high buyer types trade faster with posted prices. Low buyer types also trade faster with posted prices, either because they trade in the mechanism or because the inferred maximum buyer type falls when there is no trade in the mechanism. There is a middle range of buyer types who trade slower with posted prices, due to the extra delay before bids are made.

The acceleration or delay of trade affects the buyer and seller equally. However, pooling of buyer types in the mechanism leads to better prices for the buyer and worse prices for
Figure 8: Total Welfare Gain
This plots the extra expected discounted total gains from trade, relative to the model without the possibility of trading at a posted price, as a fraction of the welfare loss of that model compared to the efficient outcome. The model without posted-price trading generates total expected gains from trade of $13/16$ with a welfare loss of $3/16$, so the plot is 

\[
\frac{\text{'expected discounted total gains from trade' } - 13/16}{3/16}.
\]

the seller, because trade takes place only if $q < b/2$. Hence, all buyer types benefit on average from posted prices, even those who trade slower on average with posted prices. Figure 10 shows the gains achieved by the buyer.

Figure 10 shows expected discounted gains as a fraction of the buyer’s valuation. In the complete information Rubinstein benchmark, trade takes immediately at a price of $b/2$, so the buyer gains half of his valuation. As the figure shows, buyers with high valuations do better in this model than in the Rubinstein benchmark, because they trade at the seller’s initial offer price. Recall that the buyer types who trade at the initial offer are those for which $p < b/2$. Buyers with valuations such that $p > b/2$ do worse than in the complete information setting. Figure 10 shows the gains achieved by the buyer in this model, relative to the $b/2$ benchmark.
Figure 9: Expected Discount Factors
This plots the ratio of $E[e^{-rt} | b]$ for the model with posted prices ($\lambda > 0$) to the same expectation without posted prices ($\lambda = 0$), where $t$ denotes the date of trade—either at date 0, after date 0 but before the mechanism arrives, in the mechanism, or after the mechanism. In this figure, $r = 1$.

The seller actually does a little worse when there is a possibility of trading at a posted price. The seller would prefer to be able to make a take-it-or-leave-it offer to the buyer. The more alternatives the buyer has to accepting the seller’s offer, the weaker is the seller’s position. Expanding the buyer’s opportunities by including the posted price opportunity therefore lowers the seller’s expected gains from trade. However, what the buyer gains from the posted price opportunity more than offsets what the seller loses. Total gains from trade seems to be the right variable to focus on, because traders will sometimes find themselves in the role of the seller and sometimes in the role of the buyer.
Figure 10: Buyer Welfare
This is the buyer’s equilibrium expected discounted gain from trade relative to his valuation of the asset. The line at 1/2 represents the Rubinstein alternating-offer outcome, where, when the buyer’s value $b$ is common knowledge, he earns $b/2$. Buyers with values $b > 2p^*$ accept the seller’s initial offer $p^*$ and earn $b - p^* > b/2$. Buyers with values $b < 2p^*$ reject the offer and earn less than $b/2$. The possibility of trading at a posted price increases the buyer’s expected gains from trade.
6. Smarter Mechanisms

There are several simple ways to design the mechanism better, all of which lead to larger welfare gains. One possibility is to draw the price from the interval \( [x/4, x/2] \) when \( x \) is perceived as the upper bound of the buyer’s value distribution. This is motivated by Proposition 1, which shows that trade never occurs except when \( q \) is in this range. This change increases the probability that trade occurs in the mechanism. The assumption actually simplifies the calculations somewhat. Assuming \( \lambda < 4r \), the buyer type \( b \) that bids at date \( t \) is

\[
b = 2pe^{-(r-\lambda/4)t}.
\]

The inverse function is

\[
t = \frac{\log(2p/b)}{r - \lambda/4}.
\]

This is the same as the Cramton solution of the posted-orders model except that \( r \) is replaced by \( r - \lambda/4 \). This shows immediately that delay is greater when there is a possibility of trading at a posted price. The seller’s value function (13) becomes

\[
J(x) = \left( \frac{1}{6} - \frac{11}{72} \cdot \frac{\lambda}{\lambda + 6r} \right) x.
\]

The equilibrium initial offer price is

\[
p = \frac{18\lambda + 108r}{35\lambda + 144r}.
\]

It is easy to verify numerically that aggregate gains from trade are higher in this model than when the price in the mechanism is drawn from \([0,1]\).

Another simple improvement is to have the mechanism occur immediately after the seller’s initial offer. Assume the price in the mechanism is drawn uniformly from \([0,1]\). With the mechanism immediately following the seller’s offer, we can no longer conclude
that the marginal buyer type who accepts the seller’s offer is $b = 2p$. If the seller’s offer is rejected, the expected discounted gain from trade from a buyer of type $b$ when the maximum buyer type is perceived to be $x$ is $\delta(b,x)$ from (5). Thus, the marginal buyer type for the seller’s offer is the solution $x$ to $\delta(x,x) = x - p$. The solution of this equation is $x = 8(1 - \sqrt{1 - p/2})$. This is true if $p < 7/8$. There is no marginal buyer if $p > 7/8$, because the buyer always rejects such high seller offers. The seller chooses the initial offer price $p \leq 7/8$ to maximize

$$
\frac{2 - x}{2} \cdot p + \frac{x}{2} \cdot \left( \frac{x + x^2}{6} + \frac{x^2}{288} \right),
$$

where $x = 8(1 - \sqrt{1 - p/2})$. The factors in expression (15) are, respectively, the probability of the buyer accepting the seller’s offer, the seller’s gain from the offer being accepted, the probability that the buyer rejects the seller’s offer, and the expected discounted gain from trade when the mechanism begins, using Proposition 2 for the last factor. Again, it is easy to verify numerically that aggregate gains from trade are higher in this model than in the main model in the paper.

Another way to increase welfare gains is to increase the arrival rate of the mechanism. For our main results, we assumed $\lambda/r < 8$. Now, assume $\lambda/r \geq 12$. There is an equilibrium in which all buyer types who do not accept the seller’s initial offer wait for the mechanism to arrive instead of bidding prior to the mechanism. If the perceived upper bound on the buyer’s type is $x$, then a buyer of type $b \leq x$ who waits for the mechanism to arrive earns expected discounted gains from trade equal to

$$
\frac{\lambda}{\lambda + r} \delta(b,x),
$$

where $\delta$ is defined in (5). Set $k = (\lambda + r)/\lambda$. The buyer type that is indifferent about accepting the seller’s initial offer is the type $x$ that solves $\delta(x,x)/k = x - p$. The solution
to this equation is
\[ x = 8 \left( 2k - 1 - \sqrt{(2k-1)^2 - kp/2} \right). \]

Similar to the situation in the previous paragraph, the seller chooses the initial offer price \( p \) to maximize
\[ \frac{2-x}{2} \cdot p + \frac{x}{2} \cdot \frac{1}{k} \cdot \left( \frac{x}{6} + \frac{x^2}{288} \right) \]
with this definition of \( x \). A belief that supports this equilibrium is that any buyer who bids before the mechanism arrives is of type \( x \). We can show numerically for \( \lambda/r \geq 12 \) that \( x > 2p \); equivalently
\[ \frac{x}{2} < x - p = \frac{\delta(x, x)}{k}, \]
which means that deviating to bid before the mechanism arrives is suboptimal given this belief. The welfare gains are higher in this model than in our main model, and, as \( \lambda/r \to \infty \), the equilibrium converges to the equilibrium when the mechanism is held immediately after the seller’s offer.

Finally, another way to improve the mechanism is to run it immediately after the seller’s initial offer and to optimize the price in the mechanism. The optimal price is the one that maximizes the probability of trade. With the mechanism using a fixed price and following the seller’s initial offer immediately, the initial offer is essentially irrelevant. Ignoring it, the upper bound on the buyer’s value going into the mechanism is \( x = 2 \). From Proposition 1, we need \( q \geq x/4 \) to induce the seller to participate in the mechanism. Lower prices increase the probability of the buyer participating, so the price that maximizes the probability of trade is \( x/4 = 1/2 \). The buyer participates if \( b \geq 1 \). Using (3) for the expected discounted gains of the buyer and seller when \( b < 1 \), we obtain total expected discounted gains from trade of
\[ \frac{1}{2} \int_1^2 b \, db + \frac{1}{2} \left( \frac{1}{6} + \frac{1}{6} \right) = \frac{11}{12}. \]
This is higher than in any of the other models. Note that with a mechanism price of 1/2, it is useless for the seller to offer the asset at a higher price initially, and the seller does not wish to offer it at a lower price (because the seller’s optimal initial price in the posted-orders model is 3/4). So, the seller might as well offer the asset at a price of 1/2 (or anything higher).

7. Conclusion

We model the friction that trade may be delayed due to traders not wanting to post orders, because orders reveal information. An opportunity to trade at a posted price enhances welfare. The magnitude of the welfare increase is small in the model, but the mechanism in the model has not been optimized. For example, prices are drawn uniformly from [0, 1], even though trade never occurs in the model at prices above 3/4, and as time passes the range of prices at which trade might take place shrinks even further. As explained in the previous section, there are better designs that increase the welfare gains.

Gains from trading at a posted price are not evenly distributed in the model—the seller whose value is known at date 0 loses, and the buyer whose value is private information gains. We could have just as well assumed that the buyer’s value is known at date 0, the seller’s value is unknown, and the buyer makes a bid at date 0. The equilibrium would be symmetric to the equilibrium derived in this paper. The general result is that the trader whose value is known and who is moving first loses and the trader whose value is private information and who is moving second gains, when a possibility of trading at a posted price is introduced. In aggregate, the traders gain. Our view is that any given trader might sometimes be in the role of the trader whose value is known and is moving first and might sometimes be in the role of the trader who is moving second, so average or total gains should be the primary consideration in evaluating the market structure.

Our model is a subgame of a more general model in which the values of both parties
are private information until one party makes an offer/bid and thereby reveals his value. Our conjecture is that both parties gain from being able to trade at posted prices in this more general model, but that is a topic for future study.
Appendix A.

Proof of Proposition 1. First, consider the buyer. If the buyer does not participate, then, based on the conjectured behavior of the buyer, the seller infers that the buyer’s type is bounded above by $\min(x, 2q)$. Consider item (a). From the Cramton solution of the subgame following the mechanism, the value of that subgame to the buyer is $\frac{b^2}{4q}$ if $b \leq 2q$. Hence, it is optimal for the buyer not to participate if $\frac{b^2}{4q} > b - q$. Notice that

$$\frac{b^2}{4q} - b + q = \frac{(b - 2q)^2}{4q}.$$ 

Thus, the buyer should not participate if $b < 2q$ and is indifferent to participating if $b = 2q$. Whenever $b \geq 2q$, the optimal decision in the signaling game following the mechanism is to bid immediately at price $q$, so he is indifferent about participating in all of these cases.

In case (b), the value of the subgame following the mechanism is $\frac{b^2}{2x}$ if $b \leq x$, and we have

$$\frac{b^2}{2x} > \frac{b^2}{4q} \geq b - q,$$

so it is optimal for the buyer not to participate. If $b > x$, then the optimal decision in the signaling game following the mechanism is to bid immediately at price $x/2 < q$, so it is again optimal for the buyer not to participate.

Now, consider the seller. From the Cramton solution of the subgame following the mechanism, the value of not participating is $x/6$. We know that no buyer types $b \leq x$ will participate if $q \geq x/2$. Thus, the seller is indifferent about participating if $q \geq x/2$. Suppose $q < x/2$ and the seller participates. Buyer types $b > 2q$ participate, so the seller earns $q$ with probability $(x - 2q)/x$. With probability $2q/x$, the buyer does not participate, and the seller learns that $b \leq 2q$. The expected value to the seller in this circumstance is

35
$q/3$. Therefore, the expected value of participating is

$$\frac{(x - 2q)q}{x} + \frac{2q^2}{3x} = q - \frac{4q^2}{3x}.$$  \hfill (A.1)

Applying the quadratic formula, we see that this is greater than $x/6$ for $q$ between $x/4$ and $x/2$, and it is less than $x/6$ for $q < x/4$.

\begin{proof}

Proof of Proposition 2. From the proof of the previous proposition, the value for the seller conditional on $q$ is

$$\begin{align*}
&x/6 & \text{if } q < x/4, \\
&q - 4q^2/(3x) & \text{if } x/4 < q < x/2, \\
&x/6 & \text{if } q > x/2.
\end{align*}$$

Therefore, the value for the seller unconditional on $q$ is

$$\left(\frac{x}{4}\right) \frac{x}{6} + \int_{x/4}^{x/2} \left( q - \frac{4q^2}{3x} \right) \, dq + \left( 1 - \frac{x}{2} \right) \frac{x}{6} = \frac{x^2}{288} + \frac{x}{6}.$$ 

Now, we compute the buyer’s value. Suppose first that $b \leq x$. Then, the value for the buyer conditional on $q$, is

$$\begin{align*}
&b^2/(2x) & \text{if } q < x/4, \\
&b - q & \text{if } x/4 < q < b/2, \\
&b^2/(4q) & \text{if } b/2 < q < x/2, \\
&b^2/(2x) & \text{if } q > x/2.
\end{align*}$$

Thus, the value to the buyer unconditional on $q$ is

$$\left(\frac{x}{4}\right) \frac{b^2}{2x} + \int_{x/4}^{x/2} \left( 1_{q < b/2}(b - q) + 1_{q > b/2} \frac{b^2}{4q} \right) \, dq + \left( 1 - \frac{x}{2} \right) \frac{b^2}{2x}.$$ 

\end{proof}
If \( b < x/2 \), then the first indicator function in the integral above is identically zero over the range of integration. In this case, the expected value to the buyer is

\[
\frac{b^2}{2x} - \frac{b^2}{8} + \int_{x/4}^{x/2} \frac{b^2}{4q} \, dq = \frac{b^2}{2} \left( \frac{1}{x} + 2 \log(\frac{2}{x}) - \frac{1}{4} \right)
\]

If \( x > b > x/2 \), then the expected value to the buyer is

\[
\frac{b^2}{2x} - \frac{b^2}{8} + \int_{x/4}^{b/2} (b - q) \, dq + \int_{b/2}^{x/2} \frac{b^2}{4q} \, dq = \frac{b^2}{2} \left[ \frac{1}{x} + \frac{1}{2} + \frac{1}{2} \log \left( \frac{x}{b} \right) \right] - \frac{bx + x^2}{4} + \frac{x^2}{32}
\]

Now, suppose that \( b > x \). Then, if the seller does not participate, the buyer earns \( b - x/2 \). Hence, he earns \( b - x/2 \) if \( q < x/4 \). If \( x/4 < q < x/2 \), then the buyer participates—from part (a) of Lemma 1—and the seller also participates, so the buyer earns \( b - q \). If \( q > x/2 \), then the buyer bids \( x/2 \) immediately following the mechanism and earns \( b - x/2 \). Thus, if \( b > x \), the buyer earns

\[
\begin{cases} 
  b - x/2 & \text{if } q < x/4 \text{ or } q > x/2, \\
  b - q & \text{if } x/4 < q < x/2.
\end{cases}
\]

Integrating over \( q \) gives an expected value of

\[
b - \left( \frac{x}{4} + 1 - \frac{x}{2} \right) \frac{x}{2} - \int_{x/4}^{x/2} q \, dq = b - \frac{x}{2} + \frac{x^2}{32}
\]

This confirms that the expected value to the buyer is \( \delta(b, x) \).

Proof of Proposition 3. For convenience, we drop the \( p \) from \( \theta(\cdot \mid p) \) and \( \xi(\cdot \mid p) \). First, consider \( b \leq 2p \). We will verify (12b).

Step 1a. First we show that the maximum in \( b' \) in (12b) is attained at \( b' = b \). We
have
\[ \frac{\partial L(b, b'|p)}{\partial b'} = e^{-(r+\lambda)\theta(b')} \left[ -(r + \lambda) \left( b - \frac{b'}{2} \right) \frac{d\theta(b')}{db'} + \lambda \delta(b, b') \frac{d\theta(b')}{db'} - \frac{1}{2} \right]. \] (A.2)

To prove optimality, it suffices to show that the derivative is positive for \( b' < b \) and negative for \( b' > b \). From the definition of \( \theta \), we have
\[ \frac{d\theta(x)}{dx'} = -\frac{K'(x)}{rK(x)} = -\frac{K(x)}{rx^2} = -\frac{1}{rx - rcx^2}. \]

Hence, it suffices to show that
\[ \frac{(r + \lambda)(b - x/2) - \lambda \delta(b, x)}{rx - rcx^2} - \frac{1}{2} \] (A.3)
is positive for \( x < b \) and negative for \( x > b \), where we set \( x = b' \leq 2p \). Our parametric restriction \( \lambda < 8r \) guarantees that \( x - cx^2 > 0 \), so what we need to show is that
\[ (r + \lambda) \left( b - \frac{x}{2} \right) - \lambda \delta(b, x) \begin{cases} > \frac{(rx - rcx^2)}{2} & \text{if } x < b, \\ < \frac{(rx - rcx^2)}{2} & \text{if } x > b. \end{cases} \]

Equivalently,
\[ (r + \lambda) \left( b - \frac{x}{2} \right) - \frac{rx - rcx^2}{2} \begin{cases} > \lambda \delta(b, x) & \text{if } x < b, \\ < \lambda \delta(b, x) & \text{if } x > b. \end{cases} \]

Using the definition of \( \delta \), we see that this is equivalent to
\[
\frac{r + \lambda}{\lambda} \left( b - \frac{x}{2} \right) - \frac{rx - rcx^2}{2\lambda} = \begin{cases} 
< \frac{b^2}{2} \left( \frac{1}{x} + \frac{2\log(2) - 1}{4} \right) & \text{if } b < x/2, \\
< \frac{b^2}{2} \left[ \frac{1}{x} + \frac{1}{2} + \frac{1}{2} \log \left( \frac{x}{b} \right) \right] - \frac{bx}{4} + \frac{x^2}{32} & \text{if } x/2 < b < x, \\
> b - \frac{x}{2} + \frac{x^2}{32} & \text{if } b > x,
\end{cases}
\]

For \( b < x/2 \), both terms on the left-hand side are negative, and the right-hand side is positive, so the inequality holds. To evaluate the other two cases, observe that the left-hand side can be written as

\[
b - \frac{x}{2} + \frac{r}{\lambda} \left( b - \frac{x}{2} \right) - \frac{r}{\lambda} \cdot \frac{x}{2} + \frac{rcx^2}{2\lambda} = b - \frac{x}{2} + \frac{r}{\lambda} \left( b - x \right) + \frac{x^2}{32}.
\]

The desired inequality clearly holds for \( b > x \). Now consider the case \( x/2 < b < x \). We need to show that

\[
\frac{b^2}{2x} + \frac{b^2}{4} + \frac{b^2}{4} \log \left( \frac{x}{b} \right) - \frac{bx}{4} - b + \frac{x}{2} - \frac{r}{\lambda} \left( b - x \right) > 0.
\]

This expression is zero at \( x = b \), so it suffices to show that its derivative with respect to \( b \) is negative over the range \( x/2 < b < x \). Thus, we need to show that

\[
\frac{b}{x} + \frac{b}{2} \log \left( \frac{x}{b} \right) - 1 - \frac{r}{\lambda} < 0
\]

for \( x/2 < b < x \). Set \( z = b/x \), so \( 1/2 < z < 1 \). What we need to show is that

\[
z - \frac{zx}{2} \log z - 1 - \frac{r}{\lambda} < 0.
\]
Because $x \leq 2$ and $-z \log z > 0$, we have

$$z - \frac{zx}{2} \log z - 1 - \frac{r}{\lambda} < z - z \log z - 1 - \frac{r}{\lambda}.$$ 

The right-hand side of this is negative at $z = 1$, and its derivative with respect to $z$ is $-\log z > 0$, so it is negative for $1/2 < z < 1$. This completes the proof of the desired inequality for $x/2 < b < x$.

**Step 1b.** Now, still assuming $b \leq 2p$, we need to verify that $L(b, b|p) \geq b - p$. A straightforward calculation shows that

$$\frac{dL(b, b|p)}{db} \leq 1.$$ 

Therefore, $b - p - L(b, b|p)$ is an increasing function of $b$. To show that $b - p - L(b, b|p) \leq 0$ for $b \leq 2p$, it suffices to show that $p - L(2p, 2p|p) = 0$. This follows immediately from the definition of $L$ and the fact that $\theta(2p) = 0$.

**Step 2.** Now, assume $b > 2p$. We will verify (12a). To do this, we compare the buyer of type $b$ to the buyer of type $2p$. From (12b), we know that the buyer of type $2p$ finds it optimal to accept the seller’s initial offer, earning a gain of $p$. Consider any buyer type $b' \leq 2p$ that the buyer might mimic. Let $\phi$ denote the random transaction date and $\pi$ the random price that the buyer of type $b'$ realizes. Because the buyer of type $2p$ does not find it optimal to mimic, we know that

$$2p - p \geq E[e^{-r\phi}(2p - \pi)].$$
Hence,

\[ b - p = (b - 2p) + (2p - p) \geq b - 2p + \mathbb{E}[e^{-r\phi}(2p - \pi)] \]
\[ = \mathbb{E}[e^{-r\phi}(b - \pi)] + \mathbb{E}[(1 - e^{-r\phi})(b - 2p)] \]
\[ > \mathbb{E}[e^{-r\phi}(b - \pi)]. \]

Therefore, the buyer of type \( b \) is better off accepting the seller’s initial offer rather than mimicking any buyer of type \( b' \leq 2p \).

\( \square \)

**Proof of Proposition 4.** The seller’s offer \( p \) is accepted and he earns \( p \) if \( b > 2p \), which occurs with probability \( 1 - p \). With the complementary probability \( p \) the offer is rejected and the signaling game begins with the buyer’s value believed to be \( b \leq \min(2p, 1) \). We want to compute the value of the signaling game to the seller when the supremum of \( b \) at date 0 is any \( y \leq 2 \).

We need the density of the first arrival of the buyer bidding and the mechanism; that is, we want to compute the density from the seller’s point of view of the random time \( \tau \land \theta(b) \), assuming \( b \) is regarded as uniformly distributed on \([0, y]\) at date 0. Define \( \xi \) from (11) replacing \( 2p \) with \( y \). For any date \( t \),

\[ \text{prob}(\tau \land \theta(b) \leq t) = 1 - \text{prob}(\tau > t) \text{prob}(\theta(b) > t) \]
\[ = 1 - e^{-\lambda \xi(t)} \]

Hence, the density is
\[ \frac{e^{-\lambda(t)\xi(t)}(\lambda - \xi'(t))}{y} \]

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Conditional on the arrival time being \( t \), the probability that the mechanism arrived is

\[
\frac{\lambda}{\lambda - \xi'(t)/\xi(t)}
\]

and the probability that the buyer bid is

\[
\frac{-\xi'(t)/\xi(t)}{\lambda - \xi'(t)/\xi(t)}.
\]

Given that the value to the seller of the mechanism is \( \xi/6 + \xi^2/288 \), and the value to the seller of the buyer bidding is \( \xi/2 \), the value of the signaling game to the seller is

\[
\int_0^\infty e^{-(r+\lambda)t} \frac{\xi(t)}{y} \left[ \lambda \left( \frac{\xi(t)}{6} + \frac{\xi(t)^2}{288} \right) + \left( \frac{-\xi'(t)}{\xi(t)} \right) \left( \frac{\xi(t)}{2} \right) \right] \, dt \\
= \frac{1}{y} \int_0^\infty e^{-(r+\lambda)t} \left[ \frac{\lambda \xi(t)^2}{6} + \frac{\lambda \xi(t)^3}{288} + \frac{r \xi(t)^2}{2} - \frac{r c \xi(t)^3}{2} \right] \, dt.
\]

We used the fact that \(-\xi'/\xi = r - r c \xi\) to obtain the last line. We can simplify these integrals by making the change of variables \( u = \xi(t) \) to compute the integrals of \( \xi^2 \) and the change of variables \( u = \xi(t)^2 \) to compute the integrals of \( \xi^3 \). Given \( u = \xi(t) \), we have

\[
\xi' = -r \xi(1 - c \xi) = -r e^{rt} \xi^2 / K(y) \quad \Rightarrow \quad -\frac{K(y)}{r} e^{-rt} \, du = \xi(t)^2 \, dt.
\]

Furthermore, \( \xi(0) = y \), and \( \xi(\infty) = 0 \), and the inverse of \( \xi \) is \( \theta \)—defined in (10) with
\[ x^* = y \text{— so} \]

\[
\int_0^\infty e^{-(r+\lambda)t} \xi(t)^2 \, dt = \frac{K(y)}{r} \int_0^y e^{-(2r+\lambda)\theta(u)} \, du \\
= \frac{K(y)}{r} \int_0^y \left( \frac{K(u)}{K(y)} \right)^{2+\lambda/r} \, du \\
= \frac{1}{rK(y)^{1+\lambda/r}} \int_0^y K(u)^{2+\lambda/r} \, du
\]

Likewise, the change of variables \( u = \xi(t)^2 \) implies

\[
-\frac{K(y)}{2r} e^{-rt} \, du = \xi(t)^3 \, dt
\]

which produces

\[
\int_0^\infty e^{-(r+\lambda)t} \xi(t)^3 \, dt = \frac{K(y)}{2r} \int_0^{y^2} e^{-(2r+\lambda)\theta(\sqrt{u})} \, du \\
= \frac{1}{2rK(y)^{1+\lambda/r}} \int_0^{y^2} K(\sqrt{u})^{2+\lambda/r} \, du
\]

We conclude that the value of the signaling game to the seller—given that \( y \) is the supremum of the value of \( b \) at date 0—is

\[
\frac{1}{y} \left( \frac{\lambda}{6} + \frac{r}{2} \right) \left( \frac{1}{rK(y)^{1+\lambda/r}} \right) \int_0^y K(u)^{2+\lambda/r} \, du \\
+ \frac{1}{y} \left( \frac{\lambda}{288} - \frac{rc}{2} \right) \left( \frac{1}{2rK(y)^{1+\lambda/r}} \right) \int_0^{y^2} K(\sqrt{u})^{2+\lambda/r} \, du \quad (A.4)
\]

This simplifies to \( J(y) \) defined in (13). Therefore, the value of the game to the seller is \( J(2p) \) if \( p \geq 1 \) and \((1 - p)p + pJ(2p)\) if \( p < 1 \).
References


