Online Supplement

Proofs of Propositions for Let the Pirates Patch? An Economic Analysis of Software Security Patch Restrictions

Terrence August and Tunay I. Tunca
GSB, Stanford University

Before we proceed with the proofs of the propositions, we first present the following three lemmas that will be extensively used in the proofs.

Lemma B.1 *Define*

$$\overline{p} \triangleq (1-c_p)\left(1-\frac{c_p}{\pi_a\alpha}\right) \quad \text{and} \tag{B.1}$$

$$\overline{p}_1 \triangleq \frac{1-\pi_a\alpha\nu+\sqrt{(1-\pi_a\alpha\nu)^2+4\pi_a\alpha\nu\pi_d c_d}}{2}. \tag{B.2}$$

Suppose $p>\pi_d c_d$. *If* $\pi_a\alpha<c_p$ *or both* $\pi_a\alpha\geq c_p$ *and* $\pi_d c_d\geq\overline{p}$, *where* $\overline{p}$ *is as defined in (B.1) then*

(i) if $p\leq\overline{p}_1$, *then* $v_{sp}=v_p=1$,

$$v_s=\frac{\pi_d c_d\left(\sqrt{(1-\pi_a\alpha)^2+4\pi_a\alpha(\nu\pi_d c_d+(1-\nu)p)}-(1-\pi_a\alpha)\right)}{2\pi_a\alpha(\nu\pi_d c_d+(1-\nu)p)}, \tag{B.3}$$

and

$$v_b=\frac{pv_s}{\pi_d c_d}; \tag{B.4}$$

(ii) if $\overline{p}_1<p\leq 1$, *then* $v_{sp}=v_p=v_b=1$ *and*

$$v_s=\frac{\sqrt{(1-\pi_a\alpha\nu)^2+4\pi_a\alpha\nu\pi_d c_d}-(1-\pi_a\alpha\nu)}{2\pi_a\alpha\nu}. \tag{B.5}$$

Proof of Lemma B.1: For the sake of clarity in exposition, we will defer the proofs for the statements $v_{sp}=v_p=1$ when $\pi_a\alpha\geq c_p$ and $\pi_d c_d\geq\overline{p}$ to Lemmas B.2 and B.3. When $\pi_a\alpha<c_p$, since $u\leq 1$ and $p>\pi_d c_d$, under both policies l and nl, we have $v_{sp}=v_p=1$. This is because (A.3), (A.12), and (A.17) will not hold for all $v\in\mathcal{V}$. Now by (A.2) and (A.16), the equilibrium unpatched population is in the form

$$u(\sigma^*)=\nu(v_{sp}-v_s)+(1-\nu)(v_p-v_b). \tag{B.6}$$

When $v_{sp}=v_p=1$, (B.6) implies $u(\sigma^*)=\nu(1-v_s)+(1-\nu)(1-v_b)$ under either policy. Suppose $v_b<1$. Then, v_b satisfies (A.10) which by substituting into (A.14) yields (B.4) which, in turn, by substituting into (A.10) gives

$$v_s-\pi_a\alpha v_s\left(\nu(1-v_s)+(1-\nu)\left(1-\frac{pv_s}{\pi_d c_d}\right)\right)-\pi_d c_d=0. \tag{B.7}$$

(B.7) has a single root greater than $\pi_d c_d$ and this root is given in (B.3). For $v_b\leq 1$ to hold, by (B.4), we must have $v_s\leq\pi_d c_d/p$. Plugging this into (B.7), we see that $v_b\leq 1$ if and only if $p\leq\overline{p}_1$. For $p>\overline{p}_1$, substituting $v_b=1$ into (A.14) and solving the resulting quadratic equation, we obtain (B.5). □

Lemma B.2 *Suppose* $p > \pi_d c_d$*,* $\pi_a \alpha \geq c_p$ *and* $\pi_d c_d < \overline{p}$ *and that software pirates are allowed to patch vulnerabilities (i.e.,* $\rho = l$*). Define*

$$\hat{p}_1 \triangleq \sup\{p \,|\, p\, c_p + \pi_a \alpha \nu (\pi_d c_d - p)(p + c_p)^2 = 0\} \text{ and} \quad \text{(B.8)}$$

$$\overline{p}_2 \triangleq \frac{(1 - c_p)\left(1 - \frac{c_p}{\pi_a \alpha}\right) - \pi_d c_d \nu}{1 - \nu}. \quad \text{(B.9)}$$

Then

(i) if $p \leq \min(\hat{p}_1, \overline{p}_2)$*, then,*

$$v_b = \sup\left\{ v_b \,\middle|\, (v_b - p)^2 - \pi_a \alpha v_b^2 \left(\nu \left(c_p - \frac{\pi_d c_d (v_b - p)}{p} \right) + (1 - \nu)(c_p - v_b + p) \right) = 0 \right\}, \quad \text{(B.10)}$$

$$v_p = \frac{c_p v_b}{v_b - p}, \quad \text{(B.11)}$$

$$v_s = \frac{\pi_d c_d v_b}{p}, \quad \text{(B.12)}$$

and

$$v_{sp} = \frac{c_p v_s}{v_s - \pi_d c_d}; \quad \text{(B.13)}$$

(ii) if $\overline{p}_2 \leq \hat{p}_1$ *and* $\overline{p}_2 < p \leq \overline{p}_1$*, then* v_{sp}*,* v_p*,* v_b*, and* v_s *are as given in part (i) of Lemma B.1;*

(iii) if $\overline{p}_2 \leq \hat{p}_1$ *and* $\overline{p}_1 < p \leq 1$*, then* v_{sp}*,* v_p*,* v_b*, and* v_s *are as given in part (ii) of Lemma B.1;*

(iv) if $\overline{p}_2 > \hat{p}_1$ *and* $\hat{p}_1 < p \leq 1$*, then* v_{sp} *satisfies (B.13),* $v_b = v_p$*,*

$$v_p = \min(p + c_p, 1), \quad \text{(B.14)}$$

and

$$v_s = \sup\left\{ v_s \,|\, (v_s - \pi_d c_d)^2 - \nu \pi_a \alpha v_s^2 (c_p - v_s + \pi_d c_d) = 0 \right\}. \quad \text{(B.15)}$$

Proof of Lemma B.2: Suppose $v_b < v_p < 1$ and $v_{sp} < 1$. Then, by Lemma A.1, v_b, v_p, v_s, and v_{sp} satisfy (A.10), (A.11), (A.14), (A.15), respectively. Substituting (A.11) into (A.10) yields (B.11), (A.10) into (A.14) yields (B.12), and (A.15) into (A.14) yields (B.13). Substituting (B.11), (B.6), and (B.12) into (A.10) gives (B.10). Now, for $v_p \leq 1$ to hold, where v_p is given by (B.11), we must have $v_b \geq p/(1 - c_p)$. Substituting this quantity into (B.10), we obtain that $v_p \leq 1$ if and only if $p \leq \overline{p}_2$. By (A.11) and (A.15), we have $v_p = v_{sp}$. In order to satisfy $v_p \leq 1$, and $p > \pi_d c_d$, $\overline{p}_2 > \pi_d c_d$ has to hold, which is satisfied if and only if $\pi_d c_d < \overline{p}$. Therefore, under policy $\rho = l$, if $\pi_d c_d \geq \overline{p}$, then $v_p = v_{sp} = 1$ as also indicated in Lemma B.1. From (B.10), we define $f(v_b) \triangleq (v_b - p)^2 - \pi_a \alpha v_b^2 (\nu(c_p - \pi_d c_d (v_b - p)/p) + (1 - \nu)(c_p - v_b + p))$. Then, $f(p + c_p) = (c_p/p)(p c_p + \pi_a \alpha \nu (\pi_d c_d - p)(p + c_p)^2)$, and therefore v_b as defined in (B.10) which solves $f(v_b) = 0$ falls to the left of $p + c_p$, i.e., $v_b \leq p + c_p$, if and only if

$$p c_p + \pi_a \alpha \nu (\pi_d c_d - p)(p + c_p)^2 \geq 0. \quad \text{(B.16)}$$

By (B.8), (B.16) is satisfied whenever $p \leq \hat{p}_1$. This proves part (i). The proofs of parts (ii) and (iii) are very similar to that of Lemma B.1.

For part (iv), suppose $\overline{p}_2 > \hat{p}_1$ and $\hat{p}_1 < p \leq 1$. Then, $v_b \leq p + c_p$ can no longer be maintained while still satisfying (B.10). Hence $v_b = v_p$ and both satisfy (B.14). Substituting $v_b = v_p$ and (B.13) into (A.14) we obtain (B.15). Finally, from (B.15), we obtain that for $v_{sp} \leq 1$, we need $\pi_a \alpha \nu > c_p$ and $\pi_d c_d < (1 - c_p)(1 - c_p/(\pi_a \alpha \nu))$, which hold if and only if $\overline{p}_2 > \hat{p}_1$. This completes the proof. □

Lemma B.3 *Suppose $p > \pi_d c_d$. If $\pi_a \alpha \geq c_p$ and $\pi_d c_d < \overline{p}$ and that software pirates are <u>not</u> allowed to patch vulnerabilities (i.e., $\rho = nl$). Define*

$$\hat{p}_2 \triangleq \sup\left\{p \in \mathbb{R} \,|\, p\, c_p^2 - \pi_a \alpha \nu (p + c_p)^3 (p - \pi_d c_d) = 0\right\}, \tag{B.17}$$

$$\hat{p}_3 \triangleq \sup\left\{p \,|\, p\, c_p + \pi_a \alpha \nu (p + c_p)(-p + \pi_d c_d (p + c_p)) = 0\right\}. \tag{B.18}$$

$$\begin{aligned}\overline{p}_3 \triangleq\ & \sup\{p < 1 + \pi_d c_d - c_p \,|\, \pi_a \alpha (c_p^2 - c_p(1 + \pi_d c_d - 2p) + (\pi_d c_d - p)(\nu - p)) \\ & +(1 - c_p + \pi_d c_d - p)(c_p - \pi_d c_d + p)^2 = 0\},\end{aligned} \tag{B.19}$$

$$\overline{p}_4 \triangleq \frac{1 - 2c_p + 2\pi_d c_d + \pi_a \alpha \nu - \sqrt{(1 - \pi_a \alpha \nu)^2 + 4 \pi_a \alpha \nu \pi_d c_d}}{2}, \tag{B.20}$$

Then,

(i) if $p \leq \min(\hat{p}_2, \overline{p}_3)$, then v_p satisfies (B.11), v_s satisfies (B.12),

$$v_{sp} = \frac{(p - \pi_d c_d + c_p) v_s}{v_s - \pi_d c_d}, \tag{B.21}$$

and

$$v_b = \sup\left\{v_b \,\middle|\, (v_b - p)^2 - \pi_a \alpha v_b^2 \left(\nu \left(p - \pi_d c_d + c_p - \frac{\pi_d c_d (v_b - p)}{p}\right) + (1 - \nu)(c_p - v_b + p)\right) = 0\right\}; \tag{B.22}$$

(ii) if $\overline{p}_3 \leq \hat{p}_2$ and $\overline{p}_3 < p \leq \min(\overline{p}_2, \hat{p}_3)$, then $v_{sp} = 1$, v_p satisfies (B.11), v_s satisfies (B.12) and

$$v_b = \sup\left\{v_b \,\middle|\, (v_b - p)^2 - \pi_a \alpha v_b \nu (v_b - p)\left(1 - \frac{\pi_d c_d v_b}{p}\right) - \pi_a \alpha v_b^2 (1 - \nu)(c_p - v_b + p) = 0\right\}; \tag{B.23}$$

(iii) if $\overline{p}_3 \leq \hat{p}_2$, $\hat{p}_3 \leq \overline{p}_2$, and $\hat{p}_3 < p \leq 1$; or if $\overline{p}_3 > \hat{p}_2$, $\overline{p}_4 < p \leq 1$, then $v_{sp} = 1$, $v_b = v_p$, v_s satisfies (B.5) and v_p satisfies (B.14);

(iv) if $\overline{p}_3 \leq \hat{p}_2$, $\hat{p}_3 > \overline{p}_2$, and $\overline{p}_2 < p \leq \overline{p}_1$, then v_{sp}, v_p, v_b, and v_s are as given in part (i) of Lemma B.1;

(v) if $\overline{p}_3 \leq \hat{p}_2$, $\hat{p}_3 > \overline{p}_2$, and $\overline{p}_1 < p \leq 1$, then v_{sp}, v_p, v_b, and v_s are as given in part (ii) of Lemma B.1;

(vi) if $\overline{p}_3 > \hat{p}_2$, $\hat{p}_2 < p \leq \overline{p}_4$, then $v_b = v_p$, v_p satisfies (B.14), v_{sp} satisfies (B.21), and

$$v_s = \sup\left\{v_s \,|\, (v_s - \pi_d c_d)^2 - \pi_a \alpha \nu v_s^2 (p + c_p - v_s) = 0\right\}. \tag{B.24}$$

Proof of Lemma B.3: Suppose $v_b < v_p < 1$ and $v_{sp} < 1$. Then, by Lemma A.2, v_b, v_p, v_s, and v_{sp} satisfy (A.10), (A.11), (A.14), (A.19), respectively. Again, by substitution, we obtain (B.11) and (B.12). Substituting (A.19) into (A.14) yields (B.21). Substituting (B.11), (B.6), (B.12), and (B.21) into (A.10) yields (B.22). Now, for $v_{sp} \leq 1$ to hold, where v_{sp} is given by (B.21), we must have $v_b \geq p/(1 - c_p - p + \pi_d c_d)$, which, by (B.22), holds if and only if

$$\pi_a\alpha(c_p^2 - c_p(1+\pi_d c_d - 2p) + (\pi_d c_d - p)(\nu - p)) + (1 - c_p + \pi_d c_d - p)(c_p - \pi_d c_d + p)^2 \leq 0\,, \tag{B.25}$$

which is satisfied only if $\pi_d c_d \leq \overline{p}$ and $p \leq \overline{p}_3$. Now by (B.22), $v_b \leq p + c_p$ holds if and only if

$$p\, c_p^2 - \pi_a\alpha\nu(p+c_p)^3(p - \pi_d c_d) \geq 0\,, \tag{B.26}$$

which is satisfied whenever $p \leq \hat{p}_2$. This concludes the conditions for part (i).

For part (ii), making similar substitutions as in part (i) and substituting $v_{sp} = 1$ yields (B.23). For $v_p \leq 1$ to hold, where v_p is given by (B.11), we must have $v_b \geq p/(1-c_p)$ which, by (B.23), is satisified if and only if $p \leq \overline{p}_2$. Similarly, $v_b \leq p + c_p$ if and only if $p \leq \hat{p}_3$. Parts (iii) through (vi) follow the same line of proof and are omitted for conciseness. Finally, both $\overline{p}_2 > \pi_d c_d$ and $\overline{p}_3 > \pi_d c_d$ are satisfied if and only if $\pi_d c_d < \overline{p}$. If $\pi_d c_d \geq \overline{p}$, then by Lemma B.3, neither $v_{sp} < 1$ nor $v_p < 1$ can be satisfied, as stated in Lemma B.1. □

Proof of Proposition 1: Technically, we will prove that

(i) If $\pi_d c_d < (1-c_p)/2$, then there exists a $\underline{\omega} > 0$ such that if $\pi_a\alpha > \underline{\omega}$, then $\Pi_{nl}(p_{nl}^*) > \Pi_l(p_l^*)$.

(ii) There exists $\underline{\gamma}, \overline{\gamma}, \underline{\omega}, \overline{\omega}, \overline{\eta} > 0$ such that if $\underline{\gamma} < \pi_d c_d < \overline{\gamma}$, $\underline{\omega} < \pi_a\alpha < \overline{\omega}$, and $\nu < \overline{\eta}$, then $\Pi_{nl}(p_{nl}^*) > \Pi_l(p_l^*)$.

For part (i), suppose $\rho = l$. Re-arranging (B.8), we obtain $p = \pi_d c_d + p c_p/(\pi_a\alpha\nu(p+c_p)^2)$. As a result, and by (B.9), $\hat{p}_1$ converges to $\pi_d c_d$ and $\overline{p}_2$ converges to $(1 - c_p - \pi_d c_d \nu)/(1-\nu)$ as $\pi_a\alpha$ gets large. When $\pi_d c_d < (1-c_p)/2$, for sufficiently large $\pi_a\alpha$ the conditions of Lemma B.2 hold, and since $(1 - c_p - \pi_d c_d\nu)/(1-\nu) > \pi_d c_d$, we have $\overline{p}_2 > \hat{p}_1$. Then by part (iv) of Lemma B.2, $\Pi_l(p) = p(1-\nu)(1-p-c_p)$ for any $p > \pi_d c_d$, and hence

$$\max_{p > \pi_d c_d} \Pi_l(p) \leq \frac{(1-\nu)(1-c_p)^2}{4}\,. \tag{B.27}$$

On the other hand, when $p \leq \pi_d c_d$, by Lemma 1 of August and Tunca 2006, if $\pi_a\alpha \geq c_p$ and $p < \overline{p}$, then

$$v_b = \sup\{v_b \,|\, \pi_a\alpha v_b^2(v_b - c_p - p) = -(v_b - p)^2\}\,, \tag{B.28}$$

By (4), $\Pi_l(p) = p(1 - v_b)$, where by (B.28) $v_b = p + c_p - (v_b - p)^2/(\pi_a\alpha v_b^2)$, and by writing terms in orders of $1/\pi_a\alpha$ we obtain

$$v_b = p + c_p - \frac{c_p^2}{\pi_a\alpha(p+c_p)^2} + O\left(\frac{1}{(\pi_a\alpha)^2}\right)\,, \tag{B.29}$$

where O is the common order notation, implying $f = O(g(x))$ if for sufficiently large x, $f(x)/g(x)$ is bounded.[13]

By differentiating $\Pi_l(\cdot)$, computing dv_b/dp by the implicit function theorem using (B.28), and substituting (B.29) into the resulting expression, we obtain

$$\frac{d\Pi_l}{dp} = 1 - c_p - 2p - \frac{c_p^2(p - c_p)}{\pi_a\alpha(p+c_p)^3} + O\left(\frac{1}{(\pi_a\alpha)^2}\right)\,. \tag{B.30}$$

[13]For a general definition, see Knuth, D. E., *Fundamental Algorithms*, Vol. 1 of *The Art of Computer Programming*, Addison-Wesley, 1968.

Hence, $p(1-v_b)$ has an unconstrained maximizer at $\hat{p}$ which satisfies

$$\hat{p}=\frac{1-c_p}{2}+\frac{2c_p^2(3c_p-1)}{\pi_a\alpha(1+c_p)^3}+O\left(\frac{1}{(\pi_a\alpha)^2}\right). \tag{B.31}$$

Since $\pi_d c_d<(1-c_p)/2$, it then follows that for sufficiently large $\pi_a\alpha$, $\hat{p}>\pi_d c_d$ is satisfied, and hence

$$\max_{p\,\le\,\pi_d c_d}\Pi_l(p)=\pi_d c_d(1-v_b(\pi_d c_d)). \tag{B.32}$$

Substituting (B.29) into (B.32), we obtain

$$\max_{p\,\le\,\pi_d c_d}\Pi_l(p)=\pi_d c_d(1-\pi_d c_d-c_p)+\frac{c_p^2\pi_d c_d}{\pi_a\alpha(c_p+\pi_d c_d)^2}+O\left(\frac{1}{(\pi_a\alpha)^2}\right), \tag{B.33}$$

and therefore, by (B.27) and (B.33), it follows that

$$\max_p \Pi_l(p) \quad\le\quad \max\left\{\frac{(1-\nu)(1-c_p)^2}{4},\pi_d c_d(1-\pi_d c_d-c_p)+\frac{c_p^2\pi_d c_d}{\pi_a\alpha(c_p+\pi_d c_d)^2}+O\left(\frac{1}{(\pi_a\alpha)^2}\right)\right\} \tag{B.34}$$

For $\rho=nl$, by (B.19), we have

$$c_p^2-c_p(1+\pi_d c_d-2p)+(\pi_d c_d-p)(\nu-p)=\frac{(p+c_p-1-\pi_d c_d)(c_p-\pi_d c_d+p)^2}{\pi_a\alpha}, \tag{B.35}$$

and hence, $\overline{p}_3$ approaches $(\pi_d c_d+\nu-2c_p+\sqrt{(\pi_d c_d-\nu)^2+4c_p(1-\nu)})/2$ as $\pi_a\alpha$ gets large. Now, $(\pi_d c_d+\nu-2c_p+\sqrt{(\pi_d c_d-\nu)^2+4c_p(1-\nu)})/2>\pi_d c_d$ is always satisfied when $\pi_d c_d+2c_p-\nu\le 0$. However, if $\pi_d c_d+2c_p-\nu>0$, then it is satisfied if and only if $(\sqrt{(\pi_d c_d-\nu)^2+4c_p(1-\nu)})^2>(\pi_d c_d+2c_p-\nu)^2$ which holds whenever $\pi_d c_d<1-c_p$. Hence, for sufficiently large $\pi_a\alpha$, $\overline{p}_3>\pi_d c_d$. Also by (B.17), we have $p=\pi_d c_d+pc_p^2/(\pi_a\alpha\nu(p+c_p)^3)$, and hence

$$\hat{p}_2=\pi_d c_d+\frac{\pi_d c_d c_p^2}{\pi_a\alpha\nu(\pi_d c_d+c_p)^3}+O\left(\frac{1}{(\pi_a\alpha)^2}\right). \tag{B.36}$$

As a result, for sufficiently large $\pi_a\alpha$, $\overline{p}_3>\hat{p}_2$. By (B.20), $\overline{p}_4$ is the larger root of the quadratic equation $p=1-c_p+(p-\pi_d c_d+c_p-1)(p-\pi_d c_d+c_p)/(\pi_a\alpha\nu)$. Thus, we have

$$\overline{p}_4=1-c_p-\frac{\pi_d c_d(1-\pi_d c_d)}{\pi_a\alpha\nu}+O\left(\frac{1}{(\pi_a\alpha)^2}\right), \tag{B.37}$$

and hence, $\max_{\hat{p}_2<p\le\overline{p}_4}\Pi_{nl}(p)$ is given by part (vi) of Lemma B.3. Rearranging (B.24) and writing terms in orders of $1/\pi_a\alpha$, we obtain

$$v_s=p+c_p-\frac{(p+c_p-\pi_d c_d)^2}{\pi_a\alpha\nu(p+c_p)^2}+O\left(\frac{1}{(\pi_a\alpha)^2}\right), \tag{B.38}$$

and by differentiating (5) and substituting (B.14), (B.21), and (B.38) into the resulting expression for the derivative, we obtain

$$\frac{d\Pi_{nl}}{dp} = 1 - c_p - 2p + \frac{c_p \pi_d c_d(\pi_d c_d - c_p) - p\pi_d c_d(\pi_d c_d + c_p)}{\pi_a \alpha (p + c_p)^3} + O\left(\frac{1}{(\pi_a \alpha)^2}\right) . \tag{B.39}$$

Equating (B.39) to zero and solving for p, the uconstrained maximizer $\tilde{p}$ satisfies

$$\tilde{p} = \frac{1 - c_p}{2} - \frac{2\pi_d c_d(c_p(1 + c_p) + \pi_d c_d(1 - 3c_p))}{\pi_a \alpha (1 + c_p)^3} + O\left(\frac{1}{(\pi_a \alpha)^2}\right) . \tag{B.40}$$

By (B.36) and (B.37), for sufficiently large $\pi_a \alpha$, $\hat{p}_2 < \tilde{p} < \overline{p}_4$, and hence, $\tilde{p} = \text{argmax}_{\hat{p}_2 < p \leq \overline{p}_4} \Pi_{nl}(p)$. Therefore,

$$\max_p \Pi_{nl}(p) \geq \max_{\hat{p}_2 < p \leq \overline{p}_4} \Pi_{nl}(p) = \frac{(1 - c_p)^2}{4} - \frac{\pi_d c_d(1 - c_p)(1 + c_p - 2\pi_d c_d)}{\pi_a \alpha (1 + c_p)^2} + O\left(\frac{1}{(\pi_a \alpha)^2}\right) . \tag{B.41}$$

By (B.33), (B.41), and since $\pi_d c_d < (1 - c_p)/2$, there exists $\underline{\omega}$ such that for all $\pi_a \alpha > \underline{\omega}$, we have $\max_p \Pi_{nl}(p) > \max_p \Pi_l(p)$.

For part (ii), define

$$\overline{k} \triangleq \min\left(\frac{c_p^2}{(1 - \pi_d c_d)(1 + c_p)^3}, \frac{c_p}{(1 - \pi_d c_d(1 + c_p))(1 + c_p)}\right) , \tag{B.42}$$

and $\overline{\omega} \triangleq \overline{k}/\nu$. By (B.17) and (B.18), $\pi_a \alpha < \overline{\omega}$ implies that $\hat{p}_2 > 1$ and $\hat{p}_3 > 1$. Let v_b satisfy (B.22) and $\pi_a \alpha = k/\sqrt{\nu}$ for $0 < k < \overline{k}/\sqrt{\nu}$. Then we have

$$v_b = p + c_p - \frac{c_p^2 \sqrt{\nu}}{k(p + c_p)^2} + \left(p + c_p - \pi_d c_d(1 + c_p/p) + \frac{2c_p^3 p}{k^2(p + c_p)^5}\right)\nu + O\left(\nu^{3/2}\right) . \tag{B.43}$$

Now, applying the implicit function theorem to (B.22) to obtain dv_b/dv_p, we have

$$\frac{dv_b}{dp} = \frac{2p^3 - \pi_a \alpha \pi_d c_d \nu v_b^3 - p^2 v_b(2 + \pi_a \alpha v_b)}{p\left(2p^2 + 2v_b p(\pi_a \alpha(p + c_p) - 1) - 3\pi_a \alpha v_b^2(\nu \pi_d c_d + (1 - \nu)p)\right)} . \tag{B.44}$$

By taking the derivative with respect to p in (5), and substituting (B.43), (B.44), and $\pi_a \alpha = k/\sqrt{\nu}$ into the resulting expression, we have

$$\frac{d\Pi_{nl}}{dp} = 1 - c_p - 2p - \frac{c_p^2(p - c_p)\sqrt{\nu}}{k(p + c_p)^3} + \kappa_1 \nu + O(\nu^{3/2}) , \tag{B.45}$$

where $\kappa_1 \in \mathbb{R}$ is a constant. For $p > \overline{p}_2$, parts (iv) and (v) of Lemma B.3 apply, and the vendor's profit approaches zero when ν is sufficiently small. Equating (B.45) to zero, solving for p and writing the terms in orders of ν, we obtain

$$p_{nl}^* = \frac{1 - c_p}{2} + \frac{2c_p^2(3c_p - 1)\sqrt{\nu}}{k(1 + c_p)^3} + O(\nu) . \tag{B.46}$$

Hence, by (B.9) and (B.46), $p_{nl}^* < \overline{p}_2$. As ν becomes small, by (B.19), $\overline{p}_3$ converges to the larger root of the equation $(c_p^2 - c_p(1 + \pi_d c_d - 2p) - p(\pi_d c_d - p) = 0$, and hence, $\overline{p}_3 > (1 - c_p)/2$ if and only if $\pi_d c_d > (1 -$

$c_p)^2/(2(1-c_p))$. Since $(1-c_p)^2/(2(1-c_p)) < (1-c_p)/2$ is always satisfied, whenever $\pi_d c_d \in ((1-c_p)^2/(2(1-c_p)), (1-c_p)/2)$, we have $p^*_{nl} < \min(\hat{p}_2, \overline{p}_3)$. Hence, substituting (B.43) and (B.46) into (5), and by part (i) of Lemma B.3, we obtain

$$\Pi_{nl}(p^*_{nl}) = \frac{(1-c_p)^2}{4} + \frac{2c_p^2(1-c_p)\sqrt{\nu}}{k(1+c_p)^2} + z_1\nu + O\left(\nu^{3/2}\right), \tag{B.47}$$

where $z_1 \in \mathbb{R}$ is a constant satisfying

$$z_1 = \frac{4c_p^3(c_p^2(5c_p-2)+5c_p-4)}{k^2(1+c_p)^6} + \frac{(1+c_p)^2(c_p+2\pi_d c_d-1)}{8c_p}. \tag{B.48}$$

Similarly, under policy $\rho = l$, it can shown that p^*_l satisfies part (i) of Lemma B.2. Then, by (B.10),

$$v_b = p + c_p - \frac{c_p^2\sqrt{\nu}}{k(p+c_p)^2} + \left(\frac{c_p}{p+c_p} - \frac{\pi_d c_d c_p}{p} - \frac{2c_p^4}{k^2(p+c_p)^5} + \frac{2c_p^3}{k^2(p+c_p)^4}\right)\nu + O\left(\nu^{3/2}\right). \tag{B.49}$$

By the implicit function theorem and (B.10),

$$\frac{dv_b}{dp} = \frac{2p(v_b-p) + \pi_a\alpha v_b^2\left(\frac{\nu\pi_d c_d v_b}{p} + (1-\nu)p\right)}{2p(v_b-p) + \pi_a\alpha v_b\left((\nu\pi_d c_d + (1-\nu)p)(v_b + 2(v_b-p)) - 2c_p\right)}. \tag{B.50}$$

By (4) and substituting $\pi_a\alpha = k/\sqrt{\nu}$, (B.49), and (B.50) we have

$$\frac{d\Pi_l}{dp} = 1 - c_p - 2p - \frac{c_p^2(p-c_p)\sqrt{\nu}}{k(p+c_p)^3} + \kappa_2\nu + O(\nu^{3/2}), \tag{B.51}$$

and hence,

$$\Pi_l(p^*_l) = \frac{(1-c_p)^2}{4} + \frac{2c_p^2(1-c_p)\sqrt{\nu}}{k(1+c_p)^2} + z_2\nu + O\left(\nu^{3/2}\right), \tag{B.52}$$

where $\kappa_2, z_2 \in \mathbb{R}$ are again constants with z_2 satisfying

$$z_2 = \frac{4c_p^3(c_p^2(5c_p-2)+5c_p-4)}{k^2(1+c_p)^6} + \frac{c_p(c_p+4\pi_d c_d)-1}{4}. \tag{B.53}$$

Comparing (B.47) and (B.52), $\Pi_{nl}(p^*_{nl}) > \Pi_l(p^*_l)$ if and only if $z_1 > z_2$ which, by comparing (B.48) and (B.53) and carrying out the algebra, is satisfied if and only if $\pi_d c_d > (1-c_p^2)/(2(1+3c_p))$. Since

$$\frac{(1-c_p)^2}{2(1+c_p)} < \frac{1-c_p^2}{2(1+3c_p)} < \frac{1-c_p}{2}, \tag{B.54}$$

there exist $\underline{\gamma} < \pi_d c_d < \overline{\gamma}$, where $\Pi_{nl}(p^*_{nl}) > \Pi_l(p^*_l)$. ■

Proof of Proposition 2: We will show that

(i) There exists $\overline{\gamma}, \underline{\omega}, \overline{\omega}, \overline{\eta} > 0$ such that if $\pi_d c_d < \overline{\gamma}$, $\underline{\omega} < \pi_a\alpha < \overline{\omega}$, and $\nu < \overline{\eta}$, then $\Pi_l(p^*_l) > \Pi_{nl}(p^*_{nl})$.

(ii) There exist $\underline{\theta}, \overline{\gamma}, \underline{\omega}, \overline{\omega} > 0$ such that if $\pi_d c_d < \overline{\gamma}$, $c_p < \underline{\theta}$, and $\underline{\omega} \le \pi_a\alpha \le \overline{\omega}$, then $\Pi_l(p^*_l) > \Pi_{nl}(p^*_{nl})$.

For part (i), for sufficiently small ν, by the proof of part (ii) of Proposition 1, $p_{nl}^* < \overline{p}_2$, and by (B.19), $\overline{p}_3 < (1-c_p)/2$ if and only if $\pi_d c_d < (1-c_p)^2/(2(1-c_p))$. Hence, v_b satisfies (B.23), and when $\pi_a\alpha = k/\sqrt{\nu}$ for $0 < k < \overline{k}/\sqrt{\nu}$, where $\overline{k}$ is given by (B.42), we have

$$v_b = p + c_p - \frac{c_p^2\sqrt{\nu}}{k(p+c_p)^2} + \left(\frac{c_p}{p+c_p} + \frac{2c_p^3}{k^2(p+c_p)^4} + \frac{2c_p^4}{k^2(p+c_p)^5} - \frac{\pi_d c_d c_p}{p}\right)\nu + O\left(\nu^{3/2}\right). \tag{B.55}$$

By (B.23) and the implicit function theorem,

$$\frac{dv_b}{dp} = \frac{\pi_a\alpha\pi_d c_d \nu v_b^3 - 2p^3 + p^2 v_b(2 - \pi_a\alpha\nu + \pi_a\alpha v_b(1-\nu))}{p^3(\pi_a\alpha\nu - 2) + 3p\pi_a\alpha v_b^2(\pi_d c_d\nu + p - p\nu) + 2p^2 v_b(1 - \pi_a\alpha(c_p(1-\nu) + \nu + \nu\pi_d c_d + p - p\nu))}. \tag{B.56}$$

By part (ii) of Lemma B.3, (5), and substituting $\pi_a\alpha = k/\sqrt{\nu}$, (B.55), and (B.56) we have

$$\frac{d\Pi_{nl}}{dp} = 1 - c_p - 2p - \frac{c_p^2(p-c_p)\sqrt{\nu}}{k(p+c_p)^3} + \kappa_3\nu + O(\nu^{3/2}), \tag{B.57}$$

and hence

$$\Pi_{nl}(p_{nl}^*) = \frac{(1-c_p)^2}{4} + \frac{2c_p^2(1-c_p)\sqrt{\nu}}{k(1+c_p)^2} + z_3\nu + O\left(\nu^{3/2}\right), \tag{B.58}$$

where $\kappa_3, z_3 \in \mathbb{R}$ are again constants. Comparing (B.52) and (B.58) we see that $\Pi_l(p_l^*) > \Pi_{nl}(p_{nl}^*)$ if and only if $z_2 > z_3$, which is always satisfied.

To see part (ii), let $1/\nu < k < 4/\nu$ and suppose

$$0 < \pi_d c_d < \min\left(1 - \frac{1}{k\nu}, \frac{1-\sqrt{\nu}}{2}\right). \tag{B.59}$$

Since $k > 1$ and by (B.59), for $\pi_a\alpha = kc_p$, there exists $\varepsilon > 0$ such that when $c_p < \varepsilon$, $\pi_a\alpha \geq c_p$ and $\pi_d c_d < (1-c_p)(1 - c_p/(\pi_a\alpha))$, are satisfied. Then by (B.8),

$$pc_p + \pi_a\alpha\nu(\pi_d c_d - p)(p+c_p)^2 = (p + \pi_d c_d k\nu p^2 - k\nu p^3)c_p + O(c_p^2), \tag{B.60}$$

and hence, $\hat{p}_1$ approaches $(\pi_d c_d + \sqrt{4/(k\nu) + (\pi_d c_d)^2})/2$, and by (B.9), $\overline{p}_2$ approaches $(1 - k + \pi_d c_d k\nu)/(-k(1-\nu))$ for sufficiently small c_p. Since $1 - 1/(k\nu) < 1/2 + (k-2)/(2k\nu)$, by (B.59), $\pi_d c_d < 1/2 + (k-2)/(2k\nu)$, which is satisfied if and only if $\overline{p}_2 > 1/2$. Further, $\hat{p}_1 > 1/2$ is satisfied if and only if $\pi_d c_d > 1/2 - 2/(k\nu)$, which is always satisfied since $k\nu < 4$. By (B.2), $\overline{p}_1$ approaches 1 as c_p gets small. Thus, by part (iii) of Lemma B.2, for all $\delta > 0$, there exists an $\varepsilon > 0$ such that when $c_p < \varepsilon$, and $0 < p < 1 - \delta$, $v_b < 1$. Then by (B.4), (B.10) and (B.14) v_b approaches p for sufficiently small c_p. It follows that $p_l^* = \operatorname{argmax}_{0 \leq p \leq 1}\Pi_l(p)$ approaches 1/2. Then, since $\min(\overline{p}_2, \hat{p}_1) > 1/2$, by part (i) of Lemma B.2, when $p = p_l^*$, v_b satisfies (B.10).

By (4),

$$\frac{d\Pi_l}{dp} = 1 - v_b - p \cdot \frac{dv_b}{dp}. \tag{B.61}$$

Substituting (B.50) into (B.61), and by (B.10), $v_b = p + z_1 c_p + O(c_p^2)$, and hence

$$p_l^* = 1/2 + \frac{(-8z_1^2 - 2k + kz_1(1-\nu))c_p}{2(k + k\nu(2\pi_d c_d - 1) + 8z_1)} + O(c_p^2), \tag{B.62}$$

where

$$z_1 = \frac{1}{8}\left(\sqrt{16k + (k(1-\nu) + 2\pi_d c_d k\nu)^2} - (k(1-\nu) + 2\pi_d c_d k\nu)\right) . \tag{B.63}$$

By (4), (B.62), and again since $v_b = p + z_1 c_p + O(c_p^2)$, we have

$$\Pi_l(p_l^*) = \frac{1-\nu}{4} - \frac{(1-\nu)z_1 c_p}{2} + O(c_p^2) . \tag{B.64}$$

Finally, $\pi_d c_d(1-\pi_d c_d) < (1-\nu)/4$ is satisfied if and only if $\pi_d c_d < (1-\sqrt{\nu})/2$, which holds by (B.59), and hence the vendor will not set $p \le \pi_d c_d$, which verifies the optimality of (B.62).

Now, by (B.19), $\overline{p}_3 = \pi_d c_d + z_2 c_p + O(c_p^2)$ where z_2 is the larger root of

$$z_2^2 + (2 + k(\pi_d c_d - \nu))z_2 + 1 - k + \pi_d c_d k = 0 , \tag{B.65}$$

and, by (B.17), $\hat{p}_2 = \pi_d c_d + c_p/(k\nu(\pi_d c_d)^2) + O(c_p^2)$. Substituting $z_2 = 1/(k\nu(\pi_d c_d)^2)$ into (B.65), it follows that $\overline{p}_3 \le \hat{p}_2$ if and only if $1 + k\nu\pi_d c_d(\pi_d c_d - 1) \ge 0$, which is satisfied since $k\nu \le 4$. By (B.18), $\hat{p}_3$ approaches $1/(k\nu(1-\pi_d c_d))$ as c_p gets small, and by (B.59), $\hat{p}_3 < \overline{p}_2$. Further, for any $\delta > 0$, there exists a $\varepsilon > 0$ such that when $0 < c_p < \varepsilon$, for $p > \pi_d c_d + \delta$, by parts (*ii*) and (*iii*) of Lemma B.3, $v_{sp} = 1$ and $\nu_p < 1$.

When $v_b < v_p$ by (B.23), v_b satisfies

$$(v_b - p)^2 - \pi_a \alpha v_b \nu (v_b - p)\left(1 - \frac{\pi_d c_d v_b}{p}\right) - \pi_a \alpha v_b^2 (1-\nu)(c_p - v_b + p) = 0 , \tag{B.66}$$

and by the implicit function theorem, dv_b/dp satisfies (B.56). By part (*ii*) of Lemma B.3, $v_{sp} = 1$, and hence by (5),

$$\frac{d\Pi_{nl}}{dp} = 1 - v_b - p \cdot \frac{dv_b}{dp} . \tag{B.67}$$

Substituting (B.56) into (B.67), and by (B.66), $v_b = p + z_3 c_p + O(c_p^2)$. Substituting again into (B.67), the unconstrained maximizer of Π_{nl} when $v_b < v_p$ then satisfies

$$p = 1/2 + \frac{(k(1-\nu)(z_3 - 2) - 8z_3^2)c_p}{2(k + k\nu(2\pi_d c_d - 3) + 8z_3)} + O(c_p^2) , \tag{B.68}$$

where

$$z_3 = \frac{1}{8}\left(\sqrt{16k(1-\nu) + (k - 3k\nu + 2k\nu\pi_d c_d)^2} - (k - 3k\nu + 2k\nu\pi_d c_d)\right) , \tag{B.69}$$

and hence,

$$\Pi_{nl}^{ii} \le \frac{1-\nu}{4} - \frac{(1-\nu)z_3 c_p}{2} + O(c_p^2) , \tag{B.70}$$

where Π_{nl}^{ii} is the maximum profit attained when $\overline{p}_3 \le \hat{p}_2$ and $\overline{p}_3 < p \le \min(\overline{p}_2, \hat{p}_3)$. By (B.64) and (B.70), $\Pi_l(p_l^*) > \Pi_{nl}^{ii}$ if $z_1 < z_3$ which holds if and only if $\pi_d c_d < 1/2 + (k-2)/(2k\nu)$, which is satisfied by (B.59).

For the case when $v_b = v_p$, by part (*iii*) of Lemma B.3 and (5), $\Pi_{nl}(p) = p(1-\nu)(1-p-c_p)$, which has an unconstrained maximizer at $p = (1-c_p)/2$, and therefore,

$$\Pi_{nl}^{iii} \le \frac{1-\nu}{4} - \frac{(1-\nu)c_p}{2} + O(c_p^2) , \tag{B.71}$$

where where Π_{nl}^{iii} is the maximum profit attained when $\overline{p}_3 \leq \hat{p}_2$, $\hat{p}_3 \leq \overline{p}_2$ and $\hat{p}_3 < p \leq 1$. By (B.64) and (B.71), $\Pi_l(p_l^*) > \Pi_{nl}^{iii}$ if $z_1 < 1$ which holds if and only if $\pi_d c_d > 1/2 - 2/(k\nu)$, which is always satisfied since $k\nu < 4$. Therefore $\Pi_{nl}(p_{nl}^*) \leq \max\left(\Pi_{nl}^{ii}, \Pi_{nl}^{iii}\right)$, and the proof is complete. ■

Proof of Proposition 3: We will prove that if $\pi_d c_d < (1 - \sqrt{c_p(2-c_p)})/2$, there exist $0 < \underline{\pi_a \alpha} < \overline{\pi_a \alpha}$ such that if $\nu > c_p(2-c_p)$, $0 < \omega_1 < \underline{\pi_a \alpha}$ and $\omega_2 > \overline{\pi_a \alpha}$ then

$$\Pi_{nl}(p_{nl}^*)|_{\pi_a\alpha=\omega_2} = \max_{\rho\in\{l,nl\}} \Pi_\rho(p_\rho^*)\Big|_{\pi_a\alpha=\omega_2} > \max_{\rho\in\{l,nl\}} \Pi_\rho(p_\rho^*)\Big|_{\pi_a\alpha=\omega_1}. \tag{B.72}$$

When $p \leq \pi_d c_d$, by Lemma 1 of August and Tunca 2006, if $\pi_a\alpha < c_p$ or both $\pi_a\alpha \geq c_p$ and $p \geq \overline{p}$, then $v_p = 1$ and

$$v_b = -\frac{1-\pi_a\alpha}{2\pi_a\alpha} + \frac{1}{2\pi_a\alpha}\sqrt{(1-\pi_a\alpha)^2 + 4\pi_a\alpha p}\,. \tag{B.73}$$

By this fact and Lemma B.1, $v_{sp} = 1$ for sufficiently small $\pi_a\alpha$, and hence, by (4) and (5), $\Pi_l(p) = \Pi_{nl}(p)$. Further, $\Pi_{nl}(p) > 0$ if and only if $p < \overline{p}_1$. Hence, by (B.73) and (B.4), $v_b = p + O(\pi_a\alpha)$. Since $\pi_d c_d < (1 - \sqrt{c_p(2-c_p)})/2$,

$$\max_{0<p\leq\pi_d c_d} \Pi_{nl}(p) < \left(\frac{1-\sqrt{c_p(2-c_p)}}{2}\right)\left(\frac{1+\sqrt{c_p(2-c_p)}}{2}\right) = \frac{(1-c_p)^2}{4}, \tag{B.74}$$

for sufficiently small $\pi_a\alpha$. By (5) and since $v_b = p + O(\pi_a\alpha)$, we also have

$$\max_{\pi_d c_d<p\leq 1} \Pi_{nl}(p) < \frac{1-\nu}{4}. \tag{B.75}$$

On the other hand, by the proof of Proposition 1, when $\pi_a\alpha$ is sufficiently large, $\max_p \Pi_{nl}(p) > \max_p \Pi_l(p)$, where $\max_p \Pi_{nl}(p)$ approaches $(1-c_p)^2/4$. Then, by (B.74), (B.75), and since $(1-c_p)^2 > (1-\nu)$ holds if and only if $\nu > c_p(2-c_p)$ is satisfied, the result follows. ■

Proof of Proposition 4: We will prove that if $c_p < 1/3$ and $\pi_a\alpha \geq \underline{\omega}$,

(i) $\frac{d\Pi_{\rho^*}(p^*)}{d(\pi_d c_d)} < 0$ if $\pi_d c_d < \frac{1+c_p}{4}$, and

(ii) $\frac{d\Pi_{\rho^*}(p^*)}{d(\pi_d c_d)} > 0$ if $\frac{1+c_p}{4} < \pi_d c_d < \frac{1-c_p}{2}$.

Let $\xi \triangleq 1/(\pi_a\alpha)$. For sufficiently large $\pi_a\alpha$, by part (vi) of Lemma B.3, the proof of part (i) of Proposition 1, and (B.24), we obtain

$$v_s = p + c_p - \frac{(p + c_p - \pi_d c_d)^2\xi}{\nu(p+c_p)^2} + O(\xi^2)\,, \tag{B.76}$$

and by (B.21) and (B.76),

$$v_{sp} = p + c_p + \frac{\pi_d c_d(p + c_p - \pi_d c_d)\xi}{\nu(p+c_p)^2} + O(\xi^2)\,. \tag{B.77}$$

By part (vi) of Lemma B.3 and (B.14), $v_b = p + c_p$ for $0 \le p \le 1 - c_p$ as $\pi_a\alpha$ gets large. Substituting v_b and (B.77) into (5) and differentiating yields

$$\frac{d\Pi_{nl}(p)}{dp} = 1 - c_p - 2p + \frac{\pi_d c_d(c_p(c_p - \pi_d c_d) + p(c_p + \pi_d c_d))\xi}{(p + c_p)^3} + O(\xi^2)\,. \tag{B.78}$$

Equating (B.78) to zero yields the first order condition, solving which we obtain the optimal price as

$$p^* = \frac{1 - c_p}{2} - \frac{2\pi_d c_d(c_p(1 + c_p) + \pi_d c_d(1 - 3c_p))\xi}{(1 + c_p)^3} + O(\xi^2)\,. \tag{B.79}$$

Hence it follows that

$$\Pi_{nl}(p^*) = \frac{(1 - c_p)^2}{4} - \frac{\pi_d c_d(1 - c_p)(1 + c_p - 2\pi_d c_d)\xi}{(1 + c_p)^2} + O(\xi^2)\,. \tag{B.80}$$

By (B.80), we obtain $d\Pi_{nl}(p^*)/d(\pi_d c_d) = (1-c_p)(1+c_p)^{-2}(4\pi_d c_d - 1 - c_p)\xi + O(\xi^2)$. Therefore, for sufficiently large $\pi_a\alpha$ and by Proposition 1, $d\Pi_{nl}(p^*)/d(\pi_d c_d) < 0$ for $\pi_d c_d < (1 + c_p)/4$ and $d\Pi_{nl}(p^*)/d(\pi_d c_d) > 0$ for $(1 + c_p)/4 < \pi_d c_d < (1 - c_p)/2$. Since $\rho^* = nl$, by Proposition 1, the result follows. ■

Proof of Proposition 5: Technically, we will first show that there exist threshold values $\underline{\omega} > 0$ and $\hat{\gamma} < (1 - c_p)/2$ such that if $\pi_a\alpha \ge \underline{\omega}$, then

$$\lim_{\pi_d c_d \to \hat{\gamma}^-} W_{\rho^*}(p^*) < \lim_{\pi_d c_d \to \hat{\gamma}^+} W_{\rho^*}(p^*)\,. \tag{B.81}$$

We will then prove that

$$\lim_{\gamma \to \hat{\gamma}^-} \left.\frac{dW_{nl}(p^*_{nl})}{d(\pi_d c_d)}\right|_{\pi_d c_d = \gamma} > 0\,. \tag{B.82}$$

Let $\xi \triangleq 1/(\pi_a\alpha)$. Suppose that $p = \pi_d c_d < (1 - c_p)/2$. As $\pi_a\alpha$ grows large, by (B.28), we have

$$v_b = \pi_d c_d + c_p - \frac{c_p^2\xi}{(\pi_d c_d + c_p)^2} + O(\xi^2)\,. \tag{B.83}$$

By (4) and (5), $\Pi_{nl}(\pi_d c_d) = \Pi_l(\pi_d c_d)$ and by substituting (B.83), we obtain

$$\Pi_{nl}(\pi_d c_d) = \pi_d c_d(1 - c_p - \pi_d c_d) + \frac{\pi_d c_d c_p^2\xi}{(\pi_d c_d + c_p)^2} + O(\xi^2)\,. \tag{B.84}$$

Suppose that $p > \pi_d c_d$. By (B.24) and part (vi) of Lemma B.3,

$$v_s = p + c_p - \frac{(p - \pi_d c_d + c_p)^2\xi}{\nu(p + c_p)^2} + O(\xi^2)\,. \tag{B.85}$$

Substituting (B.14), (B.21), and (B.85) into (5) gives

$$\Pi_{nl}(p) = p(1 - p - c_p) - \frac{p\pi_d c_d(p - \pi_d c_d + c_p)\xi}{(p + c_p)^2} + O(\xi^2)\,, \tag{B.86}$$

and by taking first order conditions yields

$$p_{nl}^* = \frac{1-c_p}{2} - \frac{2\pi_d c_d(c_p(1+c_p)+\pi_d c_d(1-3c_p))\xi}{(1+c_p)^3} + O(\xi^2)\,. \tag{B.87}$$

By (B.86) and (B.87), we obtain

$$\Pi_{nl}(p_{nl}^*) = \frac{(1-c_p)^2}{4} - \frac{\pi_d c_d(1-c_p)(1+c_p-2\pi_d c_d)\xi}{(1+c_p)^2} + O(\xi^2)\,, \tag{B.88}$$

and by equating (B.84) and (B.88), it follows that $\hat{\gamma} \triangleq \{\pi_d c_d \mid \Pi_{nl}(\pi_d c_d) = \Pi_{nl}(p_{nl}^*)\}$ is given by

$$\hat{\gamma} = \frac{1-c_p}{2} - \sqrt{\frac{c_p(1-c_p)\xi}{1+c_p}} + O(\xi)\,. \tag{B.89}$$

By (B.84) and (B.88), we have

$$\frac{d[\Pi_{nl}(\pi_d c_d) - \Pi_{nl}(p_{nl}^*)]}{d(\pi_d c_d)} = 1 - c_p - 2\pi_d c_d + \left(-1 + \frac{2}{1+c_p} - \frac{4\pi_d c_d(1-c_p)}{(1+c_p)^2} + \frac{c_p^2(c_p-\pi_d c_d)}{(c_p+\pi_d c_d)^3}\right)\xi + O(\xi^2)\,, \tag{B.90}$$

and further by evaluating (B.90) at $\hat{\gamma}$, we have

$$\left.\frac{d[\Pi_{nl}(\pi_d c_d) - \Pi_{nl}(p_{nl}^*)]}{d(\pi_d c_d)}\right|_{\pi_d c_d = \hat{\gamma}} = \sqrt{\frac{4c_p(1-c_p)\xi}{1+c_p}} + O(\xi)\,, \tag{B.91}$$

and therefore $d[\Pi_{nl}(\pi_d c_d) - \Pi_{nl}(p_{nl}^*)]/d(\pi_d c_d) > 0$ for all $0 \le \pi_d c_d \le \hat{\gamma}$. This implies that at $\pi_d c_d = \hat{\gamma}$, the vendor switches price from p_{nl}^* to $\pi_d c_d$. As $\pi_d c_d \to \hat{\gamma}^-$, by (7), (B.85), (B.87), and (B.89), we obtain

$$W_{nl}(p_{nl}^*) = \frac{3(1-c_p)^2}{8} + \frac{\pi_d c_d\left(\pi_d c_d + c_p(3+4c_p+c_p^2 - \pi_d c_d(8+c_p)\right)\xi}{(1+c_p)^3} + O(\xi^{3/2})\,. \tag{B.92}$$

However, as $\pi_d c_d \to \hat{\gamma}^+$, by (7), (B.83), and (B.89), we have

$$W_{nl}(\pi_d c_d) = \frac{3(1-c_p)^2}{8} + \frac{1-c_p}{2}\sqrt{\frac{c_p(1-c_p)\xi}{1+c_p}} + O(\xi)\,, \tag{B.93}$$

which proves that $\lim_{\pi_d c_d \to \hat{\gamma}^-} W_{\rho^*}(p^*) < \lim_{\pi_d c_d \to \hat{\gamma}^+} W_{\rho^*}(p^*)$. By (B.92), we obtain

$$\lim_{\gamma \to \hat{\gamma}^-} \left.\frac{dW_{nl}(p_{nl}^*)}{d(\pi_d c_d)}\right|_{\pi_d c_d = \gamma} = \frac{(1+c_p(6+c_p)(2c_p-1))\xi}{(1+c_p)^3} + O(\xi^{3/2})\,, \tag{B.94}$$

which is positive for all $0 < c_p < 1$. This completes the proof. ■

Proof of Proposition 6: We will show that there exist $\underline{\theta}, \underline{\omega}, \overline{\omega} > 0$ such that if $c_p < \underline{\theta}$, and $\underline{\omega} \le \pi_a \alpha \le \overline{\omega}$, then

(i) there exists $\underline{\eta} > 0$ that if $0 < \pi_d c_d < (1 - \sqrt{\nu})/4$ and $\nu < \underline{\eta}$, then $W_l(p_l^*) \ge W_{nl}(p_{nl}^*)$;

(ii) there exist $0 < \underline{\lambda} < \overline{\lambda}$ that if $\underline{\lambda} < \pi_d c_d < \overline{\lambda}$, then $W_{nl}(p_{nl}^*) > W_l(p_l^*)$.

For convenience in exposition, define $v_{s,\rho}$, $v_{sp,\rho}$, $v_{b,\rho}$ and $v_{p,\rho}$ as the corresponding threshold values under policy ρ. Since $\pi_d c_d < (1-\sqrt{\nu})/4$ and by part (ii) of the proof of Proposition 2, both p_l^* and p_{nl}^* approach $1/2$ for sufficiently small c_p. Again, by part (ii) of the proof of Proposition 2, since $\min(\overline{p}_2, \hat{p}_1) > 1/2$, by part (i) of Lemma B.2, we have $v_{sp,l} < 1$. Now, suppose $u_l > u_{nl}$, where u_l and u_{nl} are the sizes of the unpatched populations under policy l and nl as given by (B.6), respectively. By (A.14), $v_{s,l} = \pi_d c_d/(1 - \pi_a \alpha u_l)$ and $v_{s,nl} = \pi_d c_d/(1 - \pi_a \alpha u_{nl})$, hence $v_{s,l} > v_{s,nl}$. By (A.15), $v_{sp,l} = c_p/(\pi_a \alpha u_l)$, and by (A.19), $v_{sp,nl} = \min(1, (p + c_p - \pi_d c_d)/(\pi_a \alpha u_{nl}))$, and hence $v_{sp,nl} > v_{sp,l}$. Now, defining u_ρ^L and u_ρ^H as the sizes of the unpatched populations under policy ρ in the Type L and Type H consumer populations, respectively, it follows that $u_\rho = u_\rho^L + u_\rho^H$. Since $v_{sp,nl} > v_{sp,l}$ and $v_{s,l} > v_{s,nl}$, we obtain $u_{nl}^H > u_l^H$, and since $u_l > u_{nl}$, we have $u_{nl}^L < u_l^L$. By part (i) of Lemma B.2, $v_{b,l} < v_{p,l}$, and by (A.5), (A.8), and (A.9), we have $v_{b,l} = p/(1 - \pi_a \alpha u_l)$ and $v_{p,l} = c_p/(\pi_a \alpha u_l)$. Since $v_{b,l} < p + c_p$ and $u_l > u_{nl}$, we obtain $v_{b,nl} = p/(1 - \pi_a \alpha u) < v_{b,l} < p + c_p$. By (A.5) and (A.8), $v_{p,nl} = \min(1, c_p/(\pi_a \alpha u_{nl}))$, and hence, $v_{p,nl} > v_{p,l}$. It follows that $u_{nl}^L > u_l^L$, which is a contradiction. Therefore, $u_l \le u_{nl}$, and hence, $v_{s,l} \le v_{s,nl}$.

Now, by (B.18), we know that for sufficiently small c_p, $\hat{p}_3 > 1/2$ for $k < 2/\nu$. Then, by part (ii) of Proposition 2, $v_p < 1$ and p_{nl}^* satisfies (B.68). Comparing with (B.62), it then follows that $p_{nl}^* - p_l^* = \nu c_p/2 + O(\nu^2)$ and therefore, for sufficiently small ν, $p_{nl}^* > p_l^*$ and hence $v_{b,l} \le v_{b,nl}$. As a result, $(v - C(v, \theta, \sigma^*))^+$ is greater under $\rho = l$ than $\rho = nl$ for each consumer and by (7), it follows that $W_l(p_l^*) \ge W_{nl}(p_{nl}^*)$. This proves (i).

For (ii), first notice that $(1-\sqrt{\nu})/2 < 1 - 1/(k\nu)$ if and only if $k > 2/(\nu(1+\sqrt{\nu}))$ and that $2/(\nu(1+\sqrt{\nu})) > 1/\nu$ for all $\nu \in (0,1)$. Let $2/(\nu(1+\sqrt{\nu})) < k < 4/\nu$ and $0 < \pi_d c_d < (1-\sqrt{\nu})/2$. Then (B.59) is satisfied and since $k > 1$, for $\pi_a \alpha = k c_p$, there exists $\varepsilon > 0$ such that when $c_p < \varepsilon$, $\pi_a \alpha \ge c_p$ and $\pi_d c_d < (1 - c_p)(1 - c_p/(\pi_a \alpha))$. By (B.64), (B.70) and (B.71), we then have $\Pi_l(p_l^*) > \max_{p > \pi_d c_d} \Pi_{nl}(p)$. By (4), (5), and (B.28),

$$\Pi_l(\pi_d c_d) = \Pi_{nl}(\pi_d c_d) = \pi_d c_d (1 - \pi_d c_d) - z_0 c_p + O(c_p^2)\,, \tag{B.95}$$

where $z_0 = (\pi_d c_d)^2 (k \pi_d c_d - \sqrt{4k + (k \pi_d c_d)^2})/2$. Carrying out the algebra, it follows that $z_0 < z_1$ as given in (B.63). Therefore, by (B.64), and (B.95), there exists $\varepsilon > 0$ such that for any $c_p < \varepsilon$, there exists $\delta > 0$ such that if $(1-\sqrt{\nu})/2 - \delta < \pi_d c_d < (1-\sqrt{\nu})/2$ then $\Pi_l(\pi_d c_d) = \Pi_{nl}(\pi_d c_d) > \Pi_l(p_l^*)$. Consequently, by (B.64) and the continuity of $\Pi_l(\pi_d c_d)$ as in (B.95), there exist $\underline{\lambda}, \overline{\lambda} \in (0, (1-\sqrt{\nu})/2)$ such that when $\underline{\lambda} < \pi_d c_d < \overline{\lambda}$, $\max_{p > \pi_d c_d} \Pi_{nl}(p) < \Pi_l(\pi_d c_d) = \Pi_{nl}(\pi_d c_d) < \Pi_l(p_l^*)$. Hence $p_{nl}^* = \pi_d c_d$ and p_l^* is characterized by (B.62). But by (7), for sufficiently small c_p, $W_{nl}(\pi_d c_d)$ approaches $(1 - (\pi_d c_d)^2)/2$ and by (B.62), $W_l(p_l^*)$ approaches $(3 + \nu(1 - 4(\pi_d c_d)^2))/8$. Since $\pi_d c_d < 1/2$, it follows that $\Pi_l(p_l^*) > \Pi_{nl}(p_{nl}^*)$ and $W_{nl}^* > W_l^*$. This completes the proof. ■