Online Supplement for ISR Manuscript
"Cloud Implications on Software Network Structure and Security Risks"

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Appendix

Proof of Lemma 1: We first provide the complete parameter regions and the corresponding consumer equilibrium outcome:

(I) Unpatched On-premises only ($0 < v_u < 1$):

- (A) $\delta \le 1 - c_p$
  - (i) $p \le p_s(1-c_p)/\delta$ and $\alpha \le \alpha_B$;
  - (ii) $p_s(1-c_p)/\delta < p \le p_s/\delta$ and $\alpha \le \alpha_A$;
- (B) $\delta > 1 - c_p$
  - (i) $p \le 1 - c_p - \delta + p_s$ and $\alpha \le \alpha_B$;
  - (ii) $1 - c_p - \delta + p_s < p \le p_s$ and $\alpha \le \alpha_C$;
  - (iii) $p_s < p \le p_s/\delta$ and $\alpha \le \alpha_A$;

(II) SaaS only ($0 < v_d < 1$): $p > 1 + p_s - \delta$ and $\alpha \le \alpha_D$

(III) No Patching On-premises with Unpatched On-premises at the bottom ($0 < v_u < v_d < 1$):

- (A) $\delta > 1 - c_p$
  - (a) $p_s \le 1 - c_p$
    - (i) $1 + p_s - \delta - c_p < p \le p_s$ and $\alpha_C < \alpha \le \hat{\alpha}_2$;
  - (b) $p_s > 1 - c_p$
    - (i) $1 + p_s - \delta - c_p < p \le 1 - c_p$ and $\alpha_C < \alpha \le \hat{\alpha}_2$;
    - (ii) $1 - c_p \le p \le p_s$ and $\alpha_C < \alpha$;

(IV) No Patching On-premises with Unpatched On-premises at the top ($0 < v_d < v_u < 1$):

- (A) $\delta \le 1 - c_p$
  - (a) $p_s \le \delta - c_p$
    - (i) $p_s(1-c_p)/\delta < p \le p_s/\delta$ and $\alpha_A < \alpha \le \hat{\alpha}$;
    - (ii) $p_s/\delta < p \le 1 + p_s - \delta$ and $\alpha \le \hat{\alpha}$;
    - (iii) $1 + p_s - \delta < p \le 1 - c_p$ and $\alpha_D < \alpha \le \hat{\alpha}$;
    - (iv) $1 - c_p \le p$ and $\alpha_D < \alpha$;
  - (b) $\delta - c_p < p_s \le \delta(1 - c_p)$

(i) $p_s(1-c_p)/\delta < p \le p_s/\delta$ and $\alpha_A < \alpha \le \hat{\alpha}$;
(ii) $p_s/\delta < p \le 1-c_p$ and $\alpha \le \hat{\alpha}$;
(iii) $1-c_p < p \le 1+p_s-\delta$ and all α;
(iv) $1+p_s-\delta < p$ and $\alpha_D < \alpha$;

(c) $\delta(1-c_p) < p_s$

(i) $p_s(1-c_p)/\delta < p \le 1-c_p$ and $\alpha_A < \alpha \le \hat{\alpha}$;
(ii) $1-c_p < p \le p_s/\delta$ and $\alpha_A < \alpha$;
(iii) $p_s/\delta < p \le 1+p_s-\delta$ and all α;
(iv) $1+p_s-\delta < p$ and $\alpha_D < \alpha$;

(B) $\delta > 1-c_p$

(a) $p_s \le \delta - c_p$

(i) $p_s < p \le p_s/\delta$ and $\alpha_A < \alpha \le \hat{\alpha}$;
(ii) $p_s/\delta < p \le 1+p_s-\delta$ and $\alpha \le \hat{\alpha}$;
(iii) $1+p_s-\delta < p \le 1-c_p$ and $\alpha_D < \alpha \le \hat{\alpha}$;
(iv) $1-c_p < p$ and $\alpha_D < \alpha$;

(b) $\delta - c_p < p_s \le \delta(1-c_p)$

(i) $p_s < p \le p_s/\delta$ and $\alpha_A < \alpha \le \hat{\alpha}$;
(ii) $p_s/\delta < p \le 1-c_p$ and $\alpha \le \hat{\alpha}$;
(iii) $1-c_p < p \le 1+p_s-\delta$ and all α;
(iv) $1+p_s-\delta < p$ and $\alpha_D < \alpha$;

(c) $\delta(1-c_p) < p_s \le 1-c_p$

(i) $p_s < p \le 1-c_p$ and $\alpha_A < \alpha \le \hat{\alpha}$;
(ii) $1-c_p < p \le p_s/\delta$ and $\alpha_A < \alpha$;
(iii) $p_s/\delta < p \le 1+p_s-\delta$ and all α;
(iv) $1+p_s-\delta < p$ and $\alpha_D < \alpha$;

(d) $1-c_p < p_s$

(i) $p_s < p \le p_s/\delta$ and $\alpha_A < \alpha$;
(ii) $p_s/\delta < p \le 1+p_s-\delta$ and all α;
(iii) $1+p_s-\delta < p$ and $\alpha_D < \alpha$;

(V) No SaaS: $0 < v_u < v_p < 1$

(A) $\delta \le 1-c_p$

(a) $p_s \le \delta c_p/(1-\delta)$

(i) $p \le p_s/\delta - c_p$ and $\alpha > \alpha_B$;
(ii) $p_s/\delta - c_p < p \le p_s$ and $\alpha_B < \alpha \le \alpha_F$;
(iii) $p_s < p \le p_s(1-c_p)/\delta$ and $\alpha_B < \alpha \le \alpha_E$;

(b) $p_s > \delta c_p/(1-\delta)$

(i) $p \leq p_s/\delta - c_p$ and $\alpha > \alpha_B$;

(ii) $p_s/\delta - c_p < p \leq p_s(1-c_p)/\delta$ and $\alpha_B < \alpha \leq \alpha_E$;

(B) $\delta > 1 - c_p$

(i) $p \leq p_s/\delta - c_p$ and $\alpha > \alpha_B$;

(ii) $p_s/\delta - c_p < p \leq 1 + p_s - \delta - c_p$ and $\alpha_B < \alpha \leq \alpha_F$;

(VI) SaaS in the bottom tier: $0 < v_d < v_u < v_p < 1$

(A) $\delta \leq 1 - c_p$

(a) $p_s \leq \delta c_p/(1-\delta)$

(i) $p_s < p \leq p_s(1-c_p)/\delta$ and $\alpha > \alpha_E$;

(ii) $p_s(1-c_p)/\delta < p < 1 - c_p$ and $\alpha > \hat{\alpha}_1$;

(b) $p_s > \delta c_p/(1-\delta)$

(i) $p_s/\delta - c_p < p \leq p_s(1-c_p)/\delta$ and $\alpha > \alpha_E$;

(ii) $p_s(1-c_p)/\delta < p < 1 - c_p$ and $\alpha > \hat{\alpha}_1$;

(B) $\delta > 1 - c_p$

(a) $p_s \leq 1 - c_p$

(i) $p_s < p < 1 - c_p$ and $\alpha > \hat{\alpha}_1$;

(VII) SaaS in the middle tier: $0 < v_u < v_d < v_p < 1$

(A) $\delta \leq 1 - c_p$

(a) $p_s \leq \delta c_p/(1-\delta)$

(i) $p_s/\delta - c_p < p \leq p_s$ and $\alpha > \alpha_F$;

(B) $\delta > 1 - c_p$

(a) $p_s \leq 1 - c_p$

(i) $p_s/\delta - c_p < p \leq 1 + p_s - \delta - c_p$ and $\alpha > \alpha_F$;

(ii) $1 + p_s - \delta - c_p < p \leq p_s$ and $\alpha > \hat{\alpha}_2$;

(b) $p_s > 1 - c_p$

(i) $p_s/\delta - c_p < p \leq 1 + p_s - \delta - c_p$ and $\alpha > \alpha_F$;

(ii) $1 + p_s - \delta - c_p < p < 1 - c_p$ and $\alpha > \hat{\alpha}_2$.

First, we establish the general threshold-type equilibrium structure. Given the size of unpatched user population of the on-premises version u and the size of user population of the SaaS version d, the net payoff of the consumer with type v for different strategy profiles σ is written as

$$R(v,\sigma) \triangleq \begin{cases} v - p - c_p & if \quad \sigma(v) = (OP, P)\,; \\ v - p - \pi_u u \alpha v & if \quad \sigma(v) = (OP, NP)\,; \\ \delta v - p_s - \pi_d d \alpha \delta v & if \quad \sigma(v) = (SaaS, ND)\,; \\ 0 & if \quad \sigma(v) = (N, ND)\,. \end{cases} \tag{A.1}$$

First, $\sigma(v) = (OP, P)$, if and only if

$$\begin{aligned}
v - p - c_p \geq v - p - \pi_u u\alpha v &\iff v \geq \frac{c_p}{\pi_u u\alpha}, \text{ and} \\
v - p - c_p \geq \delta v - p_s - \pi_d d\alpha\delta v &\iff v \geq \frac{p + c_p - p_s}{1 - \delta(1 - \pi_d \alpha d)}, \text{ and} \\
v - p - c_p \geq 0 &\iff v \geq p + c_p\,,
\end{aligned}$$

which can be summarized as

$$v \geq \max\left(\frac{c_p}{\pi_u \alpha u}, \frac{p + c_p - p_s}{1 - \delta(1 - \pi_d \alpha d)}, p + c_p\right). \tag{A.2}$$

By (A.2), in equilibrium, if a consumer with valuation v_0 buys and patches the on-premises alternative, then every consumer with valuation $v > v_0$ will also do so. Hence, there exists a threshold $v_p \in (0, 1]$ such that for all $v \in \mathcal{V}$, $\sigma^*(v) = (OP, P)$ if and only if $v \geq v_p$. Similarly, $\sigma(v) \in \{(OP, P),$ (OP, NP), $(SaaS, ND)\}$, i.e., the consumer purchases one of the alternatives, if and only if

$$\begin{aligned}
v - p - c_p \geq 0 &\iff v \geq p + c_p, \text{ or} \\
v - p - \pi_u u\alpha v \geq 0 &\iff v \geq \frac{p}{1 - \pi_u \alpha u}, \text{ or} \\
\delta v - p_s - \pi_d d\alpha\delta v \geq 0 &\iff v \geq \frac{p_s}{\delta(1 - \pi_d \alpha d)}\,,
\end{aligned}$$

which can be rewritten as

$$v \geq \min\left(p + c_p, \frac{p}{1 - \pi_u \alpha u}, \frac{p_s}{\delta(1 - \pi_d \alpha d)}\right). \tag{A.3}$$

Let $0 < v_1 \leq 1$ and $\sigma^*(v_1) \in \{(OP, P), (OP, NP), (SaaS, ND)\}$, then by (A.3), for all $v > v_1$, $\sigma^*(v) \in \{(OP, P), (OP, NP), (SaaS, ND)\}$, and hence there exists a $\underline{v} \in (0, 1]$, such that a consumer with valuation $v \in \mathcal{V}$ will purchase if and only if $v \geq \underline{v}$.
By (A.2) and (A.3), $\underline{v} \leq v_p$ holds. Moreover, consumers with types in $[\underline{v},\ v_p]$ choose either (OP, NP) or $(SaaS, ND)$. A purchasing consumer with valuation v will prefer (OP, NP) over $(SaaS, ND)$ if and only if

$$v - p - \pi_u u\alpha v > \delta v - p_s - \pi_d d\alpha\delta v \iff v[(1 - \pi_u \alpha u) - \delta(1 - \pi_d \alpha d)] > p - p_s\,. \tag{A.4}$$

This inequality can be either $v > (p - p_s)/((1 - \pi_u \alpha u) - \delta(1 - \pi_d \alpha d))$ or $v < (p - p_s)/((1 - \pi_u \alpha u) - \delta(1 - \pi_d \alpha d))$, depending on the sign of $(1 - \pi_u \alpha u) - \delta(1 - \pi_d \alpha d)$. Consequently, there can be two cases for (OP, NP) and $(SaaS, ND)$ in equilibrium: first, there exists $v_u \in [\underline{v}, v_p]$ such that $\sigma(v) = (OP, NP)$ for all $v \in [v_u, v_p)$, and $\sigma(v) = (SaaS, ND)$ for all $v \in [v_d, v_u)$ where $v_d = \underline{v}$. Second, there exists $v_d \in [\underline{v}, v_p]$ such that $\sigma(v) = (SaaS, ND)$ for all $v \in [v_d, v_p)$, and $\sigma(v) = (OP, NP)$ for all $v \in [v_u, v_d)$ where $v_u = \underline{v}$. If $(1 - \pi_u \alpha u) - \delta(1 - \pi_d \alpha d) = 0$, then depending on the sign of $p - p_s$, all consumers unilaterally prefer either (OP, NP) or $(SaaS, ND)$; e.g., if $p > p_s$, all consumers prefer $(SaaS, ND)$, and if $p < p_s$, then all consumers prefer (OP, NP). Finally, if $p = p_s$, all consumers are indifferent between (OP, NP) and $(SaaS, ND)$, in which case only the sizes of consumer population u and d matter in equilibrium, i.e., $(1 - \pi_u \alpha u) - \delta(1 - \pi_d \alpha d) = 0$ in equilibrium. Technically, there are multiple equilibria in this case; however, utility of each consumer and the vendor's profit are the same in all equilibria. So, without loss of generality, we focus on the threshold-type equilibrium in this case. We also provide detail discussion for this case when we investigate case (VII) below. In summary, we have established the threshold-type equilibrium structure in (6) and (7).

Next, we characterize in more detail case (VII) in which SaaS arises in the middle tier in equilibrium, i.e., $0 < v_u < v_d < v_p < 1$, as well as the corresponding parameter regions. Based on this threshold structure, it follows that $d = v_p - v_d$ and $u = v_d - v_u$. In this case, we prove the following claim first:

Claim: The equilibrium that corresponds to case (VII) arises if and only if the following conditions are satisfied:

$$p \le p_s < p + c_p < 1 \quad \text{and} \quad p_s < \delta(p + c_p) \quad \text{and}$$
$$\left\{ \left(p \le 1 - c_p - (\delta - p_s) \text{ and } \alpha > \alpha_F \right) \quad \text{or} \quad \left(p > 1 - c_p - (\delta - p_s) \text{ and } \alpha > \hat{\alpha}_2 \right) \right\}. \quad \text{(A.5)}$$

Proof: First, we prove that there exists the unique positive $\hat{\alpha}_2$ that satisfies $g_2(\alpha) = 0$, where function g_2 was defined in (14). Taking the derivative of g_2 with respect to α, we obtain

$$\frac{dg_2(\alpha)}{d\alpha} = \frac{2\delta\pi_d(p_s - p) + \Phi}{(\Phi - \alpha\delta\pi_d)^2} + \pi_u \left(1 + \frac{\Phi\pi_u + \delta\pi_d(1 - 2p - \alpha\pi_u)}{\delta\pi_d\sqrt{4p\alpha\pi_u + \left(1 - \alpha\pi_u + \frac{\Phi\pi_u}{\delta\pi_d}\right)^2}} \right). \quad \text{(A.6)}$$

The first term in (A.6) is positive because $p \le p_s$ and $\Phi > 0$, i.e., $p > 1 - c_p - (\delta - p_s)$, in the relevant region of $\hat{\alpha}_2$. Furthermore, we have

$$(\delta\pi_d)^2 \left(4p\alpha\pi_u + \left(1 - \alpha\pi_u + \frac{\Phi\pi_u}{\delta\pi_d} \right)^2 \right) - (\Phi\pi_u + \delta\pi_d(1 - 2p - \alpha\pi_u))^2$$
$$= 4p\delta\pi_d\left((1 - p)\delta\pi_d + \Phi\pi_u\right) > 0. \quad \text{(A.7)}$$

Hence, the second term in (A.6) is also positive because $p < 1 - c_p$ and $\Phi > 0$. As a result, (A.6) is positive, i.e., g_2 is increasing in α. In addition,

$$g_2(0) = -2\left(p + c_p - p_s + \frac{\Phi\pi_u}{\delta\pi_d} \right) < 0, \quad \text{(A.8)}$$

and

$$\lim_{\alpha\to\infty} g_2(\alpha) = 2(1 - p - c_p) > 0, \quad \text{(A.9)}$$

under the parameter region of $p + c_p > p_s$ and $1 - p > c_p$ from (A.5). From (14) and (A.6) - (A.9), there exists a unique $\hat{\alpha}_2 > 0$ that satisfies $g_2(\hat{\alpha}_2) = 0$.

Next, we investigate the characterization of the three thresholds, v_p, v_d and v_u. The consumer with type v_p is indifferent between (OP, P) and $(SaaS, ND)$; hence, it satisfies

$$v_p - p - c_p = \delta v_p - p_s - \pi_d(v_p - v_d)\alpha\delta v_p \Longleftrightarrow v_p(1 - \delta(1 - \pi_d\alpha(v_p - v_d))) = p + c_p - p_s\,. \quad \text{(A.10)}$$

Similarly, the consumer with type v_d is indifferent between $(SaaS, ND)$ and (OP, NP):

$$\delta v_d - p_s - \pi_d(v_p - v_d)\alpha\delta v_d = v_d - p - \pi_u(v_d - v_u)\alpha v_d$$
$$\Longleftrightarrow v_d(\delta(1 - \pi_d\alpha(v_p - v_d)) - (1 - \pi_u\alpha(v_d - v_u))) = p_s - p\,. \quad \text{(A.11)}$$

Finally, the consumer with type v_u is indifferent between (OP, NP) and (N, ND), which can be written as

$$v_u - p - \pi_u(v_d - v_u)\alpha v_u = 0 \Longleftrightarrow v_u(1 - \pi_u\alpha(v_d - v_u)) = p\,. \quad \text{(A.12)}$$

Next, we prove that under the parameter conditions given in (A.5), there exist the unique set of solutions v_u, v_d, and v_p with $0 < v_u < v_d < v_p < 1$ that satisfy (A.10), (A.11) and (A.12).
First, (A.10) can be rewritten as

$$f_1(v_p) \triangleq \delta\pi_d\alpha v_p^2 + (1 - \delta(1 + \pi_d\alpha v_d))v_p - (p + c_p - p_s) = 0\,. \quad \text{(A.13)}$$

Then, $v_d < v_p < 1$ becomes equivalent to $f_1(v_d) < 0$ and $f_1(1) > 0$. The inequality $f_1(v_d) < 0$ can be simplified to

$$v_d < \frac{p + c_p - p_s}{1 - \delta}\,. \quad \text{(A.14)}$$

For this inequality to hold, $p + c_p > p_s$ should be satisfied, i.e., $p + c_p > p_s$ is a necessary condition. In addition, (A.14) matters only when $(p + c_p - p_s)/(1 - \delta) \leq 1$, i.e., $\Phi \leq 0$. Otherwise, i.e., if $\Phi > 0$, then $v_d < 1$. Similarly, the inequality $f_1(1) > 0$ is simplified to

$$v_d < \frac{1 + p_s - (c_p + p + \delta - \alpha\delta\pi_d)}{\alpha\delta\pi_d}\,, \quad \text{(A.15)}$$

and for this inequality to hold, $\alpha > \Phi/(\delta\pi_d)$. In addition, this condition matters only when $\Phi > 0$. Otherwise, i.e., $\Phi \leq 0$, then $v_d < 1$. As a result, if $\Phi \leq 0$, then (A.14) should hold; otherwise, (A.15) should be satisfied. Furthermore, solving $f_1(v_p) = 0$, we obtain

$$v_p = \frac{\sqrt{(1 - \delta - v_d\alpha\delta\pi_d)^2 + 4\alpha\delta\pi_d(p + c_p - p_s)} - (1 - \delta) + v_d\alpha\delta\pi_d}{2\alpha\delta\pi_d}\,. \quad \text{(A.16)}$$

Next, (A.12) can be rewritten as

$$f_2(v_u) \triangleq \pi_u\alpha v_u^2 + (1 - \pi_u\alpha v_d)v_u - p = 0\,. \quad \text{(A.17)}$$

From this equation, it follows that $f_2(0) = -p < 0$; consequently, $0 < v_u < v_d$ is equivalent to $f_2(v_d) > 0$, which is simplified to $v_d > p$. Moreover, solving (A.17), we obtain

$$v_u = \frac{\sqrt{4p\alpha\pi_u + (1 - v_d\alpha\pi_u)^2} - 1 + v_d\alpha\pi_u}{2\alpha\pi_u}\,. \quad \text{(A.18)}$$

By plugging (A.16) and (A.18) into (A.11) and simplifying, (A.11) can be rewritten as $f_3(v_d) = 0$ where

$$f_3(v_d) \triangleq \frac{2(p-p_s)}{v_d} + \delta + v_d\alpha\delta\pi_d + v_d\alpha\pi_u - \sqrt{4\alpha\delta\pi_d(p+c_p-p_s) + (1-\delta-v_d\alpha\delta\pi_d)^2} - \sqrt{4p\alpha\pi_u + (1-v_d\alpha\pi_u)^2}\,. \quad \text{(A.19)}$$

Furthermore, consumer types $v \in (v_d,\, v_p)$ should prefer $(SaaS, ND)$ over (OP, NP) in equilibrium, i.e.,

$$v(\delta(1-\pi_d\alpha(v_p-v_d)) - (1-\pi_u\alpha(v_d-v_u))) > p_s - p\,, \quad \text{(A.20)}$$

for $v \in (v_d,\, v_p)$. Then, from (A.11), it implies that both $p \le p_s$ and $\delta(1-\pi_d\alpha(v_p-v_d)) > 1 - \pi_u\alpha(v_d-v_u)$ should be satisfied, which is equivalent to $p \le p_s$ and $v_d \ge 0$. In addition, since we have $v_d > p$ from $f_2(v_d) > 0$, the condition $v_d \ge 0$ becomes redundant. By taking the derivative of f_3 with respect to v_d in (A.19), we obtain

$$\frac{d\,f_3(v_d)}{d\,v_d} = \frac{2(p_s-p)}{v_d^2} + \alpha\delta\pi_d\left(1 - \frac{1-\delta-v_d\alpha\delta\pi_d}{\sqrt{4\alpha\delta\pi_d(p+c_p-p_s)+(1-\delta-v_d\alpha\delta\pi_d)^2}}\right) + \alpha\pi_u\left(1 - \frac{1-v_d\alpha\pi_u}{\sqrt{4p\alpha\pi_u+(1-v_d\alpha\pi_u)^2}}\right) > 0\,, \quad \text{(A.21)}$$

for $p \le p_s$ and $p + c_p > p_s$, both of which are satisfied from (A.5). Consequently, in this parameter region, $f_3(v_d)$ is strictly increasing in v_d. We now investigate the remaining conditions to guarantee (A.14), (A.15) and $v_d > p$. We also have necessary conditions: $p \le p_s$ and $p + c_p > p_s$ from above. In addition, for consumers who optimally choose $(OP,\, P)$ to exist in equilibrium, $1 - p - c_p > 0$, i.e., $p < 1 - c_p$, should be satisfied. From the monotonicity of $f_3(v_d)$, it follows that (A.14), (A.15) and $v_d > p$ are satisfied, if and only if $f_3((p+c_p-p_s)/(1-\delta)) > 0$, $f_3((\alpha\delta\pi_d - \Phi)/(\alpha\delta\pi_d)) > 0$ and $f_3(p) < 0$ are satisfied. First, it follows that

$$f_3(p) = -\frac{2(p_s-p)}{p} - (1-\delta-p\alpha\delta\pi_d) - \sqrt{4\alpha\delta\pi_d(p+c_p-p_s)+(1-\delta-p\alpha\delta\pi_d)^2} < -\frac{2(p_s-p)}{p} - (1-\delta-p\alpha\delta\pi_d) - |1-\delta-p\alpha\delta\pi_d| \le 0\,, \quad \text{(A.22)}$$

where the first inequality follows from $p + c_p > p_s$ and the second inequality follows from $p \le p_s$ and $x + |x| \ge 0$.

Note that when $p = p_s$, $f_3(v_d)$ in (A.19) is equivalent to $\delta(1-\pi_d\alpha(v_p-v_d)) - (1-\pi_u\alpha(v_d-v_u)) = 0$ in (A.11). Consequently, from (A.20), all consumers are indifferent between $(SaaS, ND)$ and (OP, NP); in this case, in equilibrium, consumer types $v \in [v_p, 1]$ prefer (OP, P) and consumer types $v \in [0,\, v_u)$ prefer (N, ND). Moreover, consumer types $v \in [v_u, v_p)$ choose $(SaaS, ND)$ or (OP, NP) in equilibrium in such a way that the consumer population size with $(SaaS, ND)$ is $v_p - v_d$ and the population size with (OP, NP) equals $v_d - v_u$. Technically speaking, there exist multiple consumer equilibria in this case; however, we resolve this case so that it is consistent with our threshold-type equilibrium structure, i.e., consumer types $v \in [v_d, v_p)$ choose $(SaaS, ND)$, and consumer types $v \in [v_u, v_d)$ choose (OP, NP). Each consumer's utility as well as the vendor's profit are the same in all equilibria in this case; without loss of generality, we focus on this threshold-type equilibrium structure.

Next, $f_3((\alpha\delta\pi_d - \Phi)/(\alpha\delta\pi_d)) > 0$ is relevant when $\Phi > 0$ from above, and $f_3((\alpha\delta\pi_d - \Phi)/(\alpha\delta\pi_d)) > 0$ can be rewritten as $g_2(\alpha) > 0$, where $g_2(\alpha)$ is in (14). From the definition of $\hat{\alpha}_2$ and (14), and (A.6) - (A.9), $f_3((\alpha\delta\pi_d - \Phi)/(\alpha\delta\pi_d)) > 0$ holds, if and only if $\alpha > \hat{\alpha}_2$, under $p + c_p > p_s$, $1 - p > c_p$ and $\Phi > 0$. Moreover, $\hat{\alpha}_2 > \Phi/(\delta\pi_d)$ since $g_2(\alpha)$ becomes negative as α approaches $\Phi/(\delta\pi_d)$ from above and $g_2(\alpha)$ is increasing in α. Finally, $f_3((p + c_p - p_s)/(1 - \delta)) > 0$ is simplified to

$$\frac{(p + c_p - p_s)\alpha\pi_u}{1 - \delta} - \frac{c_p + p_s - p - 2c_p\delta}{p + c_p - p_s} > \sqrt{4p\alpha\pi_u + \left(1 - \frac{(p + c_p - p_s)\alpha\pi_u}{1 - \delta}\right)^2}. \tag{A.23}$$

For this inequality to hold, the left-hand side should be positive, i.e.,

$$\alpha > \frac{(1 - \delta)(c_p + p_s - p - 2c_p\delta)}{(p + c_p - p_s)^2\pi_u}. \tag{A.24}$$

Under this condition (A.24), by squaring (A.23) and simplifying, we have

$$\alpha((c_p + p)\delta - p_s) > \frac{c_p(1 - \delta)^2(p - p_s + c_p\delta)}{(p + c_p - p_s)^2\pi_u}. \tag{A.25}$$

If $(c_p + p)\delta - p_s > 0$, then it follows that

$$\frac{c_p(1 - \delta)^2(p - p_s + c_p\delta)}{(p + c_p - p_s)^2\pi_u((c_p + p)\delta - p_s)} > \frac{(1 - \delta)(c_p + p_s - p - 2c_p\delta)}{(p + c_p - p_s)^2\pi_u}, \tag{A.26}$$

under the region of $p < 1 - c_p$, $p_s < p + c_p$ and $p \leq p_s$. However, if $(c_p + p)\delta - p_s \leq 0$, then the inequality in (A.26) holds in the opposite direction. Consequently, under the parameter region of $p < 1 - c_p$, $p_s < p + c_p$ and $p \leq p_s$, $f_3((p + c_p - p_s)/(1 - \delta)) > 0$ is simplified to

$$(c_p + p)\delta - p_s > 0 \quad \text{and} \quad \alpha > \alpha_F \triangleq \frac{c_p(1 - \delta)^2(p - p_s + c_p\delta)}{(p + c_p - p_s)^2\pi_u((c_p + p)\delta - p_s)}. \tag{A.27}$$

In summary, there exist the unique solutions v_u, v_d, and v_p with $0 < v_u < v_d < v_p < 1$ that satisfy (A.10), (A.11) and (A.12), if and only if the conditions (A.5) are satisfied. □

To complete the proof of case (VII), note that $p_s/\delta - c_p < 1 - c_p - (\delta - p_s) < 1 - c_p$ holds since $p_s < \delta$. Second, if $\delta > 1 - c_p$, then $1 - c_p - (\delta - p_s) < p_s$. In this case, if $p_s \leq 1 - c_p$, then (A.5) becomes

$$\left\{\left(\frac{p_s}{\delta} - c_p < p \leq 1 - c_p - (\delta - p_s) \text{ and } \alpha > \alpha_F\right) \text{ or } \left(1 - c_p - (\delta - p_s) < p \leq p_s \text{ and } \alpha > \hat{\alpha}_2\right)\right\}, \tag{A.28}$$

whereas if $p_s > 1 - c_p$, then (A.5) becomes

$$\left\{\left(\frac{p_s}{\delta} - c_p < p \leq 1 - c_p - (\delta - p_s) \text{ and } \alpha > \alpha_F\right) \quad \text{or} \quad \left(1 - c_p - (\delta - p_s) < p < 1 - c_p \text{ and } \alpha > \hat{\alpha}_2\right)\right\}, \tag{A.29}$$

Third, if $\delta \leq 1 - c_p$, then $p_s \leq 1 - c_p - (\delta - p_s) < 1 - c_p$. For this region of $p_s/\delta - c_p < p \leq p_s$ to be non-empty, $p_s \leq \delta c_p/(1-\delta)$. Thus, in this case, (A.5) becomes

$$\left(p_s \leq \frac{\delta c_p}{1-\delta} \text{ and } \frac{p_s}{\delta} - c_p < p \leq p_s \text{ and } \alpha > \alpha_F\right). \tag{A.30}$$

Combining (A.28)–(A.30), we then obtain the presented conditions for case (VII).

Next, for case (VI), in which SaaS arises in the low tier in equilibrium, i.e., $0 < v_d < v_u < v_p < 1$, based on the threshold-type equilibrium structure, $u = v_p - v_u$ and $d = v_u - v_d$. Following similar steps to the proof of case (VII), we prove the following claim first:

Claim: The equilibrium that corresponds to case (VI) arises if and only if the following conditions are satisfied:

$$p_s \leq p < 1 - c_p \text{ and } \alpha > \hat{\alpha}_1 \text{ and } \left\{\left(\delta > \frac{p_s}{p}\right) \quad \text{or} \quad \left(\frac{p_s}{p + c_p} \leq \delta < \frac{p_s}{p} \text{ and } \alpha > \alpha_E\right)\right\}, \tag{A.31}$$

Proof: First, we prove that there exists the unique root $\hat{\alpha}_1$ greater than c_p/π_u that satisfies $g_1(\alpha) = 0$. Similar to the proof of case (VII), it follows that $g_1(\alpha)$ is decreasing in α under the parameter region of $p_s \leq p < 1 - c_p$ and $p_s/(p + c_p) \leq \delta$. Furthermore, $g_1(\alpha)$ becomes positive as α approaches c_p/π_u from above and $g_1(\alpha)$ becomes negative as α becomes large enough, which proves that there exists a unique $\hat{\alpha}_1$ greater than c_p/π_u that solves $g_1(\alpha) = 0$.
From the threshold-type equilibrium structure, in this case, we have $u = v_p - v_u$ and $d = v_u - v_d$. Moreover, the thresholds, v_p, v_u and v_p satisfy the following three equations:

$$\pi_u \alpha (v_p - v_u) v_p = c_p \,, \tag{A.32}$$

$$v_u(1 - \pi_u \alpha (v_p - v_u)) - \delta(1 - \pi_d \alpha (v_u - v_d)) = p - p_s \,, \quad \text{and} \tag{A.33}$$

$$\delta v_d (1 - \pi_d \alpha (v_u - v_d)) = p_s \,. \tag{A.34}$$

Using (A.32) and $v_p < 1$, we obtain

$$v_u < 1 - \frac{c_p}{\alpha \pi_u} \quad \text{and} \quad v_p = \frac{v_u \alpha \pi_u + \sqrt{\alpha \pi_u \left(\alpha \pi_u v_u^2 + 4 c_p\right)}}{2 \alpha \pi_u} \,. \tag{A.35}$$

In addition, from (A.34) and $v_d < v_u$, it follows that

$$v_u > \frac{p_s}{\delta} \quad \text{and} \quad v_d = \frac{\sqrt{\delta^2 (1 - \alpha \pi_d v_u)^2 + 4 p_s \alpha \delta \pi_d} - \delta(1 - \alpha \pi_d v_u)}{2 \alpha \delta \pi_d} \,. \tag{A.36}$$

From (A.35) and (A.36), we have

$$\frac{p_s}{\delta} < v_u < 1 - \frac{c_p}{\alpha \pi_u}\,, \tag{A.37}$$

and for this region of v_u to be non-empty, $\alpha > c_p\delta/(\pi_u(\delta - p_s))$. Furthermore, by substituting v_p and v_u in (A.35) and (A.36) respectively into (A.33) and simplifying, we obtain

$$f_4(v_u) \triangleq \frac{2(p-p_s)}{v_u} - \Big(2 - \delta + \alpha v_u(\pi_u + \delta\pi_d) - \sqrt{\delta^2(1-\alpha\pi_d v_u)^2 + 4p_s\alpha\delta\pi_d} - \sqrt{\alpha\pi_u\left(\alpha\pi_u v_u^2 + 4c_p\right)}\,\Big) = 0\,. \tag{A.38}$$

By taking derivative with respect to v_u, it then follows that

$$\frac{df_4(v_u)}{dv_u} = -\frac{2(p-p_s)}{v_u^2} - \alpha\pi_u\left(1 - \frac{\alpha\pi_u v_u}{\sqrt{(\alpha\pi_u v_u)^2 + 4\alpha\pi_u c_p}}\right) - \delta\alpha\pi_d\left(1 - \frac{\delta(1-\alpha\pi_d v_u)}{\sqrt{\delta^2(1-\alpha\pi_d v_u)^2 + 4p_s\alpha\delta\pi_d}}\right)\,, \tag{A.39}$$

which is negative, i.e., $f_4(v_u)$ is decreasing in v_u, when $p \geq p_s$. Consequently, (A.37) becomes equivalent to $f_4(p_s/\delta) > 0$ and $f_4(1 - c_p/(\alpha\pi_u)) < 0$. In addition, for (A.37) to be non-empty, we need $\alpha > c_p\delta/(\pi_u(\delta - p_s))$. First, $f_4(p_s/\delta) > 0$ is simplified to

$$\left(\delta > \frac{p_s}{p}\right) \quad \text{or} \quad \left(\frac{p_s}{p + c_p} \leq \delta < \frac{p_s}{p} \text{ and } \alpha > \alpha_E\right). \tag{A.40}$$

Second, $f_4(1 - c_p/(\alpha\pi_u)) < 0$ is simplified to $\alpha > \hat{\alpha}_1$, which completes the proof. □

Using (A.31) and noting that $f_4(c_p\delta/(\pi_u(\delta - p_s))) < 0$, i.e., $c_p\delta/(\pi_u(\delta - p_s)) > \hat{\alpha}_1$, if and only if $p < p_s(1 - c_p)/\delta$, and that $\alpha_E > c_p\delta/(\pi_u(\delta - p_s))$ if and only if $p < p_s(1 - c_p)/\delta$, we then obtain the complete characterization of the corresponding regions for case (VI).

For case (V) in which there is no SaaS, i.e., $0 < v_u < v_p < 1$, we prove the following claim related to the corresponding parameter regions in which case (V) arises in equilibrium:

Claim: The equilibrium that corresponds to case (V) arises if and only if the following conditions are satisfied:

$$p < 1 - c_p \text{ and } \alpha > \alpha_B \text{ and} \left\{\left(\delta \leq \frac{p_s}{p + c_p}\right) \text{ or } \left(\frac{p_s}{p + c_p} < \delta \leq \frac{p_s}{p} \text{ and } \alpha \leq \alpha_E\right)\right\} \text{ and } \left\{\left(\delta \leq \frac{p_s}{p + c_p}\right) \text{ or } \left(\frac{p_s}{p + c_p} < \delta \text{ and } p \leq 1 + p_s - \delta - c_p \text{ and } \alpha \leq \alpha_F\right)\right\}. \tag{A.41}$$

Proof: First, for (OP, P) to exist in equilibrium, the highest type consumer $v = 1$ should prefer to patch over at least (N, ND), i.e., $1 - p - c_p > 0$, or equivalently, $p < 1 - c_p$. Next, from the threshold-type equilibrium structure, in this case, we have $u = v_p - v_u$ and $d = 0$. Furthermore, the thresholds v_p and v_u satisfy the following two equations:

$$\pi_u \alpha (v_p - v_u) v_p = c_p\,, \quad \text{and} \tag{A.42}$$

$$v_u (1 - \pi_u \alpha (v_p - v_u)) = p\,. \tag{A.43}$$

Using (A.42) and $v_p < 1$, we obtain

$$v_u < 1 - \frac{c_p}{\alpha \pi_u} \quad \text{and} \quad v_p = \frac{v_u \alpha \pi_u + \sqrt{\alpha \pi_u \left(\alpha \pi_u v_u^2 + 4c_p\right)}}{2\alpha \pi_u}\,. \tag{A.44}$$

In addition, plugging v_p from (A.44) into (A.43), we simplify (A.43) to

$$f_5(v_u) \triangleq \frac{v_u}{2}\left(2 + \alpha \pi_u v_u - \sqrt{\alpha \pi_u \left(4c_p + \alpha \pi_u v_u^2\right)}\right) - p = 0\,. \tag{A.45}$$

The first term $v_u/2$ is increasing in v_u and the second term is also increasing in v_u since

$$\frac{d}{dv_u}\left(2 + \alpha \pi_u v_u - \sqrt{\alpha \pi_u \left(4c_p + \alpha \pi_u v_u^2\right)}\right) = \alpha \pi_u \left(1 - \sqrt{\frac{\alpha \pi_u v_u^2}{4c_p + \alpha \pi_u v_u^2}}\right) > 0\,. \tag{A.46}$$

Hence, $f_5(v_u)$ is increasing in v_u. Moreover, $f_5(0) = -p < 0$, and thus, the condition $v_u < 1 - c_p/(\alpha \pi_u)$ is equivalent to $f_5(1 - c_p/(\alpha \pi_u)) > 0$, which is simplified to $\alpha > \alpha_B$.

In this case, there is no SaaS consumer segment in equilibrium. Then, first, there should be no SaaS segment at the bottom, i.e., below v_u. This condition is $v_u \le p_s/\delta$, which is equivalent to $f_5(p_s/\delta) \ge 0$. By simplifying it, we obtain

$$2\delta(p_s - p\delta) + \alpha p_s^2 \pi_u \ge 0 \quad \text{and} \quad \delta(p_s - p\delta)^2 + \alpha p_s^2 \pi_u (p_s - (p + c_p)\delta) \ge 0\,. \tag{A.47}$$

The condition (A.47) can be rewritten as

$$\left\{\left(\delta \le \frac{p_s}{p + c_p}\right) \quad \text{or} \quad \left(\frac{p_s}{p + c_p} < \delta \le \frac{p_s}{p} \ \text{ and } \ \alpha \le \alpha_E\right)\right\}. \tag{A.48}$$

Second, there should be no SaaS segment in the middle-tier. In this case, if $p_s/\delta \ge p + c_p$, i.e., the consumer type who is indifferent between ($SaaS$, ND) and (N, ND) is higher than the consumer type who is indifferent between (OP, P) and (N, ND), then there cannot be any SaaS segment in the middle tier in equilibrium. However, if $p_s/\delta < p + c_p$, we need an additional condition ensuring that v_p is higher than the consumer type who is indifferent between (OP, P) and ($SaaS$, ND) without externality, i.e., $v_p \ge (p + c_p - p_s)/(1 - \delta)$. Using v_p in (A.44), this condition becomes

$$v_u \ge v_{u5} \triangleq \frac{\alpha \pi_u (p + c_p - p_s)^2 - c_p (1 - \delta)^2}{\alpha \pi_u (1 - \delta)(p + c_p - p_s)}\,. \tag{A.49}$$

This condition is equivalent to $f_5(v_{u5}) \leq 0$, which is simplified to $p \leq 1 - c_p - (\delta - p_s)$ and $\alpha \leq \alpha_F$. As a result, there should be no SaaS segment in the middle-tier if and only if

$$\left\{ \left(\delta \leq \frac{p_s}{p + c_p} \right) \quad \text{or} \quad \left(\frac{p_s}{p + c_p} < \delta \ \text{ and } \ p \leq 1 + p_s - \delta - c_p \ \text{ and } \ \alpha \leq \alpha_F \right) \right\} , \tag{A.50}$$

which completes the proof. □

Note that (A.41) can be rewritten as

$$p < 1 - c_p \ \text{ and } \ \alpha > \alpha_B \ \text{ and} \left\{ \left(p \leq \frac{p_s}{\delta} - c_p \right) \quad \text{or} \quad \left(\frac{p_s}{\delta} - c_p < p \leq \frac{p_s}{\delta} \ \text{ and } \ \alpha \leq \alpha_E \right) \right\} \ \text{ and}$$
$$\left\{ \left(p \leq \frac{p_s}{\delta} - c_p \right) \quad \text{or} \quad \left(\frac{p_s}{\delta} - c_p < p \leq 1 + p_s - \delta - c_p \ \text{ and } \ \alpha \leq \alpha_F \right) \right\} , \tag{A.51}$$

using $p_s/\delta - c_p < 1 + p_s - \delta - c_p$ from $p_s < \delta$. Moreover, it follows that $\alpha_E < \alpha_F$ if and only if $p_s < p$ in this region. In addition, $p_s/\delta - c_p > p_s$ if and only if $p_s > \delta c_p > (1 - \delta)$. Using these relationship and algebra, and simplifying (A.51), we complete the proof of case (V).

For case (IV) in which there does not exist patching for on-premise purchases and SaaS exists the low tier in equilibrium, i.e., $0 < v_d < v_u < 1$, from the threshold-type equilibrium structure, it follows that $u = 1 - v_u$ and $d = v_u - v_d$. Similarly, in this case, we first prove the following claim:

Claim: The equilibrium that corresponds to case (IV) arises if and only if the following conditions are satisfied:

$$p_s \leq p \ \text{ and} \left\{ \left(p > \frac{p_s}{\delta} \right) \quad \text{or} \quad \left(p \leq \frac{p_s}{\delta} \ \text{ and } \ \alpha > \alpha_A \right) \right\} \ \text{ and}$$
$$\left\{ \left(p \leq 1 + p_s - \delta \right) \quad \text{or} \quad \left(p > 1 + p_s - \delta \ \text{ and } \ \alpha > \alpha_D \right) \right\} \ \text{ and}$$
$$\left\{ \left(p > 1 - c_p \right) \quad \text{or} \quad \left(p \leq 1 - c_p \ \text{ and } \ \alpha \leq \hat{\alpha}_1 \right) \right\} . \tag{A.52}$$

Proof: First, the threshold types v_u and v_d satisfy the following two equations:

$$v_d \delta (1 - \pi_d \alpha (v_u - v_d)) = p_s , \tag{A.53}$$

$$v_u (1 - \pi_u \alpha (1 - v_u) - \delta (1 - \pi_d \alpha (v_u - v_d))) = p - p_s . \tag{A.54}$$

In this case, the consumers with types greater than v_u prefer (OP, NP) over $(SaaS, ND)$, i.e.,

$$v (1 - \pi_u \alpha (1 - v_u) - \delta (1 - \pi_d \alpha (v_u - v_d))) \geq p - p_s , \tag{A.55}$$

for $v \in (v_u, 1]$. Therefore, it follows that $p \geq p_s$ and $1 - \pi_u \alpha (1 - v_u) - \delta (1 - \pi_d \alpha (v_u - v_d)) \geq 0$. Using (A.53) and the condition $v_d < v_u$, we obtain

$$v_u > p_s/\delta \quad \text{and} \quad v_d = \frac{\sqrt{4 p_s \alpha \delta \pi_d + \delta^2 (1 - \alpha \pi_d v_u)^2} - \delta (1 - \alpha \pi_d v_u)}{2 \alpha \delta \pi_d} . \tag{A.56}$$

Substituting v_d from (A.56) into (A.54) and simplifying, we have

$$f_6(v_u) \triangleq \frac{2(p-p_s)}{v_u} - 2 + \delta(1-\alpha\pi_d v_u) + 2\alpha\pi_u(1-v_u) + \sqrt{\delta(4p_s\alpha\pi_d + \delta(1-\alpha\pi_d v_u)^2)} = 0\,. \tag{A.57}$$

By taking derivative with respect to v_u, it follows that

$$\frac{df_6(v_u)}{dv_u} = -\frac{2(p-p_s)}{v_u^2} - 2\alpha\pi_u - \alpha\pi_d\delta\left(1 + \frac{\delta(1-\alpha\pi_d v_u)}{\sqrt{\delta 4p_s\alpha\pi_d + \delta^2(1-\alpha\pi_d v_u)^2}}\right) < 0\,. \tag{A.58}$$

Thus, $f_6(v_u)$ is decreasing in v_u. Furthermore, from (A.56), we need $v_u > p_s/\delta$ and we also need $v_u < 1$. Those two conditions are equivalent to $f_6(p_s/\delta) > 0$ and $f_6(1) < 0$. First, $f_6(p_s/\delta) > 0$ is simplified to $\alpha > \alpha_A$. Note that $\alpha_A < 0$ if and only if $p < p_s/\delta$, in which case all non-negative α values satisfy $f_6(p_s/\delta) > 0$. Second, similarly, $f_6(1) < 0$ is simplified to $\alpha > \alpha_D$. Also, note that $\alpha_D > 0$ if and only if $p > 1 + p_s - \delta$. Otherwise, i.e., $p \leq 1 + p_s - \delta$, all non-negative α values satisfy $f_6(1) < 0$. Finally, there should not be any customers who choose (OP, P); first, if $p > 1 - c_p$, then nobody can afford to patch. Otherwise, i.e., if $p \leq 1 - c_p$, then $v = 1$ customer should not prefer to patch instead of unpatched on-premise. This condition can be written as $1 - p - c_p \leq 1 - \alpha\pi_u(1-v_u) - p$, which is simplified to $v_u \geq 1 - c_p/(\pi_u\alpha)$. Using (A.57) and (A.58), it follows that $v_u \geq 1 - c_p/(\pi_u\alpha)$ is equivalent to $\alpha \leq \hat{\alpha}_1$, which completes the proof. □

Denote $\alpha_N \triangleq c_p/(\pi_u(1-p_s/\delta))$. Then, it follows that $\alpha_A > \alpha_N$ if and only if $p < p_s(1-c_p)/\delta$. In addition, $\alpha_N > \hat{\alpha}_1$ if and only if $p < p_s(1-c_p)/\delta$. As a result, $\alpha_A > \hat{\alpha}_1$ if and only if $p < p_s(1-c_p)/\delta$. Using this relationship and comparing all p-bounds in (A.52) together with α-bounds, we then establish all conditions in case (IV).

For case (III) in which there does not exist patching for on-premise purchases and SaaS consumption is on the high tier in equilibrium, i.e., $0 < v_u < v_d < 1$, from the threshold-type equilibrium structure, it follows that $u = v_d - v_u$ and $d = 1 - v_d$. Similar to case (IV), in this case, we first prove the following claim:

Claim: The equilibrium that corresponds to case (III) arises if and only if the following conditions are satisfied:

$$p_s > p \text{ and} \left\{ \left(p > 1 + p_s - \delta\right) \text{ or } \left(p \leq 1 + p_s - \delta \text{ and } \alpha > \alpha_C\right) \right\} \text{ and}$$
$$\left\{ \left(p > 1 - c_p\right) \text{ or } \left(1 - c_p - (\delta - p_s) < p \leq 1 - c_p \text{ and } \alpha \leq \hat{\alpha}_2\right) \right\}. \tag{A.59}$$

Proof: In this case, the threshold types v_u and v_d satisfy the following two equations:

$$v_d(\delta(1-\alpha\pi_d(1-v_d)) - (1-\alpha\pi_u(v_d-v_u))) = p_s - p\,, \tag{A.60}$$

$$v_u(1-\pi_u\alpha(v_d-v_u)) = p\,. \tag{A.61}$$

First, the consumers with types greater than v_d prefer ($SaaS$, ND) over (OP, NP), i.e.,

$$v(\delta(1-\alpha\pi_d(1-v_d))-(1-\alpha\pi_u(v_d-v_u)))\geq p_s-p\,, \tag{A.62}$$

for $v\in(v_d,1]$. Therefore, it follows that $p_s\geq p$ and $\delta(1-\alpha\pi_d(1-v_d))-(1-\alpha\pi_u(v_d-v_u))>0$. The second condition is equivalent to $v_d>0$. By (A.61) and the condition $v_u<v_d$, it follows that

$$v_d>p \quad \text{and} \quad v_u=\frac{\sqrt{4p_s\alpha\pi_u+(1-\alpha\pi_u v_d)^2}-(1-\alpha\pi_u v_d)}{2\alpha\pi_u}\,. \tag{A.63}$$

Substituting v_u from (A.63) into (A.60) and simplifying, we obtain

$$\begin{aligned}f_7(v_d)\triangleq\frac{2(p_s-p)}{v_d}&+1-2\delta+2\alpha\delta\pi_d(1-v_d)-\alpha\pi_u v_d\\&+\sqrt{4p_s\alpha\pi_u+(1-\alpha\pi_u v_d)^2}=0\,.\end{aligned} \tag{A.64}$$

Similar to case (IV), by taking derivative with respect to v_d, we have

$$\frac{df_7(v_d)}{dv_d}=-\frac{2(p_s-p)}{v_d^2}-2\alpha\delta\pi_d-\alpha\pi_u\left(1+\frac{1-\alpha\pi_u v_d}{\sqrt{4p_s\alpha\pi_u+(1-\alpha\pi_u v_d)^2}}\right)<0\,. \tag{A.65}$$

Thus, $f_7(v_d)$ is decreasing in v_d. In addition, from (A.63), we need $v_d>p$ and we also need $v_d<1$. Those two conditions are equivalent to $f_7(p)>0$ and $f_7(1)<0$. First, $f_7(p)>0$ holds always since $p_s>p$. Second, $f_7(1)<0$ is simplified to $\alpha>\alpha_C$. Also, note that $\alpha_C\geq 0$ if and only if $p\leq 1+p_s-\delta$. Otherwise, i.e., if $p>1+p_s-\delta$, then all non-negative α values satisfy $f_7(1)<0$. Finally, there should not be any customers who choose (OP, P); first, if $p>1-c_p$, then nobody can afford to patch. Otherwise, i.e., if $p\leq 1-c_p$, then $v=1$ customer should not prefer to patch instead of SaaS. This condition can be written as $1-p-c_p\leq\delta-\alpha\pi_d(1-v_d)\delta-p_s$, which is simplified to $v_d>1-(p+c_p+\delta-1-p_s)/(\pi_d\alpha\delta)$. Using (A.64) and (A.65), it follows that $v_d>1-(p+c_p+\delta-1-p_s)/(\pi_d\alpha\delta)$ becomes equivalent to $\alpha\leq\hat{\alpha}_2$ and $p>1-c_p-(\delta-p_s)$, which completes the proof. □

By comparing all p-bounds in (A.59) together with α-bounds, we then establish all conditions in case (III).

Next, for case (II) in which there exists only SaaS consumption in equilibrium, i.e., $0<v_d<1$, from the threshold-type equilibrium structure, it follows that $u=0$ and $d=1-v_d$. We prove the following claim, which then proves the equilibrium characterization of case (II):

Claim: The equilibrium that corresponds to case (II) arises if and only if the following conditions are satisfied:

$$p>p_s+1-\delta \quad \text{and} \quad \alpha\leq\alpha_D\,. \tag{A.66}$$

Proof: The threshold v_d satisfies the following equation:

$$f_8(v_d) \triangleq v_d\delta(1-\alpha\pi_d(1-v_d)) - p_s\,. \tag{A.67}$$

Since $f_8(v_d)$ is quadratic in v_d, $f_8(0) = -p_s < 0$ and $f_8(1) = \delta - p_s > 0$, there exists the unique $v_d \in (0,\,1)$ that satisfies (A.67). We need to guarantee that no consumer chooses $(OP,\,NP)$ over $(SaaS,\,ND)$. Note that in this case, if the highest type consumer does not choose $(OP,\,NP)$ without security risk over $(SaaS,\,ND)$, then no consumer will choose $(OP,\,NP)$. Hence the condition becomes $\delta(1-\alpha\pi_d(1-v_d)) \geq 1-p$, which is simplified to

$$v_d \geq v_{d2} \triangleq 1 + \frac{1-p+p_s-\delta}{\alpha\delta\pi_d}\,. \tag{A.68}$$

This condition is equivalent to $f_8(v_{d2}) \leq 0$, which is simplified to $\alpha \leq \alpha_D$. Moreover, note that $\alpha_D > 0$ if and only if $p > p_s + 1 - \delta$, which completes the proof. $\square$

Finally, for case (I) in which there exists only on-premise consumption without patching in equilibrium, i.e., $0 < v_u < 1$, from the threshold-type equilibrium structure, we have $u = 1 - v_u$ and $d = 0$. First, we prove the claim:

Claim: The equilibrium that corresponds to case (I) arises, if and only if the following conditions are satisfied.

$$\Big\{\Big(p \leq p_s\Big) \quad \text{or} \quad \Big(p_s < p \leq \frac{p_s}{\delta} \quad \text{and} \quad \alpha \leq \alpha_A\Big)\Big\} \quad \text{and}$$
$$\Big\{\Big(p > 1-c_p\Big) \quad \text{or} \quad \Big(p \leq 1-c_p \quad \text{and} \quad \alpha \leq \alpha_B\Big)\Big\} \quad \text{and}$$
$$\Big\{\Big(p > p_s\Big) \quad \text{or} \quad \Big(p \leq p_s \quad \text{and} \quad \alpha \leq \alpha_C\Big)\Big\}\,. \tag{A.69}$$

Proof: The threshold v_u satisfies the following equation:

$$f_9(v_u) \triangleq v_u(1-\alpha\pi_u(1-v_u)) - p\,. \tag{A.70}$$

Since $f_9(v_u)$ is quadratic in v_u, $f_9(0) = -p < 0$ and $f_9(1) = 1-p > 0$, there exists the unique $v_u \in (0,\,1)$ that satisfies (A.70). Similar to the previous cases, first, we need to guarantee that SaaS consumption does not come below v_u, which is achieved if either $p \leq p_s$, or $p > p_s$ and $v_u \leq p_s/\delta$. The condition $v_u \leq p_s/\delta$ is equivalent to $f_9(p_s/\delta) \geq 0$, which is simplified to $\alpha \leq \alpha_A$. Moreover, $\alpha_A \geq 0$ if and only if $p \leq p_s/\delta$. Second, we need to guarantee that nobody wants to patch, which is achieved either if $p > 1-c_p$, i.e., nobody is afford to patch, or if $p \leq 1-c_p$ and $1-p-\alpha\pi_u(1-v_u) \geq 1-p-c_p$, i.e., even the highest-type ($v=1$) consumer prefers $(OP,\,NP)$ over $(OP,\,P)$. The condition $1-p-\alpha\pi_u(1-v_u) \geq 1-p-c_p$ can be rewritten as $v_u \geq 1-\frac{c_p}{\alpha\pi_u}$, which is equivalent to $f_9(1-\frac{c_p}{\alpha\pi_u}) \leq 0$; consequently, it is simplified to $\alpha \leq \alpha_B$. In addition, we also need to guarantee that SaaS consumption does not come in above v_u. If $p > p_s$, then SaaS consumption cannot arise above v_u in equilibrium. Otherwise, i.e., if $p \leq p_s$, the highest type consumer ($v=1$) should prefer $(OP,\,NP)$ over $(SaaS,\,ND)$, which can be written as $1-p-\alpha\pi_u(1-v_u) \geq \delta - p_s$. This condition can be rewritten as $v_u \geq 1-(1-p+p_s-\delta)/(\alpha\pi_u)$, which is equivalent to $f_9(1-(1-p+p_s-\delta)/(\alpha\pi_u)) \le 0$; by simplifying, this inequality becomes $\alpha \le \alpha_C$, which completes the proof. □

Note that $\alpha_B \le \alpha_C$ if and only if $p \le 1-c_p-\delta+p_s$. In addition, $\alpha_A \le \alpha_B$ if and only if $p_s(1-c_p)/\delta \le p$. Using these relationships and (A.69), we can then completely characterize the equilibrium conditions given above for case (I). ■

Proof of Corollary 1: From Lemma 1, we obtain (No SaaS) equilibrium outcome, i.e., case (V), when $p \le p_s\delta - c_p$ and $\alpha \ge \alpha_B$. Note that $p_s\delta - c_p > 0$ if and only if $p_s > \delta c_p$. Second, again, from Lemma 1, (SaaS for low tier) equilibrium, i.e., case (VI), arises when $\max(p_s, p_s/\delta - c_p) < p \le p_s(1-c_p)/\delta$ and $\alpha > \alpha_E$, or $\max(p_s(1-c_p)/\delta, p_s) < p < 1-c_p$ and $\alpha \ge \hat{\alpha}_1$. These conditions can then be simplified to $\max(p_s, p_s/\delta - c_p) < p < 1-c_p$ and $\alpha \ge \max(\alpha_E,\, \hat{\alpha}_1)$. Finally, from Lemma 1, it follows that (SaaS for middle tier) equilibrium, i.e., case (VII), occurs when $p_s/\delta - c_p < p \le \min(p_s,\, 1+p_s-\delta-c_p)$ and $\alpha \ge \alpha_F$, or $\min(p_s,\, 1+p_s-\delta-c_p) < p \le \min(p_s,\, 1-c_p)$ and $\alpha \ge \hat{\alpha}_2$. These conditions are then simplified to $p_s < \delta c_p/(1-\delta)$, $p_s/\delta - c_p < p \le \min(p_s,\, 1-c_p)$, and $\alpha \ge \max(\alpha_F,\, \hat{\alpha}_2)$, which completes the proof. ■

Proof of Proposition 1: Technically, we will prove that there exists $\overline{\alpha} > 0$ such that when $\alpha > \overline{\alpha}$, p^* and p_s^* are set so that

(i) If $c_p > 1/3$ and $\delta > \frac{2(1-c_p)}{1+c_p}$, then $\sigma^*(v)$ is characterized by $1 > v_p > v_d > v_u > 0$ and given in (7) ;

(ii) Otherwise, $\sigma^*(v)$ is characterized by $1 > v_p > v_u > v_d > 0$ and given in (6) .

By Lemma 1 and Corollary 1, for sufficiently high α, either $v_p > v_u > v_d$ or $v_p > v_d > v_u$ is satisfied under optimal pricing in equilibrium. Suppose $v_p > v_d > v_u$. By (A.19), we obtain

$$v_d = p + c_p + \frac{pc_p\delta\pi_d - \pi_u(p+c_p-p_s)(\delta(p+c_p)-p_s)}{\alpha\delta\pi_d\pi_u(p+c_p)^2} + O\left(\frac{1}{\alpha^2}\right). \tag{A.71}$$

Substituting (A.71) into (A.18) and (A.16), we have

$$v_u = p + c_p - \frac{c_p^2\delta\pi_d + \pi_u(p+c_p-p_s)(\delta(p+c_p)-p_s)}{\alpha\delta\pi_d\pi_u(p+c_p)^2} + O\left(\frac{1}{\alpha^2}\right), \tag{A.72}$$

and

$$v_p = p + c_p + \frac{pc_p\delta\pi_d + \pi_u p_s(\delta(p+c_p)-p_s)}{\alpha\delta\pi_d\pi_u(p+c_p)^2} + O\left(\frac{1}{\alpha^2}\right). \tag{A.73}$$

By substituting (A.71), (A.72), and (A.73) into (15) which applies in this case, it follows that

$$\Pi(p, p_s) = p(1-p-c_p) + \frac{pc_p^2\delta\pi_d + \pi_u c_p p_s(\delta(p+c_p)-p_s)}{\alpha\delta\pi_d\pi_u(p+c_p)^2} + O\left(\frac{1}{\alpha^2}\right). \tag{A.74}$$

By (9) and (A.74), the interior maximizing prices satisfy

$$p^M = \frac{1-c_p}{2} + \frac{2c_p^2(3c_p-1)}{\pi_u\alpha(1+c_p)^3} + O\left(\frac{1}{\alpha^2}\right), \tag{A.75}$$

and

$$p_s^M = \frac{\delta(1+c_p)}{4} + \frac{A_1}{16c_p(1+c_p)^3\pi_d\pi_u\alpha} + O\left(\frac{1}{\alpha^2}\right), \tag{A.76}$$

where $A_1 \triangleq 8c_p\pi_d(3 - c_p^2(3+2\delta) + c_p(2\delta-5) + c_p^3(5+4\delta)) + \delta\pi_u(1+c_p)^3(1+c_p(2\delta-3))$. By substituting (A.75) and (A.76) into (A.74), the vendor's profits are given by

$$\Pi^M = \Pi(p^M, p_s^M) = \frac{(1-c_p)^2}{4} + \frac{c_p}{4\alpha}\left(\frac{\delta}{\pi_d} + \frac{8c_p(1-c_p)}{\pi_u(1+c_p)^2}\right) + \frac{64c_p^3\pi_d^2A_2 + (1+c_p)^3(\delta-2)\pi_uA_3}{16(1+c_p)^6\pi_d^2\pi_u^2\alpha^2} + O\left(\frac{1}{\alpha^3}\right). \tag{A.77}$$

where $A_2 \triangleq -4 + 5c_p + 5c_p^3 - 2c_p^2$ and $A_3 \triangleq 8c_p\pi_d(3-c_p)(1-c_p) + \delta\pi_u(1+c_p)^3$.

On the other hand, suppose $v_p > v_u > v_d$. Following a similar analysis to the one above and using (18), the interior maximizing prices satisfy

$$p^L = \frac{1-c_p}{2} + \frac{2c_p^2(3c_p-1)}{\pi_u\alpha(1+c_p)^3} + O\left(\frac{1}{\alpha^2}\right), \tag{A.78}$$

and

$$p_s^L = \frac{\delta(1+c_p)}{4} + \left(\frac{\delta(1+c_p(2\delta-3))}{16c_p\pi_d\alpha} - \frac{c_p^2\delta(3-c_p)}{(1+c_p)^3\pi_u\alpha}\right) + O\left(\frac{1}{\alpha^2}\right). \tag{A.79}$$

Substituting (A.78) and (A.79) into (18), we obtain

$$\Pi^L = \Pi(p^L, p_s^L) = \frac{(1-c_p)^2}{4} + \frac{c_p}{4\alpha}\left(\frac{\delta}{\pi_d} + \frac{8c_p(1-c_p)}{\pi_u(1+c_p)^2}\right) + \frac{A_4}{\pi_u\alpha^2(1+c_p)^3} + O\left(\frac{1}{\alpha^3}\right). \tag{A.80}$$

where

$$A_4 \triangleq \frac{\delta(\pi_u(\delta-2)(1+c_p)^3 + 16c_p^2\pi_d(c_p-3))}{16\pi_d^2} + \frac{4c_p^3A_2}{(1+c_p)^3\pi_u}. \tag{A.81}$$

Comparing (A.77) and (A.80), it follows that $\Pi^M > \Pi^L$ if and only if $(64c_p^3\pi_d^2A_2 + (1+c_p)^3(\delta-2)\pi_uA_3)\pi_u\alpha^2(1+c_p)^3 > 16(1+c_p)^6\pi_d^2\pi_u^2\alpha^2A_4$, which is satisfied if and only if $c_p > 1/3$ and $\delta(1+c_p) > 2(1-c_p)$. This completes the proof. ■

Proof of Proposition 2: By Proposition 1, when $c_p > 1/3$ and $\delta > \frac{2(1-c_p)}{1+c_p}$ are satisfied, then $p^* = p^M$ and $p_s^* = p_s^M$. Substituting (A.75) and (A.76) into (A.71), (A.72), and (A.73) and then subsequently into (17), we obtain

$$W^M \triangleq W = \frac{3(1-c_p)^2}{8} + \frac{1}{\alpha}\left(\frac{c_p\delta}{4\pi_d} + \frac{c_p^2(1-c_p)(3-c_p)}{\pi_u(1+c_p)^3}\right) + \frac{192(3-c_p)c_p^3\pi_d^2A_5 - \pi_u(2-\delta)(1+c_p)^3A_6}{32\pi_u^2\pi_d^2(1+c_p)^7\alpha^2} + O\left(\frac{1}{\alpha^3}\right). \tag{A.82}$$

where $A_5 \triangleq 5c_p^3 - 4c_p^2 + 5c_p - 2$ and $A_6 \triangleq 8\pi_d c_p(1-c_p)(7+3c_p^2-14c_p) + \delta\pi_u(1+c_p)^4$.

On the other hand, if it is not the case that both are $c_p > 1/3$ and $\delta > \frac{2(1-c_p)}{1+c_p}$ are satisfied, then, by Proposition 1, $p^* = p^L$ and $p_s^* = p_s^L$. Substituting (A.78) and (A.79) into (A.35), (A.36) and (A.38) and then subsequently into (17), we obtain

$$W^L \triangleq W = \frac{3(1-c_p)^2}{8} + \frac{1}{\alpha}\left(\frac{c_p\delta}{4\pi_d} + \frac{c_p^2(1-c_p)(3-c_p)}{\pi_u(1+c_p)^3}\right) + \frac{1}{\alpha^2}\left(\frac{\delta A_7}{32\pi_u\pi_d^2(1+c_p)^4} + \frac{6c_p^3(3-c_p)A_5}{\pi_u^2(1+c_p)^7}\right) + O\left(\frac{1}{\alpha^3}\right). \tag{A.83}$$

where $A_7 \triangleq \pi_u(1+c_p)^4(\delta-2) - 32c_p^2\pi_d(5-c_p)(1-c_p)$. However, comparing (A.82) and (A.83), it follows that $W^M > W^L$ if and only if $\delta > \frac{2(7-14c_p+3c_p^2)}{(7-c_p)(1+c_p)}$. For $c_p \in (0,1]$, note that $\frac{2(7-14c_p+3c_p^2)}{(7-c_p)(1+c_p)} < \frac{2(1-c_p)}{1+c_p}$ is always satisfied and $\frac{2(7-14c_p+3c_p^2)}{(7-c_p)(1+c_p)} < 1$ is satisfied whenever $c_p > (17-4\sqrt{15})/7$.

By Region III of Corollary 1, (A.75), and (A.76), $p^M \geq p_s^M/\delta - c_p$ is always satisfied and $p_s^M < \delta c_p/(1-\delta)$ is always satisfied for sufficiently high δ. Also, by (A.75) and (A.76), $p^M < p_s^M$ is satisfied if and only if

$$\frac{\delta + c_p(2+\delta) - 2}{4} + \frac{\delta(1+c_p(2\delta-3))}{16c_p\pi_d\alpha} + \frac{3 + c_p(2\delta + c_p - 5 - 2c_p\delta + c_p^2(4\delta-7))}{2(1+c_p)^3\pi_u\alpha} + O\left(\frac{1}{\alpha^2}\right) > 0. \tag{A.84}$$

Hence, there exist $\underline{c_p}$, $\underline{\eta} > 0$ such that if $\underline{c_p} < c_p < 1/3$ and $\underline{\eta} < \delta < 1$, then, by Proposition 1, $p^* = p^L$ and $p_s^* = p_s^L$, but (A.84) is satisfied; hence, p^M and p_s^M can induce Region III of Corollary 1, which completes the proof. ■

Proof of Lemma 2: Note that the benchmark case corresponds to $\delta = 0$ case. From Corollary 1, it then corresponds to (No SaaS) equilibrium outcome, i.e., $0 < v_u < v_p < 1$. Thus, for the benchmark case, under high α, v_u is given by the largest root of the polynomial equation

$$\pi_u\alpha v_u^3 + (1 - \pi_u\alpha(p+c_p))v_u^2 - 2pv_u + p^2 = 0. \tag{A.85}$$

By (A.85), for sufficiently large α, v_u satisfies

$$v_u = p + c_p - \frac{c_p^2}{\pi_u\alpha(p+c_p)^2} + \frac{2pc_p^3}{\pi_u^2\alpha^2(p+c_p)^5} + O\left(\frac{1}{\alpha^3}\right). \tag{A.86}$$

Maximizing $p(1-v_u)$, we obtain

$$p_{BM} = \frac{1-c_p}{2} + \frac{2c_p^2(3c_p-1)}{\pi_u\alpha(1+c_p)^3} - \frac{16c_p^3(8c_p^3 - 5c_p^2 + 8c_p - 3)}{\pi_u^2\alpha^2(1+c_p)^7} + O\left(\frac{1}{\alpha^3}\right). \tag{A.87}$$

Using (A.87) and similar analysis for the benchmark case, the threshold v_p, the respective optimal profit and welfare expressions are given by In addition, in this case, v_p satisfies

$$v_p = \frac{1+c_p}{2} + \frac{2c_p(1+c_p^2+2c_p^3)}{(1+c_p)^4\pi_u\alpha} - \frac{8c_p^2(1-2c_p)(1-7c_p-3c_p^2+5c_p^3)}{(1+c_p)^7\pi_u^2\alpha^2} + O\left(\frac{1}{\alpha^3}\right), \tag{A.88}$$

$$\Pi_{BM} = \frac{(1-c_p)^2}{4} + \frac{2c_p^2(1-c_p)}{\pi_u\alpha(1+c_p)^2} + \frac{4c_p^3(5c_p^3-2c_p^2+5c_p-4)}{\pi_u^2\alpha^2(1+c_p)^6} + O\left(\frac{1}{\alpha^3}\right) \tag{A.89}$$

and

$$W_{BM} = \frac{3(1-c_p)^2}{8} + \frac{c_p^2(3-c_p)(1-c_p)}{\pi_u\alpha(1+c_p)^3} - \frac{6c_p^3(5c_p^4-19c_p^3+17c_p^2-17c_p+6)}{\pi_u^2\alpha^2(1+c_p)^7} + O\left(\frac{1}{\alpha^3}\right). \tag{A.90}$$

Furthermore, using (A.86), (A.87) and (A.88), we also obtain the average per-user security losses and the consumer surplus as follows:

$$\hat{SL}_{BM} = c_p - \frac{8c_p^3}{(1-c_p)(1+c_p)^3\pi_u^2\alpha^2} + O\left(\frac{1}{\alpha^{5/2}}\right). \tag{A.91}$$

and

$$CS_{BM} = \frac{(1-c_p)^2}{8} + \frac{c_p^2(1-c_p)(1-3c_p)}{\pi_u\alpha(1+c_p)^3} - \frac{2c_p^3(25c_p^4-51c_p^3+57c_p^2-49c_p+10)}{\pi_u^2\alpha^2(1+c_p)^7} + O\left(\frac{1}{\alpha^3}\right). \tag{A.92}$$

Note that these benchmark measures and the benchmark equilibrium outcome are consistent with the benchmark equilibrium characterization in August and Tunca (2006) (see Lemma 1). ■

Proof of Proposition 3: By the proof of Proposition 1, (A.77), (A.80), and (A.89), it follows that

$$\frac{\Pi^* - \Pi_{BM}}{\Pi_{BM}} = \frac{c_p\delta}{\pi_d\alpha(1-c_p)^2} + O\left(\frac{1}{\alpha^2}\right). \tag{A.93}$$

Similarly, by the proof of Proposition 2, (A.82), (A.83), and (A.90), we obtain

$$\frac{W^* - W_{BM}}{W_{BM}} = \frac{2c_p\delta}{3\pi_d\alpha(1-c_p)^2} + O\left(\frac{1}{\alpha^2}\right), \tag{A.94}$$

which completes the proof. ■

Proof of Proposition 4: For part (i), by Proposition 1, if $c_p > 1/3$ and $\delta > \frac{2(1-c_p)}{1+c_p}$ are satisfied, then by substituting (A.71), (A.72), and (A.73) into (16) and subsequently into (23), we obtain

$$\hat{SL}^* = c_p + \frac{\delta - c_p(4-\delta)}{4\pi_d(1-c_p)\alpha} + O\left(\frac{1}{\alpha^2}\right). \tag{A.95}$$

If either $c_p > 1/3$ or $\delta > \frac{2(1-c_p)}{1+c_p}$ is not satisfied, then making analogous substitutions into (19) and (23), it follows that in this case $\hat{SL}^*$ also satisfies (A.95). On the other hand, using a similar train of logic for the benchmark case, we obtain (A.91). By (A.95) and (A.91), it follows that

$$\frac{\hat{SL}^* - \hat{SL}_{BM}}{\hat{SL}_{BM}} = \frac{\delta - c_p(4-\delta)}{4c_p\pi_d(1-c_p)\alpha} + O\left(\frac{1}{\alpha^2}\right), \tag{A.96}$$

which proves the result for part (i).

Next, for part (ii), because $c_p < 1/3$, part (ii) of Proposition 1 applies, hence by (18) and (20), consumer surplus is given by $CS = W - \Pi$. Following the proofs of Propositions 1 and 2, by (A.80) and (A.83), we obtain

$$\begin{aligned} CS^* = &\frac{(1-c_p)^2}{8} + \frac{c_p^2(1-c_p)(1-3c_p)}{\pi_u\alpha(1+c_p)^3} + \frac{\delta}{32\pi_d^2\alpha^2}\left(2 - \delta - \frac{64c_p^2\pi_d(1-c_p(4-c_p))}{\pi_u(1+c_p)^4}\right) \\ &- \frac{2c_p^3(25c_p^4 - 51c_p^3 + 57c_p^2 - 49c_p + 10)}{\pi_u^2\alpha^2(1+c_p)^7} + O\left(\frac{1}{\alpha^3}\right). \end{aligned} \tag{A.97}$$

Similarly, by the proof of Proposition 3, (A.89), and (A.90), we obtain (A.92). Comparing (A.97) and (A.92), it follows that

$$CS^* - CS_{BM} = \frac{\delta}{32\pi_d^2\alpha^2}\left(2 - \delta - \frac{64c_p^2\pi_d(1-c_p(4-c_p))}{\pi_u(1+c_p)^4}\right) + O\left(\frac{1}{\alpha^3}\right), \tag{A.98}$$

which proves the result for part (ii). ■

Proof of Corollary 2: From Lemma 1, by focusing on low α cases, i.e., cases (I), (II) and (III), and by simplifying the conditions using algebra, we obtain the presented results. ■

Proof of Proposition 5: By (1) and (2), for sufficiently small α, $c_p > \pi_u u(\sigma)\alpha v$ is satisfied for all $v \in \mathcal{V}$. Hence, $\sigma^*(v) \neq (OP, P)$ for all $v \in \mathcal{V}$, i.e., $v_p = 1$. Because $\delta < 1$ and, by (1), the security risk facing users is $O(\alpha)$, it is simple to establish that the only possible equilibrium consumer market structure characterization under optimal pricing is either $0 < v_d < v_u < 1$ or $0 < v'_u < 1$. Note that when the SaaS price is set to $p_s = \delta v'_u$, the former consumer market structure replicates the latter. Thus, we can focus attention on $0 < v_d < v_u < 1$ and examine the pricing problem in (9).

Using analysis similar to that used in the proof of Lemma 1, the equilibrium equations are given by

$$v_u = v_d + \frac{\delta v_d - p_s}{\delta v_d \pi_d \alpha}, \tag{A.99}$$

and

$$\begin{aligned} v_d = \sup\Big\{v_d \,\Big|\, &\delta\pi_d\left(p_s^2 - p_s v_d(1+\delta) + \delta v_d^2(1 + \pi_d\alpha(v_d - p))\right) + \\ &\pi_u\left(p_s - \delta v_d(1 + v_d\pi_d\alpha)\right)\left(p_s + \delta v_d(\pi_d\alpha(1-v_d) - 1)\right) = 0\Big\}. \end{aligned} \tag{A.100}$$

By (A.100), for sufficiently small α, v_d satisfies

$$v_d = \frac{p_s}{\delta} - \frac{\pi_d \alpha p_s (p_s - p\delta)}{\delta^2(1-\delta)} + O(\alpha^2)\,. \tag{A.101}$$

Substituting (A.101) into (A.99) and both expressions subsequently into (8), we obtain

$$\begin{aligned}\Pi(p, p_s) = &\frac{2pp_s\delta + p\delta(1-p-\delta) - p_s^2}{\delta(1-\delta)} - \frac{\pi_d \alpha (p_s - p\delta)^2 (2p_s\delta - p\delta - p_s)}{\delta^2(1-\delta)^3} - \\ &\frac{\delta^2 \pi_u \alpha (p - p_s)^2 (1 + p_s - p - \delta)}{\delta^2(1-\delta)^3} + O(\alpha^2)\,.\end{aligned} \tag{A.102}$$

Differentiating (A.102), the interior-maximizing prices satisfy

$$p^* = \frac{1}{2} - \frac{\pi_u \alpha}{8} + \frac{\pi_u^2 \alpha^2}{16(1-\delta)} + O(\alpha^3)\,, \tag{A.103}$$

and

$$p_s^* = \frac{\delta}{2} - \frac{\delta \pi_d \pi_u \alpha^2}{16(1-\delta)} + O(\alpha^3)\,. \tag{A.104}$$

Substituting (A.103) and (A.104) into (A.99) and (A.100) verifies $0 < v_d < v_u < 1$ under optimal pricing. Substituting (A.103) and (A.104) into (A.102) gives the corresponding profits

$$\Pi(p^*, p_s^*) = \frac{1}{4} - \frac{\pi_u \alpha}{8} + \frac{\pi_u^2 \alpha^2}{64(1-\delta)} + \frac{\pi_u (4\pi_u \pi_d (1-2\delta) - 3\pi_u^2)\alpha^3}{128(1-\delta)^2} + O(\alpha^4)\,, \tag{A.105}$$

which, by feasibility of $p_s = \delta v_u'$, exceed those obtainable under the $0 < v_u' < 1$ consumer market structure characterization. This completes the proof. ■

Proof of Lemma 3: For the benchmark measures for the low α region, similar to the proof of Lemma 2, the benchmark case corresponds to the case of $\delta = 0$, i.e., Region (i) in Corollary 2. In the benchmark case, under low α, v_u is given by

$$v_u = -\frac{1 - \pi_u \alpha}{2\pi_u \alpha} + \frac{1}{2\pi_u \alpha}\sqrt{(1 - \pi_u \alpha)^2 + 4\pi_u \alpha p}. \tag{A.106}$$

By (A.106), for sufficiently low α, v_u satisfies

$$v_u = p + \pi_u p(1-p)\alpha + \pi_u^2 p(1-p)(1-2p)\alpha^2 + O\left(\alpha^3\right)\,. \tag{A.107}$$

Maximizing $p(1 - v_u)$, we obtain

$$p_{BM} = \frac{1}{2} - \frac{\pi_u \alpha}{8} + \frac{\pi_u^2 \alpha^2}{16} + O\left(\alpha^3\right)\,. \tag{A.108}$$

Using (A.108), the respective optimal profit and welfare expressions are given by

$$\Pi_{BM} = \frac{1}{4} - \frac{\pi_u \alpha}{8} + \frac{\pi_u^2 \alpha^2}{64} + O\left(\alpha^3\right)\,. \tag{A.109}$$

and

$$W_{BM} = \frac{3}{8} - \frac{\pi_u \alpha}{4} + \frac{5\pi_u^2 \alpha^2}{128} + O\left(\alpha^3\right) . \tag{A.110}$$

Moreover, similar to the proof of Lemma 2, using (A.86), (A.87) and (A.88), we obtain the average per-user security losses and the consumer surplus:

$$\hat{SL}_{BM} = \frac{3\pi_u \alpha}{8} - \frac{\pi_u^2 \alpha^2}{16} + O\left(\alpha^3\right) , \tag{A.111}$$

and

$$CS_{BM} = \frac{1}{8} - \frac{\pi_u \alpha}{8} + \frac{3\pi_u^2 \alpha^2}{128} + O\left(\alpha^3\right) . \tag{A.112}$$

Finally, note that this case is consistent with August and Tunca (2006). ■

Proof of Proposition 6: By the proof of Proposition 5, (A.105), and (A.109), it follows that

$$\frac{\Pi^* - \Pi_{BM}}{\Pi_{BM}} = \frac{\delta \pi_u^2 \alpha^2}{16(1-\delta)} + O\left(\alpha^3\right) . \tag{A.113}$$

By (11), (A.99), (A.103), and (A.104), we obtain

$$\frac{W^* - W_{BM}}{W_{BM}} = \frac{5\delta \pi_u^2 \alpha^2}{48(1-\delta)} + O\left(\alpha^3\right) . \tag{A.114}$$

By (A.113) and (A.114), the result follows. ■

Proof of Proposition 7: Plugging (A.103) and (A.103) into $\hat{SL}^*$ under low α region, we obtain

$$\hat{SL}^* = \frac{3\pi_u \alpha}{8} - \frac{5\pi_u^2 \alpha^2}{32(1-\delta)} + O\left(\alpha^3\right) , \tag{A.115}$$

From (A.115) and (A.111), it then follows that

$$\frac{\hat{SL}^* - \hat{SL}_{BM}}{\hat{SL}_{BM}} = -\frac{(3+2\delta)\pi_u \alpha}{12(1-\delta)} + O\left(\alpha^2\right) , \tag{A.116}$$

which is negative when α is small. Next, substituting (A.103) and (A.103) into W^* and Π^* under low α region, and subtracting Π^* from W^*, we obtain

$$CS^* = \frac{1}{8} - \frac{\pi_u \alpha}{8} + \frac{3\pi_u^2 \alpha^2}{128(1-\delta)} + O\left(\alpha^3\right) , \tag{A.117}$$

Then, using (A.117) and (A.112), we have

$$CS^* - CS_{BM} = \frac{3\delta \pi_u^2 \alpha^2}{128(1-\delta)} + O\left(\alpha^3\right) , \tag{A.118}$$

which is positive when α is small, which completes the proof. ■

Proof of Proposition 8: Under low security-loss environment, i.e., under a sufficiently small α, using (A.105), we obtain the profits as

$$\Pi(\epsilon_u,\epsilon_d,\alpha)=\frac{1}{4}-\frac{\pi_{u0}(1-\epsilon_u)\alpha}{8}+\frac{\pi_{u0}^2(1-\epsilon_u)^2\alpha^2}{64(1-\delta)}-C_u(\epsilon_u)-C_d(\epsilon_d)$$
$$-\frac{\pi_{u0}(1-\epsilon_u)(4\pi_{u0}(1-\epsilon_u)\pi_{d0}(1-\epsilon_d)(2\delta-1)-4\pi_{d0}^2(1-\epsilon_d)^2+3\pi_{u0}^2(1-\epsilon_u)^2)\alpha^3}{128(1-\delta)^2}+O(\alpha^4)\,. \tag{A.119}$$

By taking a derivative of (A.119) with respect to ϵ_u, we obtain the first-order condition with respect to ϵ_u as follows:

$$\Pi_1\triangleq\frac{d\Pi}{d\epsilon_u}=\frac{\pi_{u0}\alpha}{8}-\frac{\pi_{u0}^2(1-\epsilon_u)\alpha^2}{32(1-\delta)}-C_u'(\epsilon_u)$$
$$+\frac{\pi_{u0}(9(1-\epsilon_u)^2\pi_{u0}^2+8(2\delta-1)(1-\epsilon_d)\pi_{d0}(1-\epsilon_u)\pi_{u0}-4(1-\epsilon_d)^2\pi_{d0}^2)\alpha^3}{128(1-\delta)^2}+O(\alpha^3)=0\,. \tag{A.120}$$

Similarly, by taking a derivative of (A.119) with respect to ϵ_d, we obtain the first-order condition with respect to ϵ_d as follows:

$$\Pi_2\triangleq\frac{d\Pi}{d\epsilon_d}=\frac{\pi_{d0}\pi_{u0}(1-\epsilon_u)((1-\epsilon_u)\pi_{u0}(2\delta-1)-2\pi_{d0}(1-\epsilon_d))\alpha^3}{32(1-\delta)^2}-C_d'(\epsilon_d)+O(\alpha^4)=0\,. \tag{A.121}$$

From (A.121), under small α, if $(1-\epsilon_u)\pi_{u0}(2\delta-1)-2\pi_{d0}(1-\epsilon_d)<0$, then Π_2 is negative, and hence $\epsilon_d^*=0$, in which case ϵ_d^* is a constant with respect to α. Next, we focus on the region in which $(1-\epsilon_u)\pi_{u0}(2\delta-1)-2\pi_{d0}(1-\epsilon_d)\geq 0$. Let $\Pi_{11}\triangleq d^2\Pi/d\epsilon_u^2$, $\Pi_{22}\triangleq d^2\Pi/d\epsilon_d^2$, and $\Pi_{12}\triangleq d^2\Pi/d\epsilon_u d\epsilon_d$. Then, using $C_u''>\tau$ and $C_d''>\tau$, we obtain that $\Pi_{11}<0$ and $\Pi_{22}<0$ under small α. Furthermore, we also have

$$\Pi_{12}=\frac{\pi_{d0}\pi_{u0}(\pi_{d0}(1-\epsilon_d)-(2\delta-1)\pi_{u0}(1-\epsilon_u))\alpha^3}{16(1-\delta)^2}+O(\alpha^4)\,. \tag{A.122}$$

Using Π_{12} in (A.122) together with Π_{11} and Π_{22}, and the condition $C_u''>\tau$ and $C_d''>\tau$, we obtain that the highest order term of α in $\Pi_{11}\Pi_{22}-\Pi_{12}^2>0$ under small α is a constant τ^2 which is positive. Consequently, the second order condition is satisfied, and the unique pair of optimizer (ϵ_u^*, ϵ_d^*) exists. Let $\Pi_{13}\triangleq d^2\Pi/d\epsilon_u d\alpha$, $\Pi_{23}\triangleq d^2\Pi/d\epsilon_d d\alpha$. By taking a derivative of (A.120) and (A.121) with respect to α, we obtain

$$\Pi_{11}\frac{d\epsilon_u^*}{d\alpha}+\Pi_{12}\frac{d\epsilon_d^*}{d\alpha}+\Pi_{13}=0\quad\text{and}\quad\Pi_{12}\frac{d\epsilon_u^*}{d\alpha}+\Pi_{22}\frac{d\epsilon_d^*}{d\alpha}+\Pi_{23}=0\,. \tag{A.123}$$

From (A.123), it follows that

$$\frac{d\epsilon_u^*}{d\alpha}=\frac{\Pi_{12}\Pi_{23}-\Pi_{22}\Pi_{13}}{\Pi_{11}\Pi_{22}-\Pi_{12}^2}\quad\text{and}\quad\frac{d\epsilon_d^*}{d\alpha}=\frac{\Pi_{12}\Pi_{13}-\Pi_{11}\Pi_{23}}{\Pi_{11}\Pi_{22}-\Pi_{12}^2}\,. \tag{A.124}$$

First, for the numerator of $d\epsilon_u^*/d\alpha$, the highest order term of α under low security-loss environment is a constant term greater than $\tau\pi_{u0}/8$, which is positive. Moreover the denominator of $d\epsilon_u^*/d\alpha$ is positive, by using a strict convexity of C_u and C_d. As a result, $d\epsilon_u^*/d\alpha$ is positive, i.e., $\epsilon_u^*(\alpha)$ is increasing in α. Second, for the numerator of $d\epsilon_d^*/d\alpha$, the highest order term of α under low security-loss environment is the second order term and greater than

$$\frac{3\tau(1-\epsilon_u)\pi_{u0}\pi_{d0}((1-\epsilon_u)\pi_{u0}(2\delta-1)-2\pi_{d0}(1-\epsilon_d))}{32(1-\delta)^2}\,, \tag{A.125}$$

which is positive when $\epsilon_d^* > 0$ from (A.121). Similarly, as a result, it then follows that $d\epsilon_d^*/d\alpha$ is positive, i.e., $\epsilon_d^*(\alpha)$ is increasing in α.

For the high security-loss environment, from Proposition 1, SaaS can be either in the middle tier or in the low tier in equilibrium. First, if SaaS goes into the middle tier in equilibrium, the firm's profit (except the investment costs) is given in (A.77). Following the steps for the low security-loss environment, we obtain first order conditions as follows:

$$\begin{aligned}\Pi_1 &\triangleq \frac{d\Pi}{d\epsilon_u} = \frac{c_{p0}(1-c_{p0}(1-\epsilon_u))}{2} - C_u'(\epsilon_u)\\ &+ \frac{c_{p0}}{4\alpha}\left(\frac{8c_{p0}^2(3-c_{p0}(1-\epsilon_u))(1-\epsilon_u)}{(1+c_{p0}(1-\epsilon_u))^3\pi_{u0}} - \frac{8c_{p0}(1-c_{p0}(1-\epsilon_u))}{(1+c_{p0}(1-\epsilon_u))^2\pi_{u0}} - \frac{\delta}{\pi_{d0}(1-\epsilon_d)}\right) + O\left(\frac{1}{\alpha^2}\right) = 0\,,\end{aligned} \tag{A.126}$$

and

$$\Pi_2 \triangleq \frac{d\Pi}{d\epsilon_d} = \frac{c_{p0}(1-\epsilon_u)\delta}{4\alpha(1-\epsilon_d)^2\pi_{d0}} - C_d'(\epsilon_d) + O\left(\frac{1}{\alpha^2}\right) = 0\,, \tag{A.127}$$

under large α. Moreover, from the condition $\tau > 1/2$, it follows that $\Pi_{11} \triangleq d^2\Pi/d\epsilon_u^2 < -\tau + c_{p0}^2/2 < 0$ and $\Pi_{22} \triangleq d^2\Pi/d\epsilon_d^2 < -\tau < 0$ for large α. In addition, we obtain

$$\Pi_{12} \triangleq \frac{d^2\Pi}{d\epsilon_u d\epsilon_d} = -\frac{c_{p0}\delta}{4(1-\epsilon_d)^2\pi_{d0}\alpha} + O\left(\frac{1}{\alpha^2}\right)\,. \tag{A.128}$$

Then, it follows that $\Pi_{11}\Pi_{22} - \Pi_{12}^2 > 0$ for large α, which then implies that the second order condition is satisfied, and hence there exists the unique maximizer (ϵ_u^*, ϵ_d^*). Next, in (A.124), the denominator of $d\epsilon_u^*/d\alpha$ is positive from the second order condition. Furthermore, the numerator of $d\epsilon_u^*/d\alpha$ in (A.124) has the highest order term, which is greater than

$$\frac{\tau}{4\pi_{u0}\alpha^2}\Big(\underbrace{\frac{8c_{p0}(1-3c_{p0}(1-\epsilon_u^*))}{(1+c_{p0}(1-\epsilon_u^*))^3}}_{A} + \underbrace{\frac{\pi_{u0}}{\pi_{d0}}}_{B}\cdot\underbrace{\frac{\delta}{1-\epsilon_d^*}}_{C}\Big)\,. \tag{A.129}$$

Note that A is finite and negative, and C is finite and positive in equilibrium; as a result, there exists $\hat{r}$ such that for all $B > \hat{r}$, (A.129) is positive. Consequently, under this condition, $d\epsilon_u^*/d\alpha$ is positive for large α, i.e., ϵ_u^* increases in α. Similarly, the numerator of $d\epsilon_u^*/d\alpha$ in (A.124) has the highest order term, which is greater than

$$-\frac{\delta c_{p0}(2\tau - c_{p0}^2)(1-\epsilon_u^*)}{8\pi_{d0}(1-\epsilon_d^*)^2\alpha^2}\,, \tag{A.130}$$

and it is negative for large α since $\tau > 1/2$. Hence, ϵ_d^* increases in α. Next, we show that SaaS goes into the middle tier in equilibrium. From Proposition 1, we need to show that $c_{p0}(1-\epsilon_u^*) \geq 1/3$ and $\delta > \frac{2(1-c_{p0}(1-\epsilon_u^*))}{1+c_{p0}(1-\epsilon_u^*)}$. First, if SaaS goes to the middle tier, ϵ_u^* satisfies (A.126), which can be simplified to

$$\frac{c_{p0}(1-c_{p0}(1-\epsilon_u))}{2} - C_u'(\epsilon_u) = O\left(\frac{1}{\alpha}\right)\,. \tag{A.131}$$

Note that at $\epsilon_u = 1 - \frac{2-\delta}{c_{p0}(2+\delta)}$, under large α, the left hand side of (A.131) can be written as

$$\frac{c_{p0}\delta}{2+\delta} - C_u'\left(1-\frac{2-\delta}{c_{p0}(2+\delta)}\right) + O\left(\frac{1}{\alpha}\right) < \frac{c_{p0}\delta}{2+\delta} - \frac{1}{2}\left(1-\frac{2-\delta}{c_{p0}(2+\delta)}\right) + O\left(\frac{1}{\alpha}\right)\,, \tag{A.132}$$

by $C_u'' > \tau \geq 1/2$. Using $\delta \geq \bar{\delta}$, we obtain that (A.132) is negative under large α, which implies that the left hand side of (A.131) at $\epsilon_u = 1 - \frac{2-\delta}{c_{p0}(2+\delta)}$ is negative. Hence, $\epsilon_u^* < 1 - \frac{2-\delta}{c_{p0}(2+\delta)}$, which then implies $\delta > \frac{2(1-c_{p0}(1-\epsilon_u^*))}{1+c_{p0}(1-\epsilon_u^*)}$. In addition, $c_{p0} \geq 1/2$ guarantees that $\bar{\delta} \leq 1$ and $c_{p0}(1-\epsilon_u^*) \geq 1/3$. Moreover, suppose that SaaS goes to the low tier in equilibrium, in which case the profit (except the investment costs) is given in (A.80). Following the same steps, under the conditions presented in the proposition, it follows that $c_{p0}(1-\epsilon_u^*) \geq 1/3$ and $\delta > \frac{2(1-c_{p0}(1-\epsilon_u^*))}{1+c_{p0}(1-\epsilon_u^*)}$ are still satisfied, i.e., there is no interior maximizer of investments $(\epsilon_u^*, \epsilon_d^*)$ in this case; as a result, the interior optimizer found in the previous case of SaaS in the middle tier is the global optimizer, which completes the proof. ■

Proof of Proposition 9: From the proof of Proposition 8, under a high security-loss environment, when $1/2 \leq c_{p0}$, $1/2 < \tau$, and $\delta > \bar{\delta}$, SaaS goes to the middle tier in equilibrium. Furthermore, ϵ_u^* satisfies (A.126) and ϵ_d^* satisfies (A.127). Comparing these two first order conditions, we obtain that $\epsilon_u^* > \epsilon_d^*$ under a sufficiently large α, which completes the proof. ■