Of Smiles and Smirks: A Term-Structure Perspective[1]

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Abstract

Empirical anamolies in the Black–Scholes model have been widely documented in the Finance literature. Patterns in these anamolies (for instance, the behavior of the volatility smile or of unconditional returns at different maturities) have also been widely documented. Theoretical efforts in the literature at addressing these anamolies have largely centered around extensions of the basic Black–Scholes model. Two approaches have become especially popular in this context—introducing jumps into the return process, and allowing volatility to be stochastic.

This paper employs commonly-used versions of these two classes of models to examine the extent to which the models are theoretically capable of resolving the observed anamolies. We focus especially on the possible "term-structures" of skewness, kurtosis, and the implied volatility smile that can arise under each model. Our central finding is that each model exhibits moment patterns and implied volatility smiles that are consistent with some of the observed anamolies, but not with others. In sum, neither class of models constitutes an adequate explanation of the empirical evidence, although stochastic volatility models fare better than jumps in this regard.

1 Introduction and Summary

It is widely acknowledged today that financial data from currency and equity markets differ in systematic ways from the model of Black and Scholes [9]. Two anamolies have been particularly well documented: (a) the presence of a greater degree of excess kurtosis in the unconditional returns distributions than is consistent with normality; and (b) the presence of a volatility smile (or skew) in which implied volatility estimates from otherwise identical options differ across strike prices.

The existence of the volatility smile may evidently be attributed to the presence of excess kurtosis in the *conditional* returns distribution, and any asymmetry in the smile to the presence of skewness in this distribution.[1] Building on this observation, attempts at reconciling the theory with the data in the finance literature have largely concentrated on extensions of the Black-Scholes model that exhibit excess kurtosis (and possibly also skewness) both conditionally and unconditionally. Two alternatives have become especially popular in recent years. One, the class of *jump-diffusion* models, augments the return distribution of the Black-Scholes framework with a Poisson-driven "jump" process. The other, the class of *stochastic volatility* models, augments the Black-Scholes framework by allowing the volatility of the return process to itself evolve stochastically over time.[2]

At an intuitive level, it is not difficult to see how jump-diffusion and stochastic volatility models could each lead to returns distributions that exhibit both skewness and excess kurtosis, and, indeed, a formal confirmation is not difficult (we provide one in this paper). It is plausible, therefore, that either class of models could be made consistent with observed degrees of unconditional kurtosis and implied volatilites for a *given* maturity. What is not apparent, however, is how these quantities will change in each case as maturity *varies*. To the best of our knowledge, this question—the possible "term-structures" of skewness, kurtosis, and the implied volatility smile in jump-diffusions and stochastic volatility models—has not been studied in the literature. It forms the subject matter of our paper.

An understanding of these possible term-structures in each model is important for a variety of reasons. As argued by Duffie and Pan [18], for example, knowledge of the conditional term-structure of skewness and kurtosis is central to the construction of models for estimating value-at-risk. Of much greater import is the fact that an extensive literature

[1] Kurtosis in the conditional returns distribution makes extreme observations more likely, thereby raising the value of away-from-the-money options relative to at-the-money options, and creating the volatility smile. Skewness in the conditional returns distribution accentuates one side of the smile, resulting in a volatility skew.

[2] For work on jump-diffusions, see, e.g., Ahn [1], Amin [3], Ball and Torous [6], Das and Foresi [15], Jarrow and Rosenfeld [27], or Merton [34]. On stochastic volatility, see Amin and Ng [4], Heston [22], Hull and White [25], Melino and Turnbull [33], Stein and Stein [39], and Wiggins [41].

in finance has documented the manner in which moments of the unconditional distributions and implied volatility smiles change with maturity in a variety of equity and currency markets.[3] Available evidence suggests very strongly, for instance, that the degree of excess kurtosis in unconditional returns tends to decrease as the horizon increases, and even that returns become approximately normal at long horizons. It is, similarly, a common feature of most markets that the implied volatility smile flattens out monotonically as maturity increases. Also of considerable interest is the behavior of implied volatilities of at-the-money (forward) options as maturity varies. In both equity and currency markets, this so-called "term structure of implied volatilites" displays a variety of patterns (increasing, decreasing, and sometimes even non-monotone) as maturity changes. By identifying the possible term-structures of skewness, kurtosis, and implied volatility smiles in each class of models, our paper provides a way to judge the extent to which either class of models is consistent with these empirical patterns.

Our results and their implications are described in Subsections 1.1 and 1.2 below. Section 1.3 then relates our paper to others in the literature.

1.1 Main Results: A Synopsis

In a nutshell, our central finding is that each class of models does well at addressing some of the anamolies associated with the Black–Scholes model. However, each model also carries implications that are at variance with some common empirical patterns in the data. As such, neither may be considered an adequate description of observed patterns of unconditional returns or implied volatilities, although stochastic volatility models fare, on the whole, better than jump-diffusions.

The key to analysing the consistency of either class of models with observed patterns in the data evidently lies in the term-structures of skewness and kurtosis in each model. For *unconditional* returns this is obvious, since empirical departures from Black–Scholes in this case are stated precisely in terms of these higher-order moments. Characterizing *conditional* skewness and kurtosis analytically will enable us to judge the factors that drive the way implied volatilities change with maturity in either model.

As the first step in our analysis, therefore, we derive closed-form solutions for skewness and kurtosis in both classes of models, conditionally and unconditionally. These closed-forms are used to provide an analytical characterization of the dependence of these quantities on the models' parameters, including, especially, the length of the holding period. We then derive implied volatilities in either class of models for a range of maturities and parameterizations,

[3] References may be found in Section 1.3 below.

and relate the behavior of these implied volatilities to the behavior of skewness and kurtosis in the model.

Concerning jump-diffusions, we find that such models are capable of reproducing observed patterns of skewness and kurtosis at short maturities, under reasonable parametrizations. Consistent with empirical patterns, moreover, the degree of skewness and excess kurtosis in these models declines monotonically as maturity increases. However, we show that excess kurtosis in jump-diffusions dissipates far more rapidly than would be suggested by data, either unconditional or conditional. Reflecting this behavior, the model is capable of producing realistic implied volatility smiles at short maturities, but, owing to the fall-off in excess kurtosis, the smile flattens out very quickly as maturity increases. Finally, of considerable importance, we find that the term-structure of implied at-the-money volatilities in a jump-diffusion is always an *increasing* function of the time-to-maturity, reflecting the monotone nature of excess kurtosis in this model. In particular, the model seems incapable of generating decreasing or non-monotone term-structure patterns as are frequently witnessed in practice (e.g., the current term-structure for S&P 500 index options).

Stochastic volatility models fare better on the whole, but not uniformly so. We find that, for reasonable parameterizations, such models do not generate very high levels of skewness and kurtosis, especially at short maturities. Moreover, conditional skewness and kurtosis in these models are always hump-shaped in the length of the horizon; consequently, they must both be *increasing* over short to moderate maturities. These features carry implications for the behavior of implied volatilities in this model (although the impact of kurtosis is confounded to an extent by a second force, viz., mean-reversion in volatility). For plausible parameter values, the relative lack of excess kurtosis results in implied volatility smiles that are fairly shallow, unless other factors kick in. Moreover, since excess kurtosis initially increases with the horizon, we find that regardless of the actual parametrization, the smile does *not* flatten out appreciably as maturity increases. On the question of the term-structure of implied volatilities, however, stochastic volatility models do well. We find that a variety of patterns are possible in this model: increasing, decreasing, or even non-monotone (for example, U-shaped). Thus, qualitatively speaking, these models are able to capture an important aspect of financial market data better than jump-diffusions.

1.2 Implications

Empirical examination has established very strong evidence in favor of time-varying volatility (see, for example, the survey of Bollerslev, Chou and Kroner [12]). On the other hand, a number of studies have also found that markets are characterized by the presence of jumps, especially when data at short frequencies are used (e.g., Ball and Tourus [6], Drost, Nijman, and Werker [20], Jarrow and Rosenfeld [27]). Moreover, in a direct comparison of the alternatives, Jorion [29] finds that the jump sub-model does better than its stochastic volatility counterpart, though the combined model does better still. Partially echoing these results, Bates [7] finds that the stochastic volatility sub-model can explain the volatility smile only under parameter values that are implausible given the time-series properties of implied volatilities, and that the jump sub-model does better in this regard. Our results are consistent with all of these studies. Indeed, since a primary message of our paper is that the two models speak to different (but relevant) aspects of the data, our results even offer, in a sense, a post-hoc rationalization of these findings.[4]

Taking all this into account, the "best" solution may appear to be the use of a model that combines time-varying volatility with jumps. Such a model would, for example, be able to generate adequate kurtosis at short maturities (through the jump component) and at moderate maturities (through the stochastic volatility component). Unfortunately, such a choice would not be a parsimonious one. In this context, our results offer at least two possible ways to form a preliminary inference about the suitability of either class of models.

First, it is possible to extract information on *conditional* skewness and kurtosis from option prices using a variety of techniques, and then to use our results on the sharply differing possible patterns of conditional skewness and kurtosis in the two models. One such technique is the use of Gram-Charlier approximations as described in Jarrow and Rudd [27]. Backus, Foresi, and Wu [5] carry out this procedure using data on $/DM option prices. They conclude that the resulting downward sloping kurtosis curve in their paper is consistent with a jump-diffusion, but not with a stochastic volatility model. Campa, Chang, and Rieder [14] describe and implement a number of alternative techniques in this regard; see also Jackwerth and Rubinstein [26] and Rosenberg [36].

Second, we have shown that certain term-structure patterns (decreasing or U-shaped) for implied volatilities cannot be consistent with a jump-diffusion model. This offers a direct route for judging the appropriateness of jump-diffusions in a specific context, since data on term-structures of implied volatilties for nearest-the-money options is relatively easy to

[4]An alternative, stronger interpretation of our results is also possible. Our entire analysis in this paper concerns models which directly extend the "normal returns" model of Black and Scholes [9]. This was a deliberate choice on our part, since such models are, far and away, the most commonly employed ones in the literature. However, our results are also qualified by this modeling strategy. Thus, one could also view our results as saying that returns processes in financial markets are fat-tailed and skewed in a more fundamental sense than can be captured by simply introducing these features into a model that has neither of these effects. We would not disagree with this interpretation, but would not like to push it without further research into either class of models, perhaps using other specifications.

get.[5,6]

1.3 The Related Literature

An extensive literature in finance has documented that excess kurtosis in unconditional returns declines with an increase in the length of the holding period. Early references to this phenomenon may be found in Kon [31] and elsewhere; for instance, while Fama [21] identified the presence of "fat tails" in stock returns using daily data, Blattberg and Gonedes [11] found that the normality assumption fits stock returns well if monthly data are used. More recently, Jorion [29] reports that excess kurtosis in the $/DM exchange rate and in the value weighted CRSP index are, respectively, 3.29 and 2.92 under weekly data, but fall to 1.56 and 0.89 under monthly data. Bates [7] contains references to other studies that have reported similar results.

The behavior of implied volatilities at different maturities has similarly received extensive attention. In an early indirect reference to this issue, Black [8] discusses the tendency of the Black-Scholes model to overprice short-maturity at-the-money options. More recent work on the behavior of implied volatilities as maturity varies includes Backus, Foresi, and Wu [5], Bodurtha and Courtadon [10], Campa and Chang [13], Derman and Kani [16], Rosenberg [36], Rubinstein [37], Stein [38], Xu and Taylor [42], and Zhu [43]. The Equity Derivatives Research publications of Goldman–Sachs also carry detailed information on this subject.

Although the patterns of these anamolies have been known for some time now, relatively few papers in the literature have attempted to derive theoretical implications of asset price processes at varying time frequencies, especially using higher-order moments. Two recent such papers are Backus, Foresi, and Wu [5], and Drost, Nijman, and Werker [20]. The former, which is closely related to ours in motivation, uses Gram-Charlier approximations to examine the impact of skewness and kurtosis on option prices, volatility smiles, and the term-structure of implied volatilites. The latter models security prices as evolving according to continuous-time GARCH diffusions (cf. Drost and Nijman [19]), and derives an overidenti-

[5]Our closed-form solutions also show that the patterns of skewness and kurtosis in *unconditional* returns differ across the two classes of models, but in a far less pronounced manner than under conditional returns. Given the large standard errors in estimating kurtosis, these differences do not appear useful from a practical standpoint.

[6]Two additional consequences of our paper may be of some importance, both related to the closed-form expressions we provide for the higher-order moments in the two classes of models. First, these closed-forms provide identification conditions that should enable efficient econometric testing using method of moments models. Second, these closed-forms should also find use in the construction of models for estimating value-at-risk (see Duffie and Pan [18] in this regard).

fying relationship between the variance and kurtosis that must be satisfied at any frequency. This paper is not, however, explicitly concerned with option prices, and does not derive implications for volatility smiles or term structures.

There is also a substantial literature which has focussed on an examination of the term structure of implied volatilities for at-the-money options (see, e.g., Stein [38], Heynen, Kemna and Vorst [23], Campa and Chang [13], and Xu and Taylor [42]). There are two important distinctions between our paper and this literature. The first is that each of these papers takes as given an underlying model of time-varying volatility, and is concerned with an empirical aspect of the term-structure; we are, in contrast, interested in the theoretical implications of different models for the shape of the term-structure. A second, and important difference, is that many of these papers use an approximation argument which leads them to replace implied volatility with average expected volatility over the given horizon. This has one important implication: the resulting shape of the term-structure of implied volatilities is entirely decided by the mean-reversion factor, i.e., it is upward sloping if current volatility is below its long-term mean, flat if it is at the long-term mean, and downward sloping otherwise. Our results on possible shapes of the term-structure suggest that a greater variety of shapes is possible than is implied by this approximation argument.

Finally, as discussed above, direct econometric tests for choosing between jump-diffusions and stochastic volatility models have also been attempted (Bates [7], or Jorion [29]). Interestingly, given our emphasis on different frequencies, Jorion [29] finds in his study that based on monthly data, there is not much difference between the models he considers, but that this is not true under weekly data.[7]

2 Geometric Brownian Motion: A Brief Review

Let S_0 denote the initial (time-0) price of the asset. Under Geometric Brownian Motion (GBM), the time-t price S_t evolves according to

$$S_t \;=\; S_0 \exp\{\alpha t + \sigma W_t\}, \tag{1}$$

where α and σ are given constants, and W_t is a standard Brownian motion process.

Let $h > 0$ denote the length of time between observations of the price, and let $Z_t(h) = \ln(S_{t+h}/S_t)$ denote the continuously compounded return from holding the asset over the time

[7] On a broader note, there is a literature in finance that has focussed on the problems of estimating continuous-time models with discrete-time data (e.g., Melino [32]). Especially relevant in this context is the recent work of Ait-Sahalia [2].

interval $[t, t+h]$. From (1), $Z_t(h)$ is given by

$$Z_t(h) \;=\; \alpha h + \sigma(W_{t+h} - W_t). \tag{2}$$

Since (W_τ) is a Wiener process, we have $(W_{t+h} - W_t) \sim N(0, h)$. It follows from (2), therefore, that the returns $Z_t(h)$ are independent of t, and are themselves normally distributed with a mean of αh and a variance of $\sigma^2 h$:

$$Z_t(h) \;\sim\; N\left(\alpha h, \sigma^2 h\right). \tag{3}$$

Now recall that if X is any random variable with mean m and variance $s^2 > 0$, then the skewness and kurtosis of X are defined as

$$\text{Skewness}\,(X) \;=\; \frac{E[(X-m)^3]}{s^3} \quad \text{and} \quad \text{Kurtosis}\,(X) \;=\; \frac{E[(X-m)^4]}{s^4}, \tag{4}$$

and that the *excess kurtosis* of X is given by Excess Kurtosis $(X) = [\text{Kurtosis}(X) - 3]$.

It is well known that any normal distribution has zero skewness and a kurtosis of 3. Therefore, expression (3) implies that the skewness and kurtosis of returns in a GBM are zero, conditionally and unconditionally. The systematic violation of these conditions in practice has led to the development of the two classes of models we examine over the remainder of this paper.

3 Jump–Diffusion Models

A jump–diffusion model is obtained by augmenting the return process in a GBM with a Poisson jump process. Given any date $t \geq 0$, and a holding-period of length $h > 0$, the returns $Z_t(h)$ over the period $[t, t+h]$ in such a model are given by

$$Z_t(h) \;=\; \begin{cases} x, & \text{if } K = 0 \\ x + y_1 + \cdots + y_K, & \text{if } K \geq 1. \end{cases} \tag{5}$$

where

1. $x \sim N(\alpha h, \sigma^2 h)$.

2. $y_1, y_2, \ldots$ is an i.i.d. sequence with common distribution G.

3. K is distributed Poisson with parameter λh, where $\lambda > 0$. That is, for $k = 0, 1, 2, \ldots$ we have:

$$\text{Prob}\,(K = k) \;=\; \left(\frac{e^{-\lambda h}(\lambda h)^k}{k!}\right).$$

Since $Z_t(h)$ does not depend on t in any way, conditional and unconditional returns distributions coincide in a jump-diffusion model. To simplify exposition in the sequel, we shall, therefore, write $Z(h)$ for $Z_t(h)$.

Throughout, we will require that the first four moments of the jump-size G exist. That is, letting $\nu_n = E(y^n)$ denote the n-th moment of G, we will assume that ν_n is finite for $n = 1, 2, 3, 4$. This condition is a mild one that is met under virtually all the distributions used in practice; it is also a necessary requirement in the context of any sensible discussion of higher-order moments. No other restrictions, or distributional requirements are placed on G.

3.1 Moments of the Jump-Diffusion Model

The following is the central result of this section:

Proposition 3.1 *The variance, skewness, and kurtosis of the jump-diffusion process (5) are given by:*

$$\textit{Variance}\,[Z(h)] \;=\; E\left[(Z(h) - E[Z(h)])^2\right] \;=\; h(\sigma^2 + \lambda\nu_2). \tag{6}$$

$$\textit{Skewness}\,[Z(h)] \;=\; \frac{E[(Z(h) - E[Z(h)])^3]}{[Var(Z(h))]^{3/2}} \;=\; \frac{1}{\sqrt{h}}\left[\frac{\lambda\nu_3}{(\sigma^2 + \lambda\nu_2)^{3/2}}\right]. \tag{7}$$

$$\textit{Kurtosis}\,[Z(h)] \;=\; \frac{E[(Z(h) - E[Z(h)])^4]}{[Var(Z(h))]^2} \;=\; 3 + \frac{1}{h}\left[\frac{\lambda\nu_4}{(\sigma^2 + \lambda\nu_2)^2}\right]. \tag{8}$$

Therefore:

1. *Concerning Skewness:*

   (a) *If $\nu_3 = 0$, then the skewness of the jump-diffusion is also zero.*

   (b) *If $\nu_3 > 0$, the skewness is positive and inversely proportional to $\sqrt{h}$.*

(c) If $\nu_3 < 0$, the skewness is negative and inversely proportional to $\sqrt{h}$.

2. *The kurtosis of the jump-diffusion process (5) is always strictly positive and is inversely proportional to h.*

Remark Under the assumption that $G \sim N(\mu, \gamma^2)$, Merton ([34], Section 4) and Jorion ([29], Section 5) state that the variance of $Z(h)$ is given by $(\sigma^2 + \lambda\gamma^2)$. Proposition 3.1 shows that this assertion is incorrect, and that the correct expression for the variance in their models should be $(\sigma^2 + \lambda[\gamma^2 + \mu^2])$. □

Proof See Appendix A. □

The content of Proposition 3.1 may be illustrated using a simple example using the most commonly-employed distribution for the jump size, the normal distribution. Let G be a normal distribution with mean μ and variance γ^2. Then, from (7)–(8), we have:

$$\text{Skewness}\,(Z(h)) \;=\; \frac{1}{\sqrt{h}}\left[\frac{\lambda(\mu^3 + 3\mu\gamma^2)}{(\sigma^2 + \lambda\gamma^2 + \lambda\mu^2)^{3/2}}\right]. \tag{9}$$

$$\text{Kurtosis}\,(Z(h)) \;=\; 3 + \frac{1}{h}\left[\frac{\lambda(\mu^4 + 6\mu^2\gamma^2 + 3\gamma^4)}{(\sigma^2 + \lambda\gamma^2 + \lambda\mu^2)^2}\right]. \tag{10}$$

Note that skewness in this case is positive if $\mu > 0$ and negative if $\mu < 0$.

3.2 Skewness and Kurtosis in the Jump-Diffusion Model

The closed-form expressions for skewness and kurtosis provided in Proposition 3.1 make it easy to verify that the model can match observed levels of skewness and kurtosis at specified intervals. Table 1 illustrates this point. It presents skewness and kurtosis values for the jump-diffusion for a range of parameter values, assuming $\lambda = 5$ and $G \sim N(\mu, \gamma^2)$. (The choice of parameter values is explained in the table.) As the numbers reveal, even for the intermediate and reasonable values of μ and γ, the model is capable of generating a considerable amount of skewness and excess kurtosis at weekly intervals.

However, Proposition 3.1 also indicates that the skewness and kurtosis of a jump-diffusion decrease very rapidly as the horizon increases; for instance, excess kurtosis at weekly intervals ($h = 1/52$) should be more than four times that at monthly intervals ($h = 1/12$). Reflecting this rate of decline, Table 1 shows that for reasonable parameter configurations, excess kurtosis becomes virtually negligible over a three-month horizon. This aspect of the model is troubling for a number of reasons. Certainly, the rate of decline appears much higher than would be suggested by empirical estimates of unconditional kurtosis at different frequencies. It also implies that implied volatility smiles should be nearly non-existent at three months, which is not typically the case. We will return to this issue in Subsection 6.1.

4 Conditional Returns under Stochastic Volatility

Stochastic volatility models generalize Geometric Brownian Motion by allowing the volatility of the return process to itself evolve stochastically over time. From (1), the cumulative returns $x_t = \ln(S_t/S_0)$ under a GBM obey $x_t = \alpha t + \sigma W_t$; expressing this in stochastic differential form, we have

$$dx_t \;=\; \alpha\, dt + \sigma\, dW_t, \tag{11}$$

In a stochastic volatility model, the instantaneous variance $V = \sigma^2$ is no longer required to be constant, but is allowed to change with time. That is, (11) is replaced by the joint process

$$dx_t \;=\; \alpha\, dt + \sqrt{V_t}\, dW_t \tag{12}$$

$$dV_t \;=\; \xi(t, V_t)\, dt + \beta(t, V_t)\, dB_t \tag{13}$$

where W_t and B_t are standard Brownian motion processes. For the purposes of analysis, we must choose a functional form for (13). A popular specification, and one that we adopt, is that V_t evolves according to a mean-reverting square-root process

$$dV_t \;=\; \kappa(\theta - V_t)\, dt + \eta\sqrt{V_t}\, dB_t, \tag{14}$$

where κ, θ, and η are all strictly positive constants. This specification has been widely used in the literature (see, e.g., Bates [7], Heston [22], Heynen, Kemna and Vorst [23]), and, as such, is a natural choice. It also possesses significant analytical advantages over alternatives, particularly for the pricing of options (on this point, see Bates [7]). Nonetheless, it seems important to emphasize that the qualitiative aspects of our results do not appear to depend too much on this assumption; similar conclusions obtained, for instance, when the square-root process was replaced by an Ornstein–Uhlenbeck process.

Finally, we adopt a general specification for the relationship between the Wiener processes W_t and B_t:

$$dB_t\; dW_t = \rho\, dt. \tag{15}$$

Expression (15) captures the possibility that increases in volatility could be related to the level of asset prices. Expressions (12), (14), and (15) complete the description of the model.

As earlier, let $Z_t(h) = x_{t+h} - x_t$ denote the returns from holding the asset between times t and $t+h$. It is immediate from (12) that these returns will depend on the path taken by the volatility process V_τ over this interval. Moreover, it is also immediate from (14) that this path will depend on the initial value V_t of the volatility at time t. Thus, the returns $Z_t(h)$ depend on time–t information through V_t (but not, of course, through x_t). It follows that conditional and unconditional returns distributions will not necessarily coincide in a stochastic volatility model. In this section, we examine the properties of the conditional returns distribution; the unconditional distribution is the subject of the section following. The proofs of all results obtained in this section may be found in Appendix B.

For notational ease, let $V_t = v$. Denote by $F(v,h,s) = E[\exp\{isZ_t(h)\} \mid x, v]$ the characteristic function of $Z_t(h)$ conditional on time-t information. In Appendix B, we derive a closed-form expression for this characteristic function. Differentiation of this characteristic function enables us to recover all the moments of $Z_t(h)$ in the usual way, and enables us to obtain the following:

Proposition 4.1 *Conditional on time–t values, the skewness and kurtosis of the stochastic volatility model are given by*

$$\mathit{Skewness}\,(Z_t(h)) \;=\; \left(\frac{3\eta\rho e^{\frac{1}{2}\kappa h}}{\sqrt{\kappa}}\right)\left[\frac{\theta(2-2e^{\kappa h}+\kappa h+\kappa h e^{\kappa h}) - v(1+\kappa h - e^{\kappa h})}{(\theta[1-e^{\kappa h}+\kappa h e^{\kappa h}]+v[e^{\kappa h}-1])^{3/2}}\right] \tag{16}$$

$$\mathit{Kurtosis}\,(Z_t(h)) \;=\; 3\left[1+\eta^2\left(\frac{\theta A_1 - vA_2}{B}\right)\right] \tag{17}$$

where, letting $y = \kappa h$,

$$A_1 \;=\; [1+4e^y-5e^{2y}+4ye^y+2ye^{2y}] + 4\rho^2[6e^y-6e^{2y}+4ye^y+2ye^{2y}+y^2e^y] \tag{18}$$

$$A_2 \;=\; 2[1-e^{2y}+2ye^y] + 8\rho^2[2e^y-2e^{2y}+2ye^y+y^2e^y] \tag{19}$$

$$B = 2\kappa[\theta(1-e^y+ye^y)+v(e^y-1)]^2 \tag{20}$$

Proof See Appendix B.1. □

Expressions (16)–(20) look menacing, but they turn out to be surprisingly tractable from an analytical standpoint. We use them in the next two subsections to identify properties of the skewness and kurtosis of the conditional returns $Z_t(h)$.

4.1 Skewness of the Conditional Returns

For notational simplicity, let $S_t(h)$ denote the skewness of $Z_t(h)$. When we wish to emphasize the dependence of $S_t(h)$ on the correlation ρ, we will write $S_t(h;\rho)$. We will show in this subsection that for $\rho \neq 0$, the absolute skewness $|S_t(h)|$ must necessarily be a hump-shaped function of h. The following result establishes several preliminary properties of $S_t(\cdot)$ that are also of independent interest:

Proposition 4.2 *$S_t(h)$ has the following properties:*

1. *$S_t(h)$ is positive if $\rho > 0$, zero if $\rho = 0$, and negative if $\rho < 0$.*
2. *$S_t(h;\rho) = -S_t(h;-\rho)$ for all ρ.*
3. *$\lim_{h\downarrow 0} S_t(h) = \lim_{h\uparrow\infty} S_t(h) = 0$.*

Proof See Appendix B.2. □

Properties 1 and 3 of Proposition 4.2 imply that when $\rho \neq 0$, skewness *cannot* be a monotone function of h as it was in the case of jump-diffusions. Indeed, in Appendix B.2, we provide an analytical proof that a very simple pattern must obtain, namely:

- If $\rho > 0$, skewness increases from zero to a maximum, and then decreases asymptotically back to zero.
- If $\rho < 0$, skewness decreases from zero to a minimum, and then increases asymptotically back to zero.

From a practical standpoint, this result raises an obvious question: how large is the value h at which absolute skewness is maximized? In Appendix B.2, we show that this maximizing value of h depends on only two parameters of the problem: the rate of mean-reversion κ and the ratio $a = v/\theta$ of current volatility to its long term mean. Table 2 summarizes the way in which this maximizing value changes with κ and a.

Plausible values for κ (i.e., ones typically found empirically) are in a neighborhood of unity. If the current value of the volatility v is not excessively off its long-term mean level θ (i.e., if a and a^{-1} are both not very large), then it is seen from Table 2 that the maximizing h is of the order of several months, and even years. From an empirical standpoint therefore, the increasing part of $S_t(h)$ is very relevant, creating a sharp contrast with conditional skewness in jump-diffusion models.

4.2 Kurtosis of the Conditional Returns

For notational simplicity, let $K_t(h)$ denote the kurtosis of $Z_t(h)$. As with skewness, when we wish to emphasize the dependence of kurtosis on the correlation, we shall write $K_t(h; \rho)$. The following result establishes some important properties of $K_t(\cdot)$ including symmetry in ρ:

Proposition 4.3 *$K_t(h)$ has the following properties:*

1. *Excess kurtosis is strictly positive everwhere: $K_t(h) > 3$ for all $h > 0$.*
2. *The degree of kurtosis only depends on the absolute value of ρ: $K_t(h; \rho) = K_t(h; -\rho)$.*
3. $\lim_{h \downarrow 0} K_t(h) = \lim_{h \uparrow \infty} K_t(h) = 3.$

Proof See Appendix B.3. □

Properties 1 and 3 imply, once again, that kurtosis cannot be a monotone function of h as it was for jump-diffusions. Indeed, exactly the same procedure we used earlier establishes that

- Kurtosis is hump-shaped as a function of h, increasing from zero to a maximum, and then decreasing asymptotically back to zero again.

We omit the details here, since the expressions are significantly more complicated than in the earlier case. However, carrying out the relevant computations shows that behavior of kurtosis in h depends on *three* factors: κ, a, and the degree of correlation ρ. Table 3 shows how the value of h that maximizes $K_t(h)$ depends on these parameters. It can be seen that the maximizing value of h increases with ρ and a, and decreases with κ. More importantly, as with skewness, it is again the case that for plausible parameter values, this maximizing value of h is quite large (of the order of several months, and even years), making the increasing portion of the kurtosis curve significant from an empirical standpoint.

4.3 Values of Skewness and Kurtosis

The previous subsections have shown that the stochastic volatility model has very different patterns of skewness and kurtosis from jump-diffusions. It is also possible to see, using the closed-form expressions for these quantities, that the degree of skewness and kurtosis the stochastic volatility model can generate for common parametrizations is less than that of jump-diffusions.

Table 4 summarizes the degree of conditional skewness and kurtosis for a range of possible values of the relevant parameters: $\rho \in \{-0.25, 0\}$, $\kappa \in \{1, 5\}$, $a \in \{0.75, 1, 1.25\}$, and $\eta \in \{0.1, 0.4\}$. The table reveals two important points. First, even for low mean-reversion ($\kappa = 1$), and implausibly high volatility of volatility ($\eta = 0.4$), the model does not generate a high degree of kurtosis at weekly intervals. (For the more reasonable value of $\eta = 0.1$, the model fails to generate a substantial degree of excess kurtosis at any interval.) Second, excess kurtosis is—as we have already seen—an increasing function of the horizon for small to moderate values of h. This suggests, contrary to observations, that the implied volatility smile should be more pronounced at (for instance) three months than at one month. We return to this point in Section 6.2 when discussing implied volatility smiles in the stochastic volatility model.

5 Unconditional Returns in the Stochastic Volatility Model

In Appendix C.1, we describe the derivation of the characteristic function of unconditional returns in the stochastic volatility model using the characteristic function of conditional returns. Successive differentiation of this function delivers all the moments of the unconditional returns. Letting $y = \kappa h$, this results in the following surprisingly simple expressions for unconditional skewness and kurtosis:

$$\text{Skewness}\,(Z(h)) \;=\; 3\left(\frac{\rho\eta}{\sqrt{\kappa\theta}}\right)\left[\frac{1-e^y+ye^y}{y^{3/2}e^y}\right] \tag{21}$$

$$\text{Kurtosis}\,(Z(h)) \;=\; 3\left[1+\frac{\eta^2}{\kappa\theta y^2 e^y}\left(1-e^y+ye^y+4\rho^2[2-2e^y+y+ye^y]\right)\right]. \tag{22}$$

As with conditional returns, expressions (21) and (22) may be used to derive properties of unconditional skewness and kurtosis in the stochastic volatility model. We begin with unconditional skewness.

5.1 Skewness of the Unconditional Returns

Let $S(h)$ denote the skewness of the unconditional returns $Z(h)$. As earlier, when we wish to emphasize the dependence of $S(h)$ on ρ, we will write $S(h;\rho)$. We will show that skewness of the unconditional returns behaves in qualitatively the same way as skewness of the conditional returns. The following preliminary result is important:

Proposition 5.1 *$S(h)$ has the following properties:*

1. *$S(h)$ is positive if $\rho > 0$, zero if $\rho = 0$, and negative if $\rho < 0$.*
2. *$S(h;\rho) = -S(h;-\rho)$.*
3. *$\lim_{h\downarrow 0} S(h) = \lim_{h\uparrow 0} S(h) = 0$.*

Proof See Appendix C.1. □

Properties 1 and 3 of Proposition 5.1 show that unconditional skewness cannot be monotone in h if $\rho \neq 0$. The natural question, therefore, is: how does skewness behave for $h > 0$? The key lies in the behavior of the derivative $S'(h)$. A simple computation shows that when $\rho \neq 0$, we have $S'(h) = 0$ for some $h > 0$ if and only if $y = \kappa h$ satisfies

$$3e^y - ye^y - 2y - 3 = 0. \tag{23}$$

There is only one non-zero solution to (23), which is given approximately by $y = 2.15$. The uniqueness of the solution implies that

- If $\rho \neq 0$, then absolute skewness $|S(h)|$ increases from zero to a maximum and then decreases asymptotically to zero. The maximum occurs approximately at $\kappa h = 2.15$.

A remarkable feature of this result is that the point where skewness reaches its extremum values depends on only a *single* parameter of the model, namely κ. The only part played by ρ is in determining whether this extremum is a maximum (if $\rho > 0$) or a minimum (if $\rho < 0$).

As mentioned above, reasonable values of κ (i.e., those typically found empirically) are in a neighborhood of unity. Even for a value such as $\kappa = 5$, however, the value of h that maximizes unconditional skewness is of the order of $2.15/5 = 0.43$ years, or almost five months. Therefore, for an empirically relevant interval of values of h, unconditional skewness in the stochastic volatility model is *increasing* in h.

5.2 Kurtosis of the Unconditional Returns

Let $K(h)$ denote the kurtosis of the conditional returns given h; the term $K(h;\rho)$ will have the obvious interpretation. We will show in this subsection that the behavior of unconditional kurtosis is very different from (and much more complex than) that of conditional kurtosis. We begin with some important properties of $K(h)$.

Proposition 5.2 *$K(h)$ has the following properties:*

1. *Excess kurtosis is strictly positive everywhere: $K(h) > 0$ at all $h > 0$.*
2. *Excess kurtosis only depends on the absolute value of ρ: $K(h;\rho) = K(h;-\rho)$.*
3. $\lim_{h\downarrow 0} K(h) = 3(1+b)$ *where $b = \eta^2/2\kappa\theta$, and* $\lim_{h\uparrow\infty} K(h) = 0$.

Proof See Appendix C.2. □

Two aspects of this result deserve emphasis. First, in contrast to conditional excess kurtosis, excess kurtosis of the unconditional distribution does *not* approach zero as $h \to 0$. We find this behavior puzzling, and are unable to see an intuitive explanation.

Secondly, since excess kurtosis has a strictly positive limit as $h \to 0$, but goes to zero as $h \to \infty$, Proposition 5.2 does *not* rule out the possibility that excess kurtosis could be monotone in h. To analyze further the behavior of kurtosis as a function of h, we examine its deriative $K'(h)$. Letting $y = \kappa h$, this derivative is seen to be

$$\frac{3\eta^2}{\theta}\left[\frac{(2e^y - 2 - y - ye^y) + 4\rho^2(4e^y - 4 - 3y - y^2 - ye^y)}{y^3 e^y}\right]. \tag{24}$$

Now, the sign of this derivative at $h > 0$ is the same as the sign of the numerator of the term in parenthesis. Denoting this numerator by $g(y)$, we may rewrite g as

$$g(y) = 2(1+8\rho^2)e^y - 2(1+8\rho^2) - (1+12\rho^2)y - (1+4\rho^2)ye^y - 4\rho^2 y^2. \tag{25}$$

Some simple calculation shows that $g(0) = g'(0) = g''(0) = 0$, and that

$$g'''(y) = (4\rho^2 - 1)e^y - (1+4\rho^2)ye^y. \tag{26}$$

If $4\rho^2 < 1$ (or, equivalently, if $|\rho| < \frac{1}{2}$), then both terms on the RHS of (26) are negative for all $y > 0$, so $g'''(y) < 0$ for all $y > 0$. This means, of course, that $g(y)$ is negative at all

$y > 0$. Since the other terms in (24) are all strictly positive at $y > 0$, it follows that (24) is negative at all $y > 0$. We have shown, therefore, that for $|\rho| < \frac{1}{2}$, excess kurtosis is itself strictly decreasing for all $y > 0$.

On the other hand, by considering the special case $\rho = \pm 1$ in (24), it can be shown that there is a unique value of y, denoted say y^* (and given approximately by $y^* = 1.83$), such that kurtosis is increasing in h for $h < y^*/\kappa$, and decreasing in h for $h > y^*/\kappa$.

To sum up the behavior of the kurtosis of unconditional returns:

- When $|\rho| < \frac{1}{2}$, excess kurtosis is a strictly decreasing frunction of h.
- When $|\rho|$ is sufficiently large, excess kurtosis is a hump-shaped function of h, increasing to a maximum and then decreasing asymptotically to zero.

5.3 Values of Unconditional Skewness and Kurtosis

Table 5 summarizes the degree of skewness and kurtosis generated by the unconditional returns of the stochastic volatility model, using the closed-form expressions derived above. The parameter ranges are the same as those considered for the conditional moments in Table 4. The numbers reveal that for the moderate value of $\eta = 0.1$, the stochastic volatility model is capable of generating excess kurtosis at weekly intervals only if mean-reversion is also low. In all cases, excess kurtosis dissipates very slowly, significantly slower than empirical estimates would appear to suggest.

6 The Behavior of Implied Volatilities

To price options under jumps or stochastic volatility, certain of the models' primitive variables must be replaced with their risk-adjusted counterparts (see Bates [7] for the arguments). In the case of the jump-diffusion, assuming that the jump-size distribution is $N(\mu, \gamma^2)$, the parameters that need adjustment are the jump intensity λ and the mean of the jump-size μ; the variance of the jump size γ^2 is not affected in this process. In the case of stochastic volatility, the rate of mean-reversion κ and the long-run mean level θ must both be adjusted; the volatility of volatility η remains unchanged. Lending analytical tractability to this process is the fact that the risk-adjusted parameters are, like their counterparts in the original model, also constants (again, see Bates [7]).

Rather than complicate the paper further by introducing new terms, we will continue to use the notation introduced earlier. It is to be kept in mind, however, that some of the parameters (namely, the ones identified in the last paragraph) are now to be interpreted as the risk-neutralized versions. We discuss the behavior of implied volatilities in jump models first.

6.1 Implied Volatilities under a Jump-Diffusion

Once the appropriate variables have been adjusted for risk, options under a jump-diffusion may be priced using Merton's [34] formula. Implied volatility estimates can then be backed out using the Black–Scholes model as the benchmark.

We carried out this procedure for a large number of parameter configurations. Typical results from these computations are reported in Tables 6 and 7. Table 6 presents implied volatilities for five maturities and seven strike prices when there is zero skewness ($\mu = 0$); Table 7 does likewise for the case of negative skewness ($\mu < 0$). In each case, the choice of parameter configurations is the same as in Table 1 of Section 3. As there, the intermediate value of γ is to be considered the most reasonable choice.

Implied volatility smiles at any maturity may be read off Tables 6 and 7 by considering the row corresponding to the maturity. Since we assume an interest rate of zero, the term-structure of implied volatilities for the at-the-money options is simply the column corresponding to the strike price of 100. Examination of these tables reveals several interesting points about the properties of the volatility smiles. (All the properties we discuss below were present in all the parameter configurations we tried, and not just the ones reported here.)

Concerning the structure of the smile at a fixed maturity (say, one month), we have seen that at low levels of γ, the jump-diffusion model does not produce much excess kurtosis even at monthly intervals (Table 1). Reflecting this, the implied volatility smile is shallow even at short maturities for $\gamma = 0.02$. As γ increases from 0.02 to 0.03, however, the degree of excess kurtosis increaes rapidly (see Table 1), and as a consequence, the smile becomes deeper, comparable to empirically observed levels at short maturities.

Secondly, to move from a two-sided smile to a one-sided smirk requires a substantial amount of skewness in the distribution of the jump-size. Table 7 describes the case $\mu = -0.01$, which corresponds to an average annual return from the jump-component of about -5%.[8] At this value, the smirk at short maturities begins to resemble observed levels. At smaller values of μ (corresponding to, e.g., an annual expected return from the jump component of -3%), the skew was barely noticeable, even at short maturities.

[8] Care has to be taken in interpreting these numbers. Since the value of λ has been risk-neutralized, all contributions of the jump-component to total returns or total variance are contributions in the risk-neutral world.

Thirdly, echoing the rapid decline of skewness and excess kurtosis in the model at longer horizons, the smile flattens out extremely quickly, much more rapidly, indeed, than one would expect in practice. At three months, for example, the implied volatility curve in Table 6 is almost flat unless γ is implausibly high.

The last, and perhaps most important point, concerns the term-structure of at-the-money implied volatilities. Reflecting the montone decreasing nature of excess kurtosis in this model, this term-structure is always a monotone *increasing* function of maturity. In particular, the model is unable to generate *decreasing* term-structures of the sort that have been observed in equity markets in recent years. Equally importantly, the slope of the term-structure implied by the model is tiny: even for unreasonbly high parameter values, the difference between the implied volatilities at one year and one month is only about one percentage point, compared to the two to three percentage points documented in practice (see, e.g., Campa and Chang [13], or Derman and Kani [16]). The jump-diffusion model appears, on both counts, to be an inadequate description of reality.

6.2 Implied Volatilities in the Stochastic Volatility Model

Option prices in the stochastic volatility model may be computed using the method introduced in Heston [22]; this involves a Fourier inversion of the characteristic function of the conditional returns to obtain the risk-neutral density.

This procedure was carried out for a large number of parameter configurations. Typical results are reported in the Tables 8–11. Two values each are considered for the rate of mean-reversion κ, the volatility of volatility η, the correlation ρ, and the initial value of volatility v. The long term mean volatility is fixed at 10%. The values of other parameters are provided in the tables.

Each table summarizes the results for seven strike prices and five maturities. Implied volatility smiles can be read off by considering the row corresponding to any maturity. Since interest rates are taken to be zero, the term-structure of at the money implied volatilities is simply the column corresponding to a strike of 100.

The behavior of implied volatilities in stochastic volatility models reflects the impact of two forces—the kurtosis generated by the parameters, and the impact of mean-reversion. The latter is itself, of course, dependent on two factors—the mean-reversion coefficient κ as well as the level of current volatility in relation to its long-term mean. To separate the effects, Tables 8 and 10 deal with the case where initial volatility is at its long-term mean. Thus, the effects on implied volatilities are primarily on account of kurtosis. In Tables 9 and 11, initial volatility is well below its long-term mean, so mean-reversion also plays a significant role.

An analysis of these tables reveals a number of interesting points. First, we have seen earlier that the stochastic volatility model has only a limited ability to generate excess kurtosis unless η is high. Reflecting this, it can be seen from Tables 8 and 10 that if $\eta = 0.1$, the smiles produced by the model are quite shallow. Matters change when mean-reversion also plays a role. A comparison of the otherwise identical Tables 8 and 9 (or Tables 10 and 11) shows that this factor has a substantial positive impact on the depth of the smile.

Second, the introduction of skewness into the model by considering $\rho = -0.25$ creates a significant asymmetry in the volatility smile, and also increases the depth of the smile. Once again, the impact is much greater if volatility is away from its long-term mean and there is also a significant mean-reversion effect.

Third, we have seen that conditional excess kurtosis increases in the length of the horizon for short to moderate maturities. Reflecting this, it can be seen that in all cases, the depth of the smile does not reduce appreciably as the horizon increases. Indeed, the smile remains quite pronounced even at 12 months in some cases.

Finally, the term-structure of at-the-money implied volatilities displays a variety of possible shapes in this model. When volatility starts out below its long-term mean, mean-reversion pulls it upward. This exerts an upward pull on the term-structure. However, since kurtosis is higher at longer maturities, a downward pull is also exerted on the term-structure. The final shape of the term-structure reflects these effects. When mean-reversion is weak (κ is small) and the kurtosis impact is large (η is high), the latter dominates initially, and a U-shaped term-structure results.[9] However, when mean-reversion is strong and the kurtosis effect is small, a monotone upward sloping structure obtains (see, for example, the case $\kappa = 5$ and $\eta = 0.1$ in Table 9).[10] Thus, stochastic volatility models are able to generate many of the patterns observed in reality. Things are not entirely rosy, however. In all cases, the difference between 12-month implied voaltilities and one-month implied volatilities is around just one percentage point, compared to the 2–3 percentage points that have been documented empirically.

[9] In some cases, the term-structure continues to decline even at 12 months. See Table 9 for the case $\kappa = 1$ and $\eta = 0.1$.

[10] We found analogously, that when volatility starts out above its long-term mean, the term-structure slopes downward, since both the kurtosis effect and mean-reversion work to pull it in the same direction; in the results of brevity, we do not report these results here.

7 Conclusion

This paper has emphasized the importance of looking at the theoretical implications of pricing models at different frequencies. We carried out this task for two of the most common approaches used in the finance literature to generate skewness and kurtosis in returns distributions—jump-diffusions and stochastic or time-varying volatility. We found that the models had dramatically different implications for the term-structures of skewness and kurtosis, both conditionally and unconditionally. Reflecting the differences in conditional returns, implied volatility estimates in the two models behave very differently.

In sum, neither model is able to capture all the aspects of the data adequately. The presence of jumps creates considerable kurtosis at short maturities, but not always at moderate or long maturities; this picture is reversed with stochastic volatility models. Reflecting this, implied volatility smiles tend to die out much quicker in jump-diffusions than empirical observations would appear to suggest; on the other hand, implied volatility smiles in stochastic volatility models sometimes remain quite sharply emphasized at moderate maturities. Finally, although the data presents a great variety of patterns of the term-structure of implied volatilites for at-the-money options, jump-diffusions are only able to produce a single shape (monotone increasing); stochastic volatility models fare much better in this regard.

Finally, we provide closed-form solutions in this paper for the higher-order moments of jump-diffusions and stochastic volatility models. These closed-forms played a central role in our analysis, but they may also find use in areas beyond this immediate paper. They may, for example, be used in the construction of models to estiamte value-at-risk as explained in Duffie and Pan [18]. They may also be used in method of moments estimation procedures where the moments may exploit data at different intervals. To the best of our knowledge, this has not been undertaken so far in the literature.

A Proof of Proposition 3.1

In stochastic differential form, the returns process under the jump-diffusion process of Section 3 may be represented as

$$dx \; = \; \alpha \, dt + \sigma \, dZ + J \, d\pi(\lambda),$$

where α is the drift and σ the standard deviation of the diffusion component, J is the jump size, and π is a Poisson process with intensity parameter λ. In the notation of Section 3, J has distribution G; its first four moments are finite, and are denoted by ν_m, $m = 1, 2, 3, 4$.

We first identify the characteristic function of the distribution of x_{t+h} given $x_t = x$. Denoting this characteristic function by $F(x, s)$, a standard argument shows that F is the solution to the following Kolmogorov Backward Equation, subject to the boundary condition that $F(x, 0) = e^{isx}$:

$$\alpha \frac{\partial F}{\partial x} + \frac{1}{2} \frac{\partial^2 F}{\partial x^2} - \frac{\partial F}{\partial h} + \lambda E[F(x + J) - F(x)] \; = \; 0.$$

It is readily checked that the solution to this differential equation is:

$$F(x, s) \; = \; \exp \left\{ isx + \alpha ish - \frac{1}{2} \sigma^2 s^2 h + \lambda h E(e^{isJ} - 1) \right\}.$$

Since $Z_t(h) = x_{t+h} - x_t$, it is immediate that the characteristic function of $Z_t(h)$ is simply $F^* = e^{-isx} F$. The moments of $Z_t(h)$ may now be computed from the characteristic function F^* by successive differentiation; i.e., if ζ_n represnts the n-th moment of $Z_t(h)$, then

$$\zeta_n \; = \; \frac{1}{i^n} \left[\frac{\partial^n F^*}{\partial s^n} \right] \Bigg|_{s=0}$$

Using these computed moments, it is straightforward to compute the variance, skewness, and kurtosis of the returns $Z_t(h)$ and to verify that they are indeed as provided in Proposition 3.1. □

B Proofs For Section 4

B.1 Deriving the Skewness and Kurtosis

Let x denote the time–t value of the returns process. Let $\bar{F}(x,v,h,s)$ be the characteristic function of x_{t+h} given time–t information. A standard argument (see, e.g., Duffie [17]) establishes that $\bar{F}$ may be obtained as the solution to the Kolmogorov Backward Equation

$$\alpha\frac{\partial \bar{F}}{\partial x} + \frac{1}{2}v\frac{\partial^2 \bar{F}}{\partial x^2} + \kappa(\theta - v)\frac{\partial \bar{F}}{\partial v} + \frac{1}{2}\eta^2 v\frac{\partial^2 \bar{F}}{\partial v^2} - \frac{\partial \bar{F}}{\partial h} + \rho\eta v\frac{\partial^2 \bar{F}}{\partial x \partial v} \;=\; 0, \tag{27}$$

subject to the initial condition

$$\bar{F}(x,v,0;s) \;=\; e^{isx}, \quad \text{for all } x, v, s. \tag{28}$$

Proposition B.1 *The solution of (27)–(28) has the form*

$$\bar{F}(x,v,h;s) \;=\; C(h)\exp[isx + A(h) + vB(h)]. \tag{29}$$

Letting $\gamma = \kappa - \rho\eta i s$ *and* $\phi = \sqrt{\gamma^2 + \eta^2 s^2}$, *we have* $A(h) = i\alpha s h$, *and*

$$B(h) \;=\; \frac{-s^2\left[e^{\phi h} - 1\right]}{(\phi+\gamma)\left[e^{\phi h} - 1\right] + 2\phi}, \qquad C(h) \;=\; \left[\frac{2\phi\left(e^{(\phi+\gamma)\frac{h}{2}}\right)}{(\phi+\gamma)\left[e^{\phi h} - 1\right] + 2\phi}\right]^{\frac{2\kappa\theta}{\eta^2}} \tag{30}$$

Proof This may be verified by direct substitution. □

It is an easy matter now to obtain the characteristic function $F(h,v,s)$ of $Z_t(h)$. Indeed, since $Z_t(h) = x_{t+h} - x$, F is related to the function $\bar{F}$ of Proposition B.1 through $F(v,h,s) = \bar{F}(x,v,h,s)e^{-isx}$. Substituting for $\bar{F}(x,v,h,s)$ from (29), we obtain

$$F^*(v,h,s) = C(h)\exp\{A(h) + vB(h)\}, \tag{31}$$

where $A(\cdot)$, $B(\cdot)$, and $C(\cdot)$ are as given in Proposition B.1.

Expression (31) may now be used to obtain the moments of $Z_t(h)$ in the usual way: if we let $\xi_m = E[(Z_t(h))^m]$, then we have

$$\xi_m \;=\; \frac{1}{i^m}\cdot\frac{\partial^m F^*}{\partial s^m}(v,h;s)\bigg|_{s=0}$$

Grinding through the relevant computations yields the expressions for skewness and kurtosis presented in Section 4.

B.2 Proof of Proposition 4.2

Property 1 of Proposition 4.2 will be established if we can show that the last term in parenthesis on the RHS of (16) is positive. We will first show that the denominator is positive, and then that the numerator is positive.

To show that the denominator is positive, it suffices to show that

$$\theta(1 - e^y + ye^y) + v(e^y - 1)] > 0 \quad \text{for all } y > 0. \tag{32}$$

Consider first the term $1 - e^y + ye^y$. At $y = 0$, this term is equal to zero. Its derivative at any $y > 0$ is given by the strictly positive quantity ye^y. Therefore, this term is strictly positive at all $y > 0$. The term $(e^y - 1)$ is obviously also strictly positive at all $y > 0$. Since θ and v are positive, (32) is established.

Turning to the numerator, it evidently suffices to show that that all $y > 0$, we have

$$2 - 2e^y + y + ye^y > 0 \quad \text{and} \quad 1 + y - e^y < 0. \tag{33}$$

For notational ease, let $h(y) = 2 - 2e^y + y + ye^y$. It is easily checked that $h(0) = h'(0) = 0$, and that $h''(y) = ye^y > 0$ for all $y > 0$. It follows immediately that $h(y)$ is positive for all $y > 0$, establishing the first part of (33). The other part is trivial since, by definition, $e^y = \sum_{n=0}^{\infty}(y^n/n!) > 1 + y$ if $y > 0$. This completes the proof of Property 1.

Property 2 is immediate from the definition of $S(h)$ in (16). Finally, to see property 3, let

$$g(y) = e^{\frac{1}{2}y}\left[\frac{\theta(2 - 2e^y + y + ye^y) - v(1 + y - e^y)}{(\theta(1 - e^y + ye^y) + v(e^y - 1))^{3/2}}\right].$$

The expression on the right-hand side of this equation has the form $0/0$ at $y = 0$, and ∞/∞ as $y \uparrow \infty$. However, using L'Hopital's rule repeatedly, it can be seen that $\lim_{y\downarrow 0} g(y) = 0$ and that $\lim_{y\uparrow\infty} g(y) = 0$. From (16) it is immediate that

$$\lim_{h\downarrow 0} S(h) \;=\; \frac{3\eta\rho}{\sqrt{k}} \cdot \lim_{y\downarrow 0} g(y) \quad \& \quad \lim_{h\uparrow\infty} S(h) \;=\; \frac{3\eta\rho}{\sqrt{k}} \cdot \lim_{y\uparrow\infty} g(y).$$

It follows that $S(h)$ goes to zero as $h \downarrow 0$ and $h \uparrow \infty$, completing the proof of the proposition. □

In the remainder of this subsection, we prove that conditional (absolute) skewness is a hump-shaped function of h. To see this, observe first from (16) that although other parameters affect the *level* of conditional skewness, the pattern of dependence on h only depends on only *two* variables: the rate of mean-reversion κ, and the ratio $a = v/\theta$ of current volatility to its long-term mean. Now consider the derivative $S_t'(h)$. It can be seen by direct calculation that h satisfies $S_t'(h) = 0$ if and only if $y = \kappa h$ satisfies

$$\begin{aligned}(1-a)(4-3a) + (1-a)^2 y - (8 - 8a + 3a^2)e^y \\ - 2a(2-a)ye^y + (4-a)e^{2y} - ye^{2y} - 2(1-a)y^2e^y &= 0.\end{aligned} \tag{34}$$

Equation (34) has a unique solution for each value of a. Combined with Properties 1 and 3 of Propostion 4.1, this uniqueness implies that

- If $\rho > 0$, skewness increases from zero to a maximum, and then decreases asymptotically back to zero.
- If $\rho < 0$, skewness decreases from zero to a minimum, and then increases asymptotically back to zero.

Finally, note that the value of h which maximizes $S_t(h)$ is simply the value of h for which (34) holds. Thus, the maximizing value of h only depends on two numbers: κ and $a = v/\theta$. Table 2 describes the way the maximizing value of h changes as these parameters change.

B.3 Proof of Proposition 4.3

To prove Property 1, it suffices to show that $A_1 > 0$ and $A_2 < 0$. Now, A_1 is of the form $f(y) + 4\rho^2 g(y)$, where

$$f(y) = 1 + 4e^y - 5e^{2y} + 4ye^y + 2ye^{2y},$$

and

$$g(y) = 6e^y - 6e^{2y} + 4ye^y + 2ye^{2y} + y^2e^y.$$

A simple computation shows that $f(0) = 0$ and $f'(y) = 4e^y(2 - 2e^y + y + ye^y)$. The term in parenthesis is strictly positive for $y > 0$, as we showed in the proof of Proposition 4.2 (see equation (33). Therefore, $f'(y) > 0$ for all $y > 0$, and so $f(y) > 0$ for all $y > 0$.

The argument for $g(y)$ is similar. It can be checked that $g(0) = g'(0) = g''(0) = g'''(0) = 0$, and that $g^{(4)}(y) > 0$ for all $y > 0$. Therefore, $g(y) > 0$ for all $y > 0$.

Since f and g are both strictly positive functions, A_1 is strictly positive. That $A_2 < 0$ is checked similarly. The details are omitted. This proves Property 1.

Property 2 is immediate from (17) since $K(h)$ only depends on ρ^2. This leaves Property 3. Let

$$h(y) = [\theta(1 - e^y + ye^y) + v(e^y - 1)]^2.$$

Then, for $y = \kappa h$, we have

$$K(h) \;=\; 3\left[1 + \frac{\eta^2}{2k}\left(\frac{\theta A_1 - vA_2}{h(y)}\right)\right].$$

Now, $A_1/h(y) = [f(y)/h(y)] + 4\rho^2[g(y)/h(y)]$, where f and g were defined above. Repeated use of L'Hopital's rule shows that the limits as $y \downarrow 0$ and $y \uparrow \infty$ of $[f(y)/h(y)]$ and $[g(y)/h(y)]$ are all zero. Thus, $A_1/h(y)$ tends to zero as y tends to zero or becomes unbounded. A similar argument shows that $A_2/h(y)$ also tends to zero as y tends to zero or to infinity. Since κ is fixed, $y \downarrow 0$ if and only if $h \downarrow 0$, and $y \uparrow \infty$ only if $h \uparrow \infty$, so we finally obtain

$$\lim_{h\downarrow 0} K(h) = \lim_{h\uparrow\infty} K(h) = 0.$$

This completes the proof of the proposition. □

C Proofs for Section 5

We begin with a derivation of the characteristic function of unconditional returns. In Appendix B.1, we showed that *conditional* on $V_t = v$, the characteristic function $F(v, h, s)$ of $Z_t(h)$ has the form

$$D(h, s)\exp\{vB(h, s)\}. \tag{35}$$

Now, it is well known that the square-root process (14) has a stationary density that is gamma, given by

$$g(v) \;=\; \frac{1}{\Gamma(\nu)}\omega^\nu v^{\nu-1}e^{-\omega v} \tag{36}$$

where $\omega = 2\kappa/\eta^2$ and $\nu = 2\kappa\theta/\eta^2$. Combining (35) and (36), the characteristic function of the *unconditional* returns $Z(h)$, denoted say $P(h,s)$, is given by

$$P(h,s) \;=\; \int F^*(v,h,s)g(v)dv \;=\; D\int \exp\{vB\}g(v)dv. \tag{37}$$

But this last integral on the right-hand side is simply the moment generating function $E(e^{tv})$ of the gamma distribution evaluated at $t = B$. It is well-known that this function has the form $[\omega/(\omega - B)]^\nu$. Therefore, P has the form

$$P(h,s) \;=\; D(h,s)\left(\frac{\omega}{\omega - B(h,s)}\right)^\nu. \tag{38}$$

Successive differentiation of the function P delivers all the moments of the unconditional returns, and leads, in particular, to the forms of skewness and kurtosis provided in Section 5.

C.1 Proof of Proposition 5.1

For ease of reference, we reproduce the form of skewness here (recall that $y = \kappa h$):

$$3\left(\frac{\rho\eta}{\sqrt{\kappa\theta}}\right)\left[\frac{1 - e^y + ye^y}{y^{3/2}e^y}\right]. \tag{39}$$

Property 1 of Proposition 5.1 will be proved if we can show that the term $(1 - e^y + ye^y)$ is strictly positive for all $y > 0$. But we have already done this in the proof of Proposition 4.2 above (see equation (32)).

Property 2 of Proposition 5.1 is immediate from the form of skewness (39). This leaves Property 3. Now, the limits as $h \downarrow 0$ and $h \uparrow \infty$ of $S(h)$ are clearly determined entirely by the limits as $y \downarrow 0$ and $y \uparrow \infty$ of the function $h(y)$ defined by

$$h(y) \;=\; \frac{1 - e^y + ye^y}{y^{3/2}e^y}. \tag{40}$$

Indeed, we need only show that $h(y)$ tends to zero as y tends to 0 or $+\infty$.

As $y \downarrow 0$, both the numerator and the denominator of the right-hand side of (40) approach zero. However, repeated application of L'Hopital's rule shows that $\lim_{y\downarrow 0} h(y) = 0$, establishing one part of the desired result.

To see the other part, note that $h(y)$ can be rewritten as

$$h(y) = \frac{1}{y^{3/2}e^y} - \frac{1}{y^{3/2}} + \frac{1}{y^{1/2}}. \tag{41}$$

Since each term on the right-hand side of (41) tends to zero as $y \uparrow \infty$, it is clearly the case that $h(y)$ also goes to zero as $y \uparrow \infty$. This completes the proof of Proposition 5.1

C.2 Proof of Proposition 5.2

For notational ease, define

$$g(y) \;=\; \frac{(1+8\rho^2) - (1+8\rho^2)e^y + (1+4\rho^2)ye^y + 4\rho^2 y}{y^2 e^y}.$$

Then, we have

$$K(h) \;=\; 3\left[1 + \left(\frac{\eta^2}{\kappa\theta}\right) g(\kappa h)\right].$$

To prove Property 1 now, it suffices to show that $g(y) > 0$ at all $y > 0$. A simple calculation shows that we have $g(0) = g'(0) = 0$, and that

$$g''(y) = e^y + (1+4\rho^2)ye^y > 0 \quad \text{for all } y > 0.$$

It follows that $g(y)$ is strictly positive (as is $g'(y)$) for $y > 0$, proving Property 1.

Property 2 is an immediate consequence of the fact that $K(h)$ only depends on ρ^2. Finally, to see Property 3, note that

$$\lim_{h\downarrow 0} K(h) \;=\; 3\left[1 + \frac{\eta^2}{\kappa\theta}(\lim_{y\downarrow 0} g(y))\right], \tag{42}$$

and

$$\lim_{h\uparrow\infty} K(h) \;=\; 3\left[1 + \frac{\eta^2}{\kappa\theta}(\lim_{y\uparrow\infty} g(y))\right]. \tag{43}$$

Repeated applications of L'Hopital's rule to g establishes that

$$\lim_{y\downarrow 0} g(y) = \frac{1}{2} \quad \text{and} \quad \lim_{y\uparrow\infty} g(y) = 0. \tag{44}$$

Substituting (44) into (42) and (43) completes the proof of Property 3. $\square$

Table 1: Skewness and Kurtosis in the Jump-Diffusion Model

This table presents the values of skewness and kurtosis in the jump-diffusion model when $\lambda = 5$ and $G \sim N(\mu, \gamma^2)$, for a range of values of μ and γ. To facilitate comparison across these values, the total variance of annual returns is fixed at approximately 0.02 in all cases. When $\gamma = 0.02$, this means about 10% of the annual variance comes from the jump-component; the figure rises to about 23% when $\gamma = 0.03$, and to over 60% when $\gamma = 0.05$.

Parameters			Skewness			Excess Kurtosis		
σ	μ	γ	1 week	1 month	3 months	1 week	1 month	3 months
0.134	0	0.02	0	0	0	0.312	0.072	0.024
0.134	−0.001	0.02	−0.765	−0.367	−0.212	0.313	0.072	0.024
0.134	−0.01	0.02	−7.383	−3.545	−2.045	0.452	0.104	0.035
0.125	0	0.03	0	0	0	1.580	0.365	0.122
0.125	−0.001	0.03	−1.147	−0.551	−0.318	1.582	0.365	0.122
0.125	−0.01	0.03	−11.068	−5.317	−3.070	1.844	0.425	0.142
0.087	0	0.05	0	0	0	12.188	2.813	0.938
0.087	−0.001	0.05	−1.911	−0.918	−0.530	12.190	2.813	0.938
0.087	−0.01	0.05	−18.438	−8.857	−5.114	12.534	2.892	0.964

Table 2: Values of h that Maximize Conditional Skewness

This table presents the values of the horizon h (measured in years) at which conditional (absolute) skewness $|S_t(h)|$ in the stochastic volatility model is maximized. We consider a range of values for the other two parameters that affect this maximum: the ratio a of current volatility to its long-term mean, and the rate of mean-reversion κ.

κ	a	h (years)	κ	a	h (years)
0.10	3.00	31.11	5.00	3.00	0.62
0.10	2.00	27.31	5.00	2.00	0.55
0.10	1.00	21.49	5.00	1.00	0.43
0.10	0.50	16.57	5.00	0.50	0.33
0.10	0.33	14.06	5.00	0.33	0.28
1.00	3.00	3.11	50.00	3.00	0.06
1.00	2.00	2.73	50.00	2.00	0.05
1.00	1.00	2.15	50.00	1.00	0.04
1.00	0.50	1.66	50.00	0.50	0.03
1.00	0.33	1.41	50.00	0.33	0.03

Table 3: Values of h that Maximize Conditional Kurtosis

This table presents the values of the horizon h (measured in years) at which conditional kurtosis $K_t(h)$ in the stochastic volatility model is maximized. We consider a range of values for the other three parameters that affect the location of this maximum: the mean-reversion coefficient κ of the volatility process, the correlation ρ between the returns and volatility processes, and the ratio a of current volatility to its long-term mean.

κ	a	ρ	h (years)	κ	a	ρ	h (years)
0.10	2.00	0.00	24.92	5.00	2.00	0.00	0.4984
0.10	2.00	0.25	25.47	5.00	2.00	0.25	0.5095
0.10	2.00	0.50	26.13	5.00	2.00	0.50	0.5226
0.10	1.00	0.00	18.93	5.00	1.00	0.00	0.3785
0.10	1.00	0.25	19.18	5.00	1.00	0.25	0.3826
0.10	1.00	0.50	19.52	5.00	1.00	0.50	0.3904
0.10	0.50	0.00	13.94	5.00	0.50	0.00	0.2788
0.10	0.50	0.25	13.94	5.00	0.50	0.25	0.2788
0.10	0.50	0.50	13.94	5.00	0.50	0.50	0.2788
1.00	2.00	0.00	2.49	50.00	2.00	0.00	0.0498
1.00	2.00	0.25	2.55	50.00	2.00	0.25	0.0510
1.00	2.00	0.50	2.61	50.00	2.00	0.50	0.0523
1.00	1.00	0.00	1.89	50.00	1.00	0.00	0.0379
1.00	1.00	0.25	1.91	50.00	1.00	0.25	0.0383
1.00	1.00	0.50	1.95	50.00	1.00	0.50	0.0390
1.00	0.50	0.00	1.39	50.00	0.50	0.00	0.0279
1.00	0.50	0.25	1.39	50.00	0.50	0.25	0.0279
1.00	0.50	0.50	1.39	50.00	0.50	0.50	0.0279

Table 4: Conditional Skewness and Kurtosis in the Stochastic Volatility Model

This table presents the values of conditional skewness and kurtosis at various horizons in the stochastic volatility model for a range of values of the four relevant parameters: the correlation ρ between the returns and volatility processes, the coefficient κ of mean-reversion in the volatility process, the ratio a of current volatility to its long-term mean, and the volatility of volatility η.

Parameters				Skewness			Kurtosis		
ρ	κ	a	η	1 week	1 month	3 months	1 week	1 month	3 months
0	1	0.75	0.1	0.00	0.00	0.00	0.03	0.10	0.26
0	1	1.00	0.1	0.00	0.00	0.00	0.02	0.08	0.21
0	1	1.25	0.1	0.00	0.00	0.00	0.02	0.06	0.17
0	5	0.75	0.1	0.00	0.00	0.00	0.02	0.08	0.12
0	5	1.00	0.1	0.00	0.00	0.00	0.02	0.06	0.11
0	5	1.25	0.1	0.00	0.00	0.00	0.01	0.05	0.10
-0.25	1	0.75	0.1	-0.06	-0.12	-0.19	0.03	0.13	0.33
-0.25	1	1.00	0.1	-0.05	-0.11	-0.17	0.02	0.10	0.26
-0.25	1	1.25	0.1	-0.05	-0.09	-0.16	0.02	0.08	0.22
-0.25	5	0.75	0.1	-0.06	-0.10	-0.14	0.03	0.10	0.15
-0.25	5	1.00	0.1	-0.05	-0.09	-0.13	0.02	0.08	0.14
-0.25	5	1.25	0.1	-0.05	-0.09	-0.12	0.02	0.07	0.13
0	1	0.75	0.4	0.00	0.00	0.00	0.40	1.64	4.20
0	1	1.00	0.4	0.00	0.00	0.00	0.30	1.25	3.33
0	1	1.25	0.4	0.00	0.00	0.00	0.24	1.01	2.76
0	5	0.75	0.4	0.00	0.00	0.00	0.37	1.21	1.94
0	5	1.00	0.4	0.00	0.00	0.00	0.29	0.99	1.73
0	5	1.25	0.4	0.00	0.00	0.00	0.23	0.83	1.56
-0.25	1	0.75	0.4	-0.24	-0.48	-0.77	0.50	2.05	5.27
-0.25	1	1.00	0.4	-0.21	-0.42	-0.69	0.38	1.57	4.19
-0.25	1	1.25	0.4	-0.19	-0.38	-0.63	0.30	1.27	3.47
-0.25	5	0.75	0.4	-0.23	-0.42	-0.54	0.47	1.52	2.45
-0.25	5	1.00	0.4	-0.20	-0.38	-0.52	0.36	1.25	2.21
-0.25	5	1.25	0.4	-0.18	-0.35	-0.49	0.29	1.05	2.01

Table 5: Unconditional Skewness and Kurtosis in the Stochastic Volatility Model

This table presents the values of unconditional skewness and kurtosis at various horizons in the stochastic volatility model for a range of values of the four relevant parameters: the correlation ρ between the returns and volatility processes, the coefficient κ of mean-reversion in the volatility process, and the volatility of volatility η.

Parameters			Skewness			Kurtosis		
ρ	k	η	1 week	1 month	3 months	1 week	1 month	3 months
0	1	0.1	0.00	0.00	0.00	1.49	1.46	1.38
0	5	0.1	0.00	0.00	0.00	0.29	0.26	0.21
-0.25	1	0.1	-0.05	-0.11	-0.17	1.49	1.47	1.41
-0.25	5	0.1	-0.05	-0.09	-0.13	0.29	0.27	0.22
0	1	0.4	0.00	0.00	0.00	23.85	23.35	22.12
0	5	0.4	0.00	0.00	0.00	4.65	4.20	3.30
-0.25	1	0.4	-0.21	-0.42	-0.69	23.89	23.51	22.56
-0.25	5	0.4	-0.20	-0.38	-0.52	4.69	4.33	3.57

Table 6: Implied Volatilties in the Jump-Diffusion Model

This table presents implied volatilities in the jump-diffusion model when there is zero skewness. The jump-intensity is $\lambda = 5$ and the jump-size G is distributed $N(0, \gamma^2)$. Results are presented for the same range of values of γ and σ considered in Table 1. To facilitate comparison across these values, the total variance of annual returns is fixed at approximately 0.02 in all cases. When $\gamma = 0.02$, this means about 10% of the annual variance comes from the jump-component; the figure rises to about 23% when $\gamma = 0.03$, and to over 60% when $\gamma = 0.05$. Seven strike prices are considered ranging from 85 to 115. The current price of the stock is taken to be 100, and the interest rate to be zero.

Zero Skewness: $\lambda = 5, S = 100, r = 0$										
μ	γ	σ	Months	85	90	95	100	105	110	115
0	0.02	0.1342	1	0.1480	0.1436	0.1416	0.1410	0.1416	0.1433	0.1463
0	0.02	0.1342	2	0.1428	0.1418	0.1414	0.1412	0.1414	0.1418	0.1425
0	0.02	0.1342	3	0.1420	0.1416	0.1413	0.1413	0.1414	0.1416	0.1419
0	0.02	0.1342	6	0.1415	0.1414	0.1414	0.1414	0.1414	0.1414	0.1415
0	0.02	0.1342	12	0.1414	0.1414	0.1414	0.1414	0.1414	0.1414	0.1414
0	0.03	0.1245	1	0.1631	0.1507	0.1421	0.1396	0.1422	0.1494	0.1591
0	0.03	0.1245	2	0.1476	0.1434	0.1411	0.1404	0.1412	0.1432	0.1463
0	0.03	0.1245	3	0.1440	0.1421	0.1410	0.1407	0.1411	0.1421	0.1435
0	0.03	0.1245	6	0.1419	0.1414	0.1411	0.1411	0.1412	0.1415	0.1418
0	0.03	0.1245	12	0.1414	0.1413	0.1413	0.1413	0.1413	0.1414	0.1415
0	0.05	0.0866	1	0.1995	0.1768	0.1468	0.1280	0.1475	0.1748	0.1937
0	0.05	0.0866	2	0.1677	0.1538	0.1396	0.1336	0.1406	0.1532	0.1647
0	0.05	0.0866	3	0.1558	0.1464	0.1387	0.1360	0.1395	0.1466	0.1541
0	0.05	0.0866	6	0.1448	0.1414	0.1392	0.1387	0.1397	0.1419	0.1447
0	0.05	0.0866	12	0.1415	0.1406	0.1401	0.1401	0.1404	0.1410	0.1419

Table 7: Implied Volatilities in the Jump-Diffusion Model

This table presents implied volatilities in the jump-diffusion model when there is negative skewness. The jump-intensity is $\lambda = 5$ and the jump-size G is distributed $N(\mu, \gamma^2)$, where $\mu = -0.01$. Results are presented for the same range of values of γ and σ considered in Table 1. To facilitate comparison across these values, the total variance of annual returns is fixed at approximately 0.02 in all cases. When $\gamma = 0.02$, this means about 10% of the annual variance comes from the jump-component; the figure rises to about 23% when $\gamma = 0.03$, and to over 60% when $\gamma = 0.05$. Seven strike prices are considered ranging from 85 to 115. The current price of the stock is taken to be 100, and the interest rate to be zero.

Negative Skewness: $\lambda = 5, S = 100, r = 0$										
μ	γ	σ	Months	85	90	95	100	105	110	115
-0.01	0.02	0.1342	1	0.1581	0.1505	0.1453	0.1425	0.1413	0.1410	0.1417
-0.01	0.02	0.1342	2	0.1484	0.1459	0.1440	0.1428	0.1420	0.1415	0.1412
-0.01	0.02	0.1342	3	0.1461	0.1447	0.1437	0.1429	0.1423	0.1419	0.1416
-0.01	0.02	0.1342	6	0.1444	0.1438	0.1434	0.1430	0.1426	0.1424	0.1421
-0.01	0.02	0.1342	12	0.1437	0.1434	0.1432	0.1430	0.1429	0.1427	0.1426
-0.01	0.03	0.1245	1	0.1749	0.1604	0.1477	0.1409	0.1397	0.1430	0.1499
-0.01	0.03	0.1245	2	0.1559	0.1495	0.1448	0.1418	0.1405	0.1405	0.1416
-0.01	0.03	0.1245	3	0.1503	0.1467	0.1440	0.1422	0.1411	0.1407	0.1408
-0.01	0.03	0.1245	6	0.1459	0.1445	0.1434	0.1426	0.1419	0.1415	0.1412
-0.01	0.03	0.1245	12	0.1442	0.1436	0.1432	0.1427	0.1424	0.1421	0.1419
-0.01	0.05	0.0866	1	0.2112	0.1877	0.1566	0.1290	0.1389	0.1645	0.1830
-0.01	0.05	0.0866	2	0.1777	0.1629	0.1466	0.1347	0.1354	0.1451	0.1557
-0.01	0.05	0.0866	3	0.1648	0.1543	0.1442	0.1372	0.1361	0.1403	0.1464
-0.01	0.05	0.0866	6	0.1515	0.1467	0.1426	0.1398	0.1385	0.1387	0.1400
-0.01	0.05	0.0866	12	0.1459	0.1440	0.1424	0.1412	0.1404	0.1400	0.1398

Table 8: Implied Volatilities in the Stochastic Volatility Model

This table presents implied volatilities in the stochastic volatility model when there is no skewness and the initial volatility is at its long-term mean level. Two levels each are considered for the mean-reversion coefficient κ and the volatility of volatility η. Seven strike prices are considered ranging from 85 to 115. The current price of the stock is 100. The interest rate r is zero.

$S = 100, r = 0, \theta = 0.01$									
Correlation $\rho = 0$, Initial Volatility $v_0 = 0.0100$									
Parameters		Maturity	Strike Prices						
k	η	Months	85	90	95	100	105	110	115
1	0.1	1	0.1000	0.1035	0.1007	0.0996	0.1006	0.1029	0.1071
1	0.1	2	0.1069	0.1031	0.1004	0.0994	0.1003	0.1025	0.1053
1	0.1	3	0.1064	0.1027	0.1001	0.0991	0.1000	0.1021	0.1049
1	0.1	6	0.1051	0.1017	0.0994	0.0985	0.0993	0.1011	0.1036
1	0.1	12	0.1029	0.1002	0.0985	0.0979	0.0985	0.0998	0.1018
1	0.4	1	0.1000	0.1187	0.1076	0.0947	0.1068	0.1197	0.1000
1	0.4	2	0.1374	0.1225	0.1044	0.0910	0.1036	0.1194	0.1324
1	0.4	3	0.1364	0.1198	0.1013	0.0883	0.1004	0.1165	0.1302
1	0.4	6	0.1276	0.1112	0.0940	0.0834	0.0932	0.1081	0.1214
1	0.4	12	0.1108	0.0977	0.0851	0.0785	0.0846	0.0953	0.1057
5	0.1	1	0.1000	0.1028	0.1006	0.0997	0.1005	0.1023	0.1000
5	0.1	2	0.1048	0.1020	0.1002	0.0996	0.1002	0.1016	0.1036
5	0.1	3	0.1037	0.1015	0.1000	0.0995	0.1000	0.1011	0.1028
5	0.1	6	0.1018	0.1005	0.0998	0.0995	0.0998	0.1004	0.1013
5	0.1	12	0.1005	0.1000	0.0997	0.0996	0.0997	0.1000	0.1003
5	0.4	1	0.1000	0.1200	0.1065	0.0959	0.1057	0.1186	0.1000
5	0.4	2	0.1321	0.1177	0.1029	0.0943	0.1023	0.1149	0.1266
5	0.4	3	0.1271	0.1135	0.1005	0.0938	0.1000	0.1110	0.1218
5	0.4	6	0.1158	0.1056	0.0974	0.0939	0.0971	0.1039	0.1117
5	0.4	12	0.1055	0.1001	0.0965	0.0952	0.0964	0.0993	0.1032

Table 9: Implied Volatilities in the Stochastic Volatility Model

This table presents implied volatilities in the stochastic volatility modelwhen there is no skewness and the initial volatility is at its long-term mean level. Two levels each are considered for the mean-reversion coefficient κ and the volatility of volatility η. Seven strike prices are considered ranging from 85 to 115. The current price of the stock is 100. The interest rate r is zero.

$S = 100, r = 0, \theta = 0.01$									
Correlation $\rho = 0$, Initial Volatility $v_0 = 0.0075$									
Parameters		Maturity	Strike Prices						
k	η	Months	85	90	95	100	105	110	115
1	0.1	1	0.1000	0.0910	0.0883	0.0867	0.0882	0.0911	0.1000
1	0.1	2	0.0969	0.0921	0.0885	0.0870	0.0883	0.0913	0.0949
1	0.1	3	0.0967	0.0921	0.0886	0.0873	0.0885	0.0913	0.0948
1	0.1	6	0.0962	0.0921	0.0892	0.0881	0.0891	0.0914	0.0945
1	0.1	12	0.0956	0.0924	0.0904	0.0896	0.0903	0.0920	0.0942
1	0.4	1	0.1000	0.0767	0.0966	0.0811	0.0959	0.0704	0.1000
1	0.4	2	0.1010	0.1132	0.0946	0.0780	0.0936	0.1105	0.1209
1	0.4	3	0.1281	0.1116	0.0918	0.0759	0.0908	0.1083	0.1223
1	0.4	6	0.1210	0.1039	0.0854	0.0731	0.0846	0.1006	0.1145
1	0.4	12	0.1057	0.0921	0.0786	0.0712	0.0781	0.0896	0.1005
5	0.1	1	0.1000	0.0926	0.0900	0.0888	0.0899	0.0922	0.1000
5	0.1	2	0.0970	0.0937	0.0915	0.0907	0.0914	0.0932	0.0956
5	0.1	3	0.0970	0.0944	0.0927	0.0921	0.0927	0.0940	0.0959
5	0.1	6	0.0974	0.0959	0.0951	0.0948	0.0951	0.0957	0.0967
5	0.1	12	0.0981	0.0975	0.0972	0.0971	0.0972	0.0975	0.0978
5	0.4	1	0.1000	0.0767	0.0974	0.0848	0.0966	0.1034	0.1000
5	0.4	2	0.1253	0.1112	0.0954	0.0853	0.0946	0.1084	0.1203
5	0.4	3	0.1220	0.1079	0.0940	0.0863	0.0934	0.1053	0.1166
5	0.4	6	0.1122	0.1017	0.0930	0.0893	0.0927	0.0999	0.1080
5	0.4	12	0.1034	0.0979	0.0941	0.0927	0.0939	0.0970	0.1011

Table 10: Implied Volatilities in the Stochastic Volatility Model

This table presents implied volatilities in the stochastic volatility model when there is negative skewness and the initial volatility is below its long-term mean level. The neggative skewness is caused by the negative correlation of $\rho = -0.25$ in changes in the volatility and returns processes. Two levels each are considered for the mean-reversion coefficient κ and the volatility of volatility η. Seven strike prices are considered ranging from 85 to 115. The current price of the stock is 100. The interest rate r is zero.

$S = 100, r = 0, \theta = 0.01$									
Correlation $\rho = -0.25$, Initial Volatility $v_0 = 0.0100$									
Parameters		Maturity	Strike Prices						
k	η	Months	85	90	95	100	105	110	115
1	0.1	1	0.1000	0.1086	0.1036	0.0996	0.0976	0.0974	0.1000
1	0.1	2	0.1135	0.1080	0.1031	0.0993	0.0973	0.0970	0.0982
1	0.1	3	0.1129	0.1075	0.1028	0.0990	0.0970	0.0967	0.0978
1	0.1	6	0.1112	0.1062	0.1018	0.0984	0.0965	0.0960	0.0968
1	0.1	12	0.1086	0.1043	0.1006	0.0977	0.0960	0.0953	0.0956
1	0.4	1	0.1644	0.1365	0.1155	0.0947	0.0982	0.1117	0.1232
1	0.4	2	0.1515	0.1327	0.1117	0.0908	0.0949	0.1083	0.1204
1	0.4	3	0.1480	0.1293	0.1081	0.0879	0.0920	0.1054	0.1176
1	0.4	6	0.1389	0.1200	0.0999	0.0829	0.0857	0.0979	0.1097
1	0.4	12	0.1265	0.1087	0.0920	0.0794	0.0798	0.0885	0.0978
5	0.1	1	0.1000	0.1076	0.1032	0.0997	0.0978	0.0973	0.1000
5	0.1	2	0.1108	0.1063	0.1025	0.0995	0.0978	0.0971	0.0974
5	0.1	3	0.1093	0.1054	0.1021	0.0995	0.0978	0.0970	0.0970
5	0.1	6	0.1062	0.1035	0.1013	0.0994	0.0981	0.0972	0.0968
5	0.1	12	0.1035	0.1019	0.1006	0.0995	0.0986	0.0979	0.0974
5	0.4	1	0.1610	0.1334	0.1138	0.0959	0.0974	0.1094	0.1308
5	0.4	2	0.1437	0.1272	0.1095	0.0941	0.0944	0.1039	0.1141
5	0.4	3	0.1379	0.1224	0.1065	0.0934	0.0926	0.1004	0.1094
5	0.4	6	0.1255	0.1133	0.1019	0.0935	0.0913	0.0947	0.1004
5	0.4	12	0.1141	0.1062	0.0996	0.0948	0.0925	0.0926	0.0945

Table 11: Implied Volatilities in the Stochastic Volatility Model

This table presents implied volatilities in the stochastic volatility model when there is negative skewness and the initial volatility is below its long-term mean level. The negative skewness is caused by negative correlation of $\rho = -0.25$ in changes in the returns and volatility processes. Two levels each are considered for the mean-reversion coefficient κ and the volatility of volatility η. Seven strike prices are considered ranging from 85 to 115. The current price of the stock is 100. The interest rate r is zero.

$S = 100, r = 0, \theta = 0.01$									
Correlation $\rho = -0.25$, Initial Volatility $v_0 = 0.0075$									
Parameters		Maturity	Strike Prices						
k	η	Months	85	90	95	100	105	110	115
1	0.1	1	0.1286	0.0978	0.0915	0.0867	0.0848	0.0859	0.1098
1	0.1	2	0.1036	0.0973	0.0915	0.0870	0.0850	0.0855	0.0876
1	0.1	3	0.1032	0.0971	0.0916	0.0872	0.0852	0.0856	0.0875
1	0.1	6	0.1024	0.0968	0.0918	0.0879	0.0860	0.0861	0.0875
1	0.1	12	0.1013	0.0966	0.0925	0.0894	0.0877	0.0872	0.0879
1	0.4	1	0.1000	0.1232	0.1052	0.0812	0.0878	0.0704	0.1000
1	0.4	2	0.1433	0.1241	0.1018	0.0778	0.0854	0.1004	0.1107
1	0.4	3	0.1404	0.1209	0.0984	0.0756	0.0828	0.0981	0.1108
1	0.4	6	0.1311	0.1120	0.0910	0.0725	0.0773	0.0910	0.1033
1	0.4	12	0.1210	0.1027	0.0854	0.0720	0.0735	0.0834	0.0934
5	0.1	1	0.1239	0.0980	0.0928	0.0888	0.0870	0.0871	0.1000
5	0.1	2	0.1031	0.0982	0.0939	0.0906	0.0888	0.0884	0.0892
5	0.1	3	0.1026	0.0984	0.0948	0.0921	0.0904	0.0897	0.0900
5	0.1	6	0.1018	0.0990	0.0966	0.0947	0.0934	0.0926	0.0922
5	0.1	12	0.1011	0.0995	0.0981	0.0970	0.0961	0.0954	0.0949
5	0.4	1	0.1574	0.1257	0.1050	0.0848	0.0886	0.1017	0.1000
5	0.4	2	0.1379	0.1206	0.1019	0.0851	0.0868	0.0979	0.1091
5	0.4	3	0.1325	0.1166	0.0999	0.0860	0.0861	0.0950	0.1046
5	0.4	6	0.1218	0.1093	0.0975	0.0888	0.0869	0.0909	0.0970
5	0.4	12	0.1119	0.1039	0.0972	0.0923	0.0901	0.0903	0.0923

References

[1] Ahn, C.M. (1992) Option Pricing when Jump Risk is Systematic, *Mathematical Finance* 2(4), 299–308.

[2] Ait-Sahalia, Y. (1996) Do Interest Rates Really follow Continuous-Time Markov Diffussions?, Working paper, University of Chicago, Graduate School of Business.

[3] Amin, K.I. (1993) Jump Diffusion Option Valuation in Discrete Time, *Journal of Finance* 48(5), 1833–1863.

[4] Amin, K.I. and V. Ng (1993) Option Valuation with Systematic Stochastic Volatility, *Journal of Finance* 48(3), 881–910.

[5] Backus, D., S. Foresi, and L. Wu (1997) "Accounting for Biases in Black-Scholes," working paper, NYU Stern School of Business.

[6] Ball, C. and W. Torous (1985) On Jumps in Common Stock Prices and Their Impact on Call Option Pricing, *Journal of Finance* 40, 155–173.

[7] Bates, D.S. (1996) Jumps and Stochastic Volatility: Exchange Rate Processes Implicit in Deutsche Mark Options, *Review of Financial Studies* 9(1), 69–107.

[8] Black, F. (1975) Fact and Fantasy in the Use of Options, *Financial Analysts Journal* 31, 36–72.

[9] Black, F. and M. Scholes (1973) The Pricing of Options and Corporate Liabilities, *Journal of Political Economy* 81(3), 637–654.

[10] Bodurtha, J. and G. Courtadon (1987) Tests of the American Option Pricing Model in the Foreign Currency Options Market, *Journal of Financial and Quantitative Analysis* 22, 153–167.

[11] Blattberg, R. and N. Gonedes (1974) A Comparison of the Stable and Student Distributions as Statistical Models for Stock Prices, *Journal of Business* 47, 244–280.

[12] Bollerslev, T., R.Y. Chou, and K.F. Kroner (1992) ARCH Modelling in Finance, *Journal of Econometrics* 52, 5–59.

[13] Campa, J. and K.H. Chang (1995) Testing the Expectations Hypothesis on the Term Structure of Volatilities, *Journal of Finance* 50, 529–547.

[14] Campa, J.; K.H. Chang, and R. Rieder (1997) Implied Exchange Rate Distributions: Evidence from OTC Options Markets, mimeo, New York University.

[15] Das, S.R. and S. Foresi (1996) Exact Prices for Bond and Option Prices with Systematic Jump Risk, *Review of Derivatives Research* 1(1), 7-24.

[16] Derman, E. and I. Kani (1994) Riding on a Smile, *Risk* 7(2), 32–39.

[17] Duffie, D. (1996) *Dynamic Asset Pricing Theory*, Second Edition, Princeton University Press, Princeton, NJ.

[18] Duffie, D. and J. Pan (1997) An Overview of Value at Risk, *Journal of Derivatives* Spring 1997, 7–49.

[19] Drost, F., and T. Nijman (1993) Temporal aggregation of GARCH models, *Econometrica* 61(4), 909–927.

[20] Drost, F., T. Nijman, and Werker (1995) Estiamtion and Testing in Models containing both Jumps and Conditional Heteroskedasticity, mimeo, Department of Economics, Tilburg University.

[21] Fama, E. (1965) The Behavior of Stock Prices, *Journal of Business* 47, 244-280.

[22] Heston, S.L. (1993) A Closed-Form Solution for Options with Stochastic Volatility with Applications to Bond and Currency Options, *Review of Financial Studies* 6(2), 327–343.

[23] Heynen, R.; A.G.Z. Kemna, and T. Vorst (1994) Analysis of the Term Strucutre of Implied Volatilities, *Journal of Financial and Quantitative Analysis* 29, 31–56.

[24] Hsieh, D.A. (1989) Modeling Heteroskedasticity in Daily Foreign-Exchange Rates, *Journal of Business and Economic Statistics* 7(3), 307–317.

[25] Hull, J. and A. White (1987) The Pricing of Options on Assets with Stochastic Volatilities, *Journal of Finance* 42, 281–300.

[26] Jackwerth, J. and M. Rubinstein (1996) Recovering Probability Distributions from Options Prices, *Journal of Finance* 51(3), 1611–1631.

[27] Jarrow, R. and E. Rosenfeld (1984) Jump Risks and the Intertemporal Capital Asset Pricing Model, *Journal of Business* 57, 337–351.

[28] Jarrow, R. and A. Rudd (1982) Approximate Option Valuation for Arbitrary Stochastic Processes, *Jpurnal of Financial Economics*, v10(3), 347-369.

[29] Jorion, P. (1988) On Jump Processes in the Foreign Exchange and Stock Markets, *Review of Financial Studies* 1(4), 427–445.

[30] Karlin, H. and S. Taylor (1975) *A Second Course in Stochastic Processes* (Second Edition), Academic Press, San Diego, CA.

[31] Kon, S. (1984) Models of Stock Returns: A Comparison, *Journal of Finance* 39, 147–165.

[32] Melino, A. (1994) Estimation of Continuous-Time Models in Finance, in C. Sims (Ed.) *Advances in Econometrics: Sixth World Congress*, Vol. II, Cambridge Unviersity Press, Cambridge and New York.

[33] Melino, A. and S.M. Turnbull (1990) Pricing Foreign Currency Options with Stochastic Volatility, *Journal of Econometrics* 45, 239–265.

[34] Merton, R.C. (1976) Option Pricing when the Underlying Process for Stock Returns is Discontinuous, *Journal of Financial Economics* 3, 124–144.

[35] Nelson, D. (1990) ARCH models as Diffusion Approximations, *Journal of Econometrics* 45, 7–38.

[36] Rosenberg, J. (1996) Pricing Multivariate Contingent Claims using Estimated Risk-Neutral Density Functions, mimeo, New York University.

[37] Rubinstein, M. (1994) Implied Binomial Trees, *Journal of Finance* 49(3), 393–440.

[38] Stein, J. (1989) Overreaction in the Options Market, *Journal of Finance*, 44(4), 1011–23.

[39] Stein, E.M. and J.C. Stein (1991) Stock Price Distributions with Stochastic Volatility: An Analytic Approach, *Review of Financial Studies* 4(4), 727–752.

[40] Taylor, S. and X. Xu (1994) The Magnitude of Implied Volatility Smiles: Theory and Empirical Evidence for Exchange Rates, *Review of Futures Markets* 13, 355–380.

[41] Wiggins, J.B. (1987) Option Values under Stochastic Volatility: Theory and Empirical Estimates, *Journal of Financial Economics* 19(2), 351–372.

[42] Xu, X. and S.J. Taylor (1994) The Term Structure of Volatility implied by Foreign Exchange Options, *Journal of Financial and Quantitative Analysis* 29(1), 57–74.

[43] Zhu, Y. (1997) *Three Essays in Mathematical Finance*, Doctoral Dissertation, Department of Mathematics, New York University.