# Today's Agenda

1. MLE–Simple Introduction
   – GARCH estimation
2. Kalman Filtering
3. The Delta Method
4. Empirical Portfolio Choice
5. Wold Decomposition of Stationary Processes

# 1 Maximum Likelihood Estimation

(Preliminaries for GARCH/Stochastic Volatility & Kalman Filtering)

- Suppose we have the series $\{Y_1, Y_{2,}...., Y_T\}$ with a joint density $f_{Y_T....Y_1}(\theta)$ that depends on some parameters $\theta$ (such as means, variances, etc.)
- We observe a realization of $Y_t$.
- If we make some functional assumptions on $f$, we can think of $f$ as the probability of having observed this particular sample, given the parameters $\theta$.
- The maximum likelihood estimate (MLE) of $\theta$ is the value of the parameters $\theta$ for which this sample is most likely to have been observed.
- In other words, $\hat{\theta}^{MLE}$ is the value that maximizes $f_{Y_T....Y_1}(\theta)$.

- Q: But, how do we know what $f$–the true density of the data–is?
- A: We don't.
- Usually, we assume that $f$ is normal, but this is strictly for simplicity. The fact that we have to make distributional assumptions limits the use of MLE in many financial applications.
- Recall that if $Y_t$ are independent over time, then

$$\begin{aligned} f_{Y_T....Y_1}(\theta) &= f_{Y_T}(\theta_T) f_{Y_{T-1}}(\theta_{T-1})...f_{Y_1}(\theta_1) \\ &= \Pi_{i=1}^{T} f_{Y_i}(\theta_i) \end{aligned}$$

- Sometimes it is more convenient to take the log of the likelihood function, then

$$\Lambda(\theta) = \log f_{Y_T....Y_1}(\theta) = \sum_{i=1}^{T} \log f_{Y_i}(\theta)$$

- However, in most time series applications, the independence assumption is untenable. Instead, we use a conditioning trick.
- Recall that
$$f_{Y_2 Y_1} = f_{Y_2|Y_1} f_{Y_1}$$
- In a similar fashion, we can write
$$f_{Y_T....Y_1}(\theta) = f_{Y_T|Y_{T-1}....Y_1}(\theta) f_{Y_{T-1}|Y_{T-2}....Y_1}(\theta) ... f_{Y_1}(\theta)$$
- The log likelihood can be expressed as
$$\Lambda(\theta) = \log f_{Y_T....Y_1}(\theta) = \sum_{i=1}^{T} \log f_{Y_i|Y_{i-1,...,}Y_1}(\theta_i)$$

- Example: The log-likelihood of an AR(1) process
$$Y_t = c + \phi Y_{t-1} + \varepsilon_t$$
- Suppose that $\varepsilon_t$ is iid $N(0, \sigma^2)$
- Recall that $E(Y_t) = \frac{c}{1-\phi}$ and $Var(Y_t) = \frac{\sigma^2}{1-\phi^2}$
- Since $Y_t$ is a linear function of the $\varepsilon_t'$s, then it is also Normal (sum of normals is a normal).
- Therefore, the density (unconditional) of $Y_t$ is Normal.
- Result: If $Y_1$ and $Y_2$ are jointly Normal, then the marginals are also normal.
- Therefore,
$$f_{Y_2|Y_1} \text{ is } N\left((c + \phi y_1), \sigma^2\right)$$
or
$$f_{Y_2|Y_1} = \frac{1}{\sqrt{2\pi\sigma^2}} \exp\left[\frac{-(y_2 - c - \phi y_1)^2}{2\sigma^2}\right]$$

- Similarly,

$$f_{Y_3|Y_2} \text{ is } N\left(\left(c + \phi y_2\right), \sigma^2\right)$$

or

$$f_{Y_3|Y_2} = \frac{1}{\sqrt{2\pi\sigma^2}} \exp\left[\frac{-\left(y_3 - c - \phi y_2\right)^2}{2\sigma^2}\right]$$

Then,the log likelihood can be written as

$$\begin{aligned}\Lambda(\theta) &= \log f_{Y_1} + \sum_{t=2}^{T} \log f_{Y_t|Y_{t-1}} \\ &= -\frac{1}{2}\log(2\pi) - \frac{1}{2}\log\left(\sigma^2/\left(1-\phi^2\right)\right) \\ &\quad -\frac{\{y_1 - (c/(1-\phi))\}^2}{2\sigma^2/\left(1-\phi^2\right)} \\ &\quad -\frac{(T-1)}{2}\log(2\pi) - \frac{(T-1)}{2}\log\left(\sigma^2\right) \\ &\quad -\sum_{t=2}^{T}\frac{\left(y_t - c - \phi y_{t-1}\right)^2}{2\sigma^2}\end{aligned}$$

- The unknown parameters are collected in $\theta = (c, \phi, \sigma)$
- We can maximize $\Lambda(\theta)$ with respect to all those parameters and find the estimates that maximize the probability of having observed such a sample.

$$\max_{\theta} \Lambda(\theta)$$

- Sometimes, we can even put constraints (such as $|\phi| < 1$)
- Q: Is it necessary to put the constraint $\sigma^2 > 0$?

- Note: If we forget the first observation, then we can write (setting $c = 0$) the FOC:

$$
\begin{aligned}
-\sum_{t=2}^{T} \frac{\partial}{\partial \phi} \frac{\left(y_t - \phi y_{t-1}\right)^2}{2\sigma^2} &= 0 \\
\sum_{t=2}^{T} y_{t-1}\left(y_t - \phi y_{t-1}\right) &= 0 \\
\hat{\phi} &= \frac{\sum_{t=2}^{T} y_{t-1} y_t}{\sum_{t=2}^{T} y_{t-1}^2}
\end{aligned}
$$

- RESULT: In the *univariate linear* regression case, OLS, GMM, MLE are equivalent!!!

- To summarize the maximum likelihood principle:
  (a) Make a distributional assumption about the data
  (b) Use the conditioning to write the joint likelihood function
  (c) For convenience, we work with the log-likelihood function
  (d) Maximize the likelihood function with respect to the parameters
- There are some subtle points.
  - We had to specify the unconditional distribution of the first observation
  - We had to make an assumption about the dependence in the series
- But sometimes, MLE is the only way to go.
- MLE is particularly appealing if we know the distribution of the series. Most other deficiencies can be circumvented.

- Now, you will ask: What are the properties of $\hat{\theta}^{MLE}$? More specifically, is it consistent? What is its distribution, where
$$\hat{\theta}^{MLE} = \arg\max \Lambda(\theta)$$
- Yes, $\hat{\theta}^{MLE}$ is a consistent estimator of $\theta$.
- As you probably expect the asymptotic distribution of $\hat{\theta}^{MLE}$ is normal.
- Result:
$$T^{1/2}\left(\hat{\theta}^{MLE} - \theta\right) \sim {}^{a}N(0, V)$$
$$V = \left[-\frac{\partial^2 \Lambda(\theta)}{\partial\theta\partial\theta'}|_{\hat{\theta}^{MLE}}\right]^{-1}$$
or
$$V = \sum_{t=1}^{T} l\left(\hat{\theta}^{MLE}, y\right) l\left(\hat{\theta}^{MLE}, y\right)$$
$$l\left(\hat{\theta}^{MLE}, y\right) = \frac{\partial f}{\partial\theta}\left(\hat{\theta}^{MLE}, y\right)$$
- But we will not dwell on proving those properties.

Another Example: The log-likelihood of an AR(1)+ARCH(1) process

$$Y_t = c + \phi Y_{t-1} + u_t$$

- where,

$$u_t = \sqrt{h_t} v_t$$

- ARCH(1) is:

$$h_t = \zeta + a u_{t-1}^2$$

where $v_t$ is iid with mean $0$, and $E\left(v_t^2\right) = 1.$

- GARCH(1,1): Suppose, we specify $h_t$ as

$$h_t = \zeta + \delta h_{t-1} + a u_{t-1}^2$$

- Recall that $E\left(Y_t\right) = \frac{c}{1-\phi}$ and $Var\left(Y_t\right) = \frac{\sigma^2}{1-\phi^2}$
- Since $Y_t$ is a linear function of the $\varepsilon_t'$s, then it is also Normal (sum of normals is a normal).
- Therefore, the density (unconditional) of $Y_t$ is Normal.
- Result: If $Y_1$ and $Y_2$ are jointly Normal, then the marginals are also normal.
- Therefore,

$$f_{Y_2|Y_1} \text{ is } N\left(\left(c + \phi y_1\right), h_2\right)$$

or for the ARCH(1)

$$f_{Y_2|Y_1} = \frac{1}{\sqrt{2\pi\left(\zeta + a u_1^2\right)}} \exp\left[\frac{-\left(y_2 - c - \phi y_1\right)^2}{2\left(\zeta + a u_1^2\right)}\right]$$

- Similarly,

$$f_{Y_3|Y_2} \text{ is } N\left(\left(c+\phi y_2\right), h_3\right)$$

or

$$f_{Y_3|Y_2} = \frac{1}{\sqrt{2\pi\left(\zeta + a u_2^2\right)}} \exp\left[\frac{-\left(y_3 - c - \phi y_2\right)^2}{2\left(\zeta + \alpha u_2^2\right)}\right]$$

Then, the conditional log likelihood can be written as

$$
\begin{aligned}
\Lambda\left(\theta|y_1\right) &= \sum_{t=2}^{T} \log f_{Y_t|Y_{t-1}} \\
&= -\frac{(T-1)}{2}\log\left(2\pi\right) - \frac{(1)}{2}\sum_{t=2}^{T}\log\left(\zeta + a u_{t-1}^2\right) \\
&\quad -\sum_{t=2}^{T}\frac{\left(y_t - c - \phi y_{t-1}\right)^2}{2\left(\zeta + \alpha u_{t-1}^2\right)}
\end{aligned}
$$

- The unknown parameters are collected in $\theta = (c, \phi, \zeta, \alpha)$
- We can maximize $\Lambda\left(\theta\right)$ with respect to all those parameters and find the estimates that maximize the probability of having observed such a sample.

$$\max_{\theta} \Lambda\left(\theta\right)$$

- Example:mle_arch.m

- Similarly for GARCH(1,1):

$$\begin{aligned}\Lambda(\theta|y_1) &= \sum_{t=2}^{T}\log f_{Y_t|Y_{t-1}} \\ &= -\frac{(T-1)}{2}\log(2\pi)-\frac{(1)}{2}\sum_{t=2}^{T}\log(h_t) \\ &\quad -\sum_{t=2}^{T}\frac{(y_t-c-\phi y_{t-1})^2}{2(h_t)}\end{aligned}$$

  where

$$h_t=\zeta+\delta h_{t-1}+\alpha u_{t-1}^2$$

- To construct $h_t$, we have to filter the $\{u_{t-1}\}$ series.
- For a given $u_t s$, $h_{0,}$ and $\zeta$, $\delta$, and $\alpha$, we construct $h_t$
- The $h_t$ will allow us to evaluate the likelihood $\Lambda(\theta|y_1)$
- Optimize $\Lambda(\theta|y_1)$ with respect to all the parameters, given the initial conditions.
- This recursive feature of the GARCH makes it harder to estimate with GMM.

## 2 Kalman Filtering

- History: Kalman (1963) paper
- Problem: We have a missile that we want to guide to its proper target.
  - The trajectory of the missile IS observable from the control center.
  - Most other circumstances, such as weather conditions, possible interception methods, etc. are NOT observable, but can be forecastable.
  - We want to guide the missile to its proper destination.
- In finance the setup is very similar, but the problem is different.
- In the missile case, the parameters of the system are known. The interest is, given those parameters, to control the missile to its proper destination.
- In finance, we want to estimate the parameters of the system. We are usually not concerned with a control problem, because there are very few instruments we can use as controls (although there are counter-examples).

### 2.1 Setup (Hamilton CH 13)

$$
\begin{aligned}
y_t &= A'x_t + H'z_t + w_t \\
z_t &= Fz_{t-1} + v_t
\end{aligned}
$$

where

- $y_t$ is the observable variable (think "returns")
  - The first equation, the $y_t$ equation is called the "space" or the "observation" equation.
- $z_t$ is the unobservable variable (think "volatility" or "state of the economy")
  - The second equation, the $z_t$ equation is called the "state" equation.
- $x_t$ is a vector of exogenous (or predetermined) variables (we can set $x_t = 0$ for now).
- $v_t$ and $w_t$ are iid and assumed to be uncorrelated at all lags
$$E\left(w_t v_t'\right) = 0$$
- Also $E\left(v_t v_t'\right) = Q$, $E\left(w_t w_t'\right) = R$
- The system of equations is known as a state-space representation.
- Any time series can be written in a state-space representation.

- In standard engineering problems, it is assumed that we know the parameters $A, H, F, Q, R$.
- The problem is to give impulses $x_t$ such that, given the states $z_t$, the missile is guided as closely to target as possible.
- In finance, we want to estimate the unknown parameters $A, H, F, Q, R$ in order to understand where the system is going, given the states $z_t$. There is little attempt at guiding the system. In fact, we usually assume that $x_t = 1$ and $A = E(Y_t)$, or even that $x_t = 0$.

- Note: Any time series can be written as a state space.
- Example: AR(2): $Y_{t+1} - \mu = \phi_1 (Y_t - \mu) + \phi_2 (Y_{t-1} - \mu) + \varepsilon_{t+1}$
- State equation:
$$\begin{bmatrix} Y_{t+1} - \mu \\ Y_t - \mu \end{bmatrix} = \begin{bmatrix} \phi_1 & \phi_2 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} Y_t - \mu \\ Y_{t-1} - \mu \end{bmatrix} + \begin{bmatrix} \varepsilon_{t+1} \\ 0 \end{bmatrix}$$
- Observation equation:
$$y_t = \mu + \begin{bmatrix} 1 & 0 \end{bmatrix} \begin{bmatrix} Y_{t+1} - \mu \\ Y_t - \mu \end{bmatrix}$$
- There are other state-space representations of $Y_t$. Can you write down another one?

- As a first step, we will assume that $A, H, F, Q, R$ are known.
- Our goal would be to find a best linear forecast of the state (unobserved) vector $z_t$. Such a forecast is needed in control problems (to take decisions) and in finance (state of the economy, forecasts of unobserved volatility).
- The forecasts will be denoted by:
$$z_{t+1|t} = E\left(z_{t+1}|y_t..., x_t....\right)$$
  and we assume that we are only taking linear projections of $z_{t+1}$ on $y_t..., x_t.....$ Nonlinear Kalman Filters exist but the results are a bit more complicated.
- The Kalman Filter calculates the forecasts $z_{t+1|t}$ recursively, starting with $z_{1|0}$, then $z_{2|1}$, ...until $z_{T|T-1}$.
- Since $z_{t|t-1}$ is a forecast, we can ask how good of a forecast it is?
- Therefore, we define $P_{t|t-1} = E\left(\left(z_t - z_{t|t-1}\right)\left(z_t - z_{t|t-1}\right)\right)$, which is the forecasting error from the recursive forecast $z_{t|t-1}$.

- The Kalman Filter can be broken down into 5 steps

1. Initialization of the recursion. We need $z_{1|0}$. Usually, we take $z_{1|0}$ to be the unconditional mean, or $z_{1|0} = E(z_1)$. (Q: how can we estimate $E(z_1)$? ) The associated error with this forecast is $P_{1|0} = E\left(\left(z_{1|0} - z_1\right)\left(z_{1|0} - z_1\right)\right)$

2. Forecasting $y_t$ (intermediate step)
   The ultimate goal is to calculate $z_{t|t-1}$, but we do that recursively. We will first need to forecast the value of $y_t$, based on available information:
$$E\left(y_t | x_t, z_t\right) = A'x_t + H'z_t$$
   From the law of iterated expectations,
$$E_{t-1}\left(E_t\left(y_t\right)\right) = E_{t-1}\left(y_t\right) = A'x_t + H'z_{t|t-1}$$
   The error from this forecast is
$$y_t - y_{t|t-1} = H'\left(z_t - z_{t|t-1}\right) + w_t$$
   with MSE
$$\begin{aligned} & E\left(y_t - y_{t|t-1}\right)\left(y_t - y_{t|t-1}\right)' \\ = \; & E\left[H'\left(z_t - z_{t|t-1}\right)\left(z_t - z_{t|t-1}\right)' H\right] + E\left[w_t w_t'\right] \\ = \; & H' P_{t|t-1} H + R \end{aligned}$$

3. Updating Step ($z_{t|t}$)

– Once we observe $y_t$, we can update our forecast of $z_t$, denoting it by $z_{t|t}$, before making the new forecast, $z_{t+1|t}$.

– We do this by calculating $E\left(z_t|y_t, x_t, ...\right) = z_{t|t}$

$$\begin{aligned} z_{t|t} &= z_{t|t-1} + E\left(\left(z_t - z_{t|t-1}\right)\left(y_t - y_{t|t-1}\right)\right) * \\ &\quad \left(E\left(y_t - y_{t|t-1}\right)\left(y_t - y_{t|t-1}\right)'\right)^{-1}\left(y_t - y_{t|t-1}\right) \end{aligned}$$

– We can write this a bit more intuitively as:

$$z_{t|t} = z_{t|t-1} + \beta\left(y_t - y_{t|t-1}\right)$$

where $\beta$ is the OLS coefficient from regressing $\left(z_t - z_{t|t-1}\right)$ on $\left(y_t - y_{t|t-1}\right)$.

– The bigger is the relationship between the two forecasting errors, the bigger the correction must be.

– It can be shown that

$$z_{t|t} = z_{t|t-1} + P_{t|t-1} H \left( H' P_{t|t-1} H + R \right)^{-1} \left( y_t - A' x_t - H' z_{t|t-1} \right)$$

– This updated forecast uses the old forecast $z_{t|t-1}$, and the just observed values of $y_t$ and $x_t$.

4. Forecast $z_{t+1|t}$.

– Once we have an update of the old forecast, we can produce a new forecast, the forecast of $z_{t+1|t}$

$$
\begin{aligned}
E_t\left(z_{t+1}\right) &= E\left(z_{t+1}|y_t, x_t, ...\right) \\
&= E\left(Fz_t + v_{t+1}|y_t, x_t, ...\right) \\
&= FE\left(z_t|y_t, x_t, ...\right) + 0 \\
&= Fz_{t|t}
\end{aligned}
$$

– We can use the above equation to write

$$
\begin{aligned}
E_t\left(z_{t+1}\right) &= F\{z_{t|t-1} \\
&\quad +P_{t|t-1}H\left(H'P_{t|t-1}H + R\right)^{-1}\left(y_t - A'x_t - H'z_{t|t-1}\right) \\
&= Fz_{t|t-1} \\
&\quad +FP_{t|t-1}H\left(H'P_{t|t-1}H + R\right)^{-1}\left(y_t - A'x_t - H'z_{t|t-
\end{aligned}
$$

– We can also derive an equation for the error in forecast as a recursion

$$
\begin{aligned}
P_{t+1|t} &= F[P_{t|t} \\
&\quad -P_{t|t-1}H\left(H'P_{t|t-1}H + R\right)^{-1}H'P_{t|t-1}]F' \\
&\quad +Q
\end{aligned}
$$

5. Go to step 2, until we reach T. Then, we are done.

- Summary: The Kalman Filter produces
  - The optimal forecasts of $z_{t+1|t}$ and $y_{t+1|t}$ (optimal within the class of linear forecasts)
  - We need some initialization assumptions
  - We need to know the parameters of the system, i.e. $A$, $H$, $F$, $Q$, $R$.
- Now, we need to find a way to estimate the parameters $A$, $H$, $F$, $Q$, $R$.
- By far, the most popular method is MLE.
- Aside: Simulations Methods–getting away from the restrictive assumptions of $\varepsilon_t$

## 2.2 Estimation of Kalman Filters (MLE)

- Suppose that $z_1$, and the shocks $(w_t, v_t)$ are jointly normally distributed.
- Under such an assumption, we can make the very strong claim that the forecasts $z_{t+1|t}$ and $y_{t+1|t}$ are optimal among any functions of $x_t$, $y_{t-1}$.... In other words, if we have normal errors, we cannot produce better forecasts using the past data than the Kalman forecasts!!!
- If the errors are normal, then all variables in the linear system have a normal distribution.
- More specifically, the distribution of $y_t$ conditional on $x_t$, and $y_{t-1}, \ldots$ is normal, or

$$y_t | x_t, y_{t-1} \ldots \sim N\left(A'x_t + H'z_{t|t-1}, \left(H'P_{t|t-1}H + R\right)\right)$$

- Therefore, we can specify the likelihood function of $y_t | x_t, y_{t-1}$ as we did above.

$$\begin{aligned} f_{y_t|x_t,y_{t-1}} &= (2\pi)^{-n/2} \left|H'P_{t|t-1}H + R\right|^{-1/2} \\ &\quad \times \exp[-\frac{1}{2}\left(y_t - A'x_t - H'z_{t|t-1}\right)' \left(H'P_{t|t-1}H + R\right)^{-1} \\ &\quad \times \left(y_t - A'x_t - H'z_{t|t-1}\right)] \end{aligned}$$

- The problem is to maximize

$$\max_{A,H,F,Q,R} \sum_{t=1}^{T} \log f_{y_t|x_t,y_{t-1}}$$

- Words of wisdom:
  - This maximization problem can easily get unmanageable to estimate, even using modern computers. The problem is that searching for global $\max$ is very tricky.
    * A possible solution is to make as many restrictions as possible and then to relax them one by one.
    * A second solution is to write a model that gives theoretical restrictions.
  - Recall that there are more than 1 state space representations of an AR process. This implies that some of the parameters in the state-space system are not identified. In other words, more than one value of the parameters (different combinations) can give rise to the same likelihood function.
    * Then, which likelihood do we choose?
    * Have to make restrictions so that we have an exactly identified problem.

## 2.3 Applications in Finance

- Anytime we have unobservable state variables
  - Filtering expected returns (Pastor and Stambaugh (JF, 2008))
  - Filtering variance (Brandt and Kang (JFE, 2007))
- Interpolation of data
  - Bernanke and Kuttner (JME?)
- Time varying parameters
  - Time-varying Betas (Ghysels (JF, 1998))

## 3 Kalman Smoother

- For purely forecating purposes, we need
$$z_{t|t-1} = E\left(z_t | I_{t-1}\right)$$
where $I_{t-1} = \{y_{t-1,} y_{t-2}, ..., y_1, x_{t-1}, ...x_1\}$ and the corresponding error $P_{t|t-1} = E\left(\left(z_t - z_{t|t-1}\right)^2\right)$

- But if we want to model a process (understand its properties), we might want to incorporate all the available information in $I_T = \{y_{T,} y_{T-1}, ..., y_1, x_T, ...x_1\}$.

- In other words, we might want to estimate
$$z_{t|T} = E\left(z_t | I_T\right)$$
There is definitely a look-ahead bias here, but that is the point. We want to include all available information in order to get a better glimpse into the properties of $z_t$!

- Recall that from the KF, we have the sequences $\{z_{t+1|t}\}$, $\{z_{t|t}\}$, $\{P_{t+1|t}\}$, $\{P_{t|t}\}$.

- Suppose someone tells you the correct value of $z_{t+1}$ at time $t$. How can you improve upon the best forecast $z_{t|t}$? It turns out that we do the same updating as we did in step 3 of the KF:

$$\begin{aligned} E\left(z_t | z_{t+1,} I_t\right) &= z_{t|t} + E\left(\left(z_t - z_{t|t}\right)\left(z_{t+1} - z_{t+1|t}\right)\right) * \\ &\quad \left(E\left(z_{t+1} - z_{t+1|t}\right)\left(z_{t+1} - z_{t+1|t}\right)'\right)^{-1}\left(z_{t+1} - z_{t-}\right. \end{aligned}$$

1. – We can write this a bit more intuitively as:

$$E\left(z_t | z_{t+1,} I_t\right) = z_{t|t} + J_t\left(z_{t+1} - z_{t+1|t}\right)$$

where

$$\begin{aligned} J_t &= E\left(\left(z_t - z_{t|t}\right)\left(z_{t+1} - z_{t+1|t}\right)\right) * \\ &\quad \left(E\left(z_{t+1} - z_{t+1|t}\right)\left(z_{t+1} - z_{t+1|t}\right)'\right)^{-1} \\ &= P_{t|t} F P_{t+1|t}^{-1} \end{aligned}$$

– Because the process is Markovian, $E\left(z_t | z_{t+1,} I_t\right) = E\left(z_t | z_{t+1,} I_T\right)$. We can't do better than that! Hence,

$$E\left(z_t | z_{t+1,} I_T\right) = z_{t|t} + J_t\left(z_{t+1} - z_{t+1|t}\right)$$

– Last step. We can show that

$$E\left(z_t | I_T\right) = z_{t|T} = z_{t|t} + J_t\left(z_{t+1|T} - z_{t+1|t}\right)$$

– Hence, the KS algorithm is, after we obtain the KF $\{z_{t+1|t}\}$, $\{z_{t|t}\}$, $\{P_{t+1|t}\}$, $\{P_{t|t}\}$

2. Start at the end, $z_{T|T}$.

3. Compute $J_{T-1} = P_{T-1|T-1} F P_{T|T-1}^{-1}$

4. Compute

$$z_{T-1|T} = z_{T-1|T-1} + J_{T-1}\left(z_{T|T} - z_{T|T-1}\right)$$

5. Use $z_{T-1|T}$ to compute $z_{T-2|T}$ and so on.

6. We can compute the associated MSE as

$$P_{t|T} = P_{t|t} + J_t\left(P_{t+1|T} - P_{t+1|t}\right) J_t^{'}$$

## 4 Time-Varying Parameters

- An example of a time varying parameter model:

$$\begin{aligned} r_{t+1} &= \alpha + \beta_t x_t + \varepsilon_{t+1} \\ \beta_{t+1} &= \gamma \beta_t + v_{t+1} \end{aligned}$$

- Q: What equations are observations and what are the state equations?
- Note that this does not fit within the KF setup:

$$\begin{aligned} y_t &= A' x_t + H' z_t + w_t \\ z_t &= F z_{t-1} + v_t \end{aligned}$$

- We need the generalization

$$\begin{aligned} y_t &= A(x_t) + H(x_t)' z_t + w_t \\ z_{t+1} &= F(x_t) z_t + v_{t+1} \end{aligned}$$

- Note that $F(x_t)$ and not $F(x_{t+1})$ in the state equation!

- Now, we have to assume that–we didn't have to do it earlier!

$$\begin{bmatrix} w_t \\ v_{t+1} \end{bmatrix} | x_t, I_{t-1} \sim N\left( \begin{bmatrix} 0 \\ 0 \end{bmatrix}, \begin{bmatrix} Q(x_t) & 0 \\ 0 & R(x_t) \end{bmatrix} \right)$$

- Before, we had linearity in all variables. Now, we don't.

- Given the conditional normal assumption, we can show that

$$\begin{bmatrix} z_t \\ y_t \end{bmatrix} | x_t, I_{t-1} \sim N\left( \begin{bmatrix} z_{t|t-1} \\ A(x_t) + H(x_t)' z_{t|t-1} \end{bmatrix}, V \right)$$

$$V = \begin{bmatrix} P_{t|t-1} & P_{t|t-1} H(x_t) \\ H'(x_t) P_{t|t-1} & H'(x_t) P_{t|t-1} H(x_t) + R(x_t) \end{bmatrix}$$

where $\left\{ z_{t|t-1} \right\}, \left\{ z_{t|t} \right\}, \left\{ P_{t|t-1} \right\}, \left\{ P_{t|t} \right\}$ are obtained from the KF procedure above.

- Notice that, conditional on $x_t$, the time varying parameters are fixed.

- Estimation is easy (MLE), given the assumption.

- TVP Example:
$$r_t \;=\; \beta_t \lambda_t + w_t$$
$$\beta_{t+1} - \bar{\beta} \;=\; F\left(\beta_t - \bar{\beta}\right) + v_{t+1}$$
We have a CAPM with TV $\beta s$ in mind.

- If we assume that
$$\left[\begin{array}{c} w_t \\ v_{t+1} \end{array} |x_t, I_{t-1}\right] \sim N\left(\left[\begin{array}{c} 0 \\ 0 \end{array}\right], \left[\begin{array}{cc} \sigma^2 & 0 \\ 0 & Q \end{array}\right]\right)$$
then we are within the KF framework.

- Substituting the state variable $z_t = \left(\beta_t - \bar{\beta}\right)$ in the space equation, we can write
$$r_t = \lambda_t \bar{\beta} + \lambda_t z_t + w_t$$

- We can plug in the MLE estimator directly.

- Note: We can allow an AR(p) dynamics in the state equation quite easily.

- Example: Wells, C., The Kalman Filter in Finance, Springer Netherlands.
- Example: Ludvigson and NG (JFE, 2007)

$$
\begin{aligned}
m_{t+1} &= a'F_t + \beta' Z_t + \varepsilon_{t+1} \\
VOL_{t+1} &= c'F_t + d'Z_t + u_{t+1}
\end{aligned}
$$

  where $VOL_{t+1}$ is the realized volatility in month $t+1$. (Observable)

## 5 Brandt and Kang (JFE, 2004):

$$
\begin{aligned}
r_{t+1} &= \mu_t + \sigma_t u_{t+1} \\
\begin{bmatrix} \ln \mu_t \\ \ln \sigma_t \end{bmatrix} &= d + A \begin{bmatrix} \ln \mu_{t-1} \\ \ln \sigma_{t-1} \end{bmatrix} + \varepsilon_t
\end{aligned}
$$

The Delta Method

- We estimate $y = \theta x + \varepsilon$, and obtain $\hat{\theta}$ but are interested in a function $g(\theta)$, where $g(.)$ is some non-linear model.
- Example: we have a forecast of the volatility, $\hat{\sigma}_t$ and want to test its economic significance.
- Statistical measure of fit: MSE=$E\left\{(\hat{\sigma}_t - \sigma_t)^2\right\}$
- Economic measure of fit: $C(\hat{\sigma}_t, S_t, K, r, T)$ and compare it to $C(\sigma_t, S_t, K, r, T)$, where $C(.)$ is the BS call-option formula.
- We want to know whether $C(\hat{\sigma}_t, S_t, K, r, T) - C(\sigma_t, S_t, K, r, T)$ is economically and statistically different from zero.

- The Delta method
- If we have a consistent, asymptotically normal estimator
$$\sqrt{T}\left(\hat{\theta}-\theta\right) \rightarrow^{d} N(0,V)$$
and $g(.)$ is differentiable, then
$$\sqrt{T}\left(g\left(\hat{\theta}\right)-g\left(\theta\right)\right) \;\rightarrow\; ^{d}N(0,D'VD)$$
$$D \;=\; \frac{\partial g}{\partial \theta}|\theta$$
- Sketch of the proof: From the Mean-Value Theorem, we can write

$$g\left(\hat{\theta}\right)=g\left(\theta\right)+\frac{\partial g'}{\partial \theta}|\theta^{M}\left(\hat{\theta}-\theta\right)$$

where $\theta^{M}$ lies between $\hat{\theta}$ and $\theta$. Since, $\hat{\theta} \rightarrow^{p} \theta$, then $\theta^{M} \rightarrow^{p} \theta$ and $\frac{\partial g'}{\partial \theta}|\theta^{M} \rightarrow^{p} \frac{\partial g'}{\partial \theta}|\theta$ (Continuous Mapping Theorem).

Then, we can write

$$\sqrt{T}\left(g\left(\hat{\theta}\right)-g\left(\theta\right)\right)=\frac{\partial g'}{\partial\theta}|\theta^{M}\sqrt{T}\left(\hat{\theta}-\theta\right)$$

- – $\frac{\partial g'}{\partial\theta}|\theta^{M} \rightarrow^{p} \frac{\partial g'}{\partial\theta}|\theta$
  – $\sqrt{T}\left(\hat{\theta}-\theta\right)\rightarrow^{d} N(0,V)$
  – Slutsky Theorem: $\frac{\partial g'}{\partial\theta}|\theta^{M}\sqrt{T}\left(\hat{\theta}-\theta\right) \rightarrow^{d} \left[\frac{\partial g'}{\partial\theta}|\theta\right] N(0,V)$
- Or

$$\sqrt{T}\left(g\left(\hat{\theta}\right)-g\left(\theta\right)\right)\rightarrow^{d} N(0,\left[\frac{\partial g'}{\partial\theta}|\theta\right] V \left[\frac{\partial g}{\partial\theta}|\theta\right])$$

- Example: We run the regression (s.e. in parentheses)

$$
\begin{aligned}
y_t &= \alpha + \beta x_t + \varepsilon_t \\
&= \underset{(0.04)}{0.1} + \underset{(0.3)}{1.1} x_t + \varepsilon_t
\end{aligned}
$$

- A test of $\beta = 1$ yields, $t = (1.1 - 1)/0.3 = 0.33$
- We are interested in $\ln(\hat{\beta})$ and testing under the null of $ln(\beta) = 0$. From the delta method, we know that

  $$\sqrt{T}\left(\ln\left(\hat{\beta}\right) - \ln(\beta)\right) \rightarrow^d N(0, D^2 V)$$

  where $D = \frac{1}{1.1} = 0.91$, $V = 0.3\hat{}2 = 0.09$,or

$$\sqrt{T}(0.095 - 0) \rightarrow^d N(0, 0.91^2 0.09)$$

and a test of $ln(\beta) = 0$ is $t = 0.095/0.2862 = 0.33.$

# 6 Empirical Portfolio Choice–Mean-Variance Implementation

- The solution to the mean-variance problem:

$$\begin{aligned} \min_x var\left(r_{p,t+1}\right) &= x'\Sigma x \\ s.t. E\left(r_p\right) &= x'\mu = \bar{\mu} \end{aligned}$$

  is

$$\begin{aligned} x^* &= \frac{\bar{\mu}}{\mu'\Sigma\mu} \times \Sigma^{-1}\mu \\ &= \lambda\Sigma^{-1}\mu \end{aligned}$$

- Now, we have to rely on econometrics, to implement the solution.
- Two step approach:
  - Solve the model
  - Estimate the parameters and plug them in!

- PLUG-IN APPROACH:
- We continue with the assumption that returns are i.i.d.
- Then, we can estimate

$$\hat{\mu} = \frac{1}{T}\sum_{t=1}^{T} r_{t+1}$$

$$\hat{\Sigma} = \frac{1}{T-N-2}\sum_{t=1}^{T}\left(r_{t+1}-\hat{\mu}\right)\left(r_{t+1}-\hat{\mu}\right)'$$

- We plug in the estimates into the optimal solution

$$\hat{x}^* = \frac{1}{\gamma}\hat{\Sigma}^{-1}\hat{\mu}$$

- Under the normality assumption, this estimator is unbiased, or

$$E\left(\hat{x}^*\right) = \frac{1}{\gamma}E\left(\hat{\Sigma}^{-1}\right)E\left(\hat{\mu}\right)$$

- In the univariate case we can show by the delta method that

$$Var\left(\hat{x}^*\right) = \frac{1}{\gamma^2}\left(\frac{\mu}{\sigma^2}\right)^2\left(\frac{var(\hat{\mu})}{\mu^2}+\frac{var(\hat{\sigma}^2)}{\sigma^4}\right)$$

- Example
- Suppose we have 10 years of monthly data, or $T = 120$.
- Suppose we have a stock with $\mu = 0.06$ and $\sigma = 0.15$.
- Suppose that $\gamma = 5$.
- Note that

$$\begin{aligned} \hat{x}^* &= \frac{1}{\gamma}\frac{\hat{\mu}}{\sigma} \\ &= \frac{0.06}{5 * 0.15\hat{}2} = 0.533 \end{aligned}$$

- Very close to the usual 60/40 advice by financial advisors!
- With i.i.d. returns, the standard errors of the mean and variance are

$$\begin{aligned} var(\hat{\mu}) &= \frac{\sigma}{\sqrt{T}} = \frac{0.15}{\sqrt{120}} = 0.014 \\ var(\hat{\sigma}^2) &= \sqrt{2}\frac{\sigma^2}{\sqrt{T}} = \sqrt{2}\frac{0.15^2}{\sqrt{120}} = 0.003 \end{aligned}$$

- Plugging all these in the formula for $Var\left(\hat{x}^*\right)$ ,we obtain

$$Var\left(\hat{x}^*\right) = 0.14$$

- We can test hypotheses as with every other parameter of interest.

- Estimating $\Sigma$ is very problematic
- Many parameters to estimate
  - Suppose we have 500 assets in the portfolio. We have 125,250 unique elements to estimate.
  - In general, for N assets, we have $N(N+1)/2$ unique elements to estimate!
- We need $\Sigma^{-1}$. Small estimation errors $\hat{\Sigma}$ results in very different $\hat{\Sigma}^{-1}$.
- Solution: Shrink the matrix

$$\hat{\Sigma}^s = \delta S + (1-\delta)\,\hat{\Sigma}$$

where

$$\begin{aligned}
\delta &\approx \frac{1}{T}\frac{A-B}{C} \\
A &= \sum_i \sum_j asy\ var\left(\sqrt{T}\hat{\sigma}_{i,j}\right) \\
B &= \sum \sum asy\ cov\left(\sqrt{T}\hat{\sigma}_{i,j}, \sqrt{T}s_{i,j}\right) \\
C &= \sum \sum \left(\hat{\sigma}_{i,j} - s_{i,j}\right)^2
\end{aligned}$$

where $S$ is often taken to be $I$. For more discussions, see Ledoit and Wolf (2003)

- We can also shrink the weights directly
$$x^s = \delta x_0 + (1 - \delta)\, x^*$$
- This approach is often used in applied work.
- Problem with shrinkage: Ad-hoc. No economic justification for it or for $\delta$.
- Bayesian framework
- Economic constraints [Jagannathan and Ma (JF, 2003)]

- Another solution: Factor models for stock $i$

$$r_{i,t} = \alpha_i + \beta_i f_m + \varepsilon_{i,t}$$

- We can take variances to show that

$$\Sigma_r = \sigma_m^2 \beta\beta' + \Sigma_r$$

  where $\beta$ is a vector of the betas and $\Sigma_r$ is a diagonal matrix with diagonal elements the variances of $\varepsilon_{i,t}$.

- Now, the problem is reduced significantly!

- What about time variation in $\mu$ and $\Sigma$!

## 7 Wold Decomposition: Stationary Processes

- Q: Isn't the AR(1) (or ARMA(p,q)) model restrictive?
- No, because of the Wold decomposition result

Wold's (1938) Theorem: Any zero-mean covariance stationary process $Y_t$ can be represented in the form

$$Y_t = \sum_{j=0}^{\infty} \psi_j \varepsilon_{t-j} + \kappa_t$$

where $\psi_0 = 1$ and $\sum_{j=0}^{\infty} \psi_j^2 < \infty$ (square summable). The term $\varepsilon_t$ is white noise and represents the linear projection error of $Y_t$ on lagged $Y_t's$

$$\varepsilon_t = Y_t - E\left(Y_t | Y_{t-1}, Y_{t-2}, ...\right).$$

The value $\kappa_t$ is uncorrelated with $\varepsilon_{t-j}$ for any $j$ and is a purely deterministic term.

- Can we estimate all $\psi_j$ in the Wold's decomposition?

- The stationary ($|\phi| < 1$) AR(1) model can be written as

$$\begin{aligned} Y_t &= \phi Y_{t-1} + \varepsilon_t \\ (1 - \phi L) Y_t &= \varepsilon_t \\ Y_t &= (1 - \phi L)^{-1} \varepsilon_t \\ &= \sum_{j=0}^{\infty} \phi^j \varepsilon_{t-j} \end{aligned}$$

or $\psi_j = \phi^j$. This is the restriction for the AR(1) model.

- The stationary ARMA(1,1) model can be written as

$$\begin{aligned} Y_t &= \phi Y_{t-1} + \varepsilon_t + \theta \varepsilon_{-1} \\ (1 - \phi L) Y_t &= (1 + \theta L) \varepsilon_t \\ Y_t &= \frac{\varepsilon_t}{(1 - \phi L)} + \frac{\theta \varepsilon_{t-1}}{(1 - \phi L)} \\ &= \varepsilon_t + \sum_{j=1}^{\infty} \phi^{j-1} (\phi + \theta) \varepsilon_{t-j} \end{aligned}$$

or $\psi_j = \phi^{j-1} (\phi + \theta)$.

- And so on.

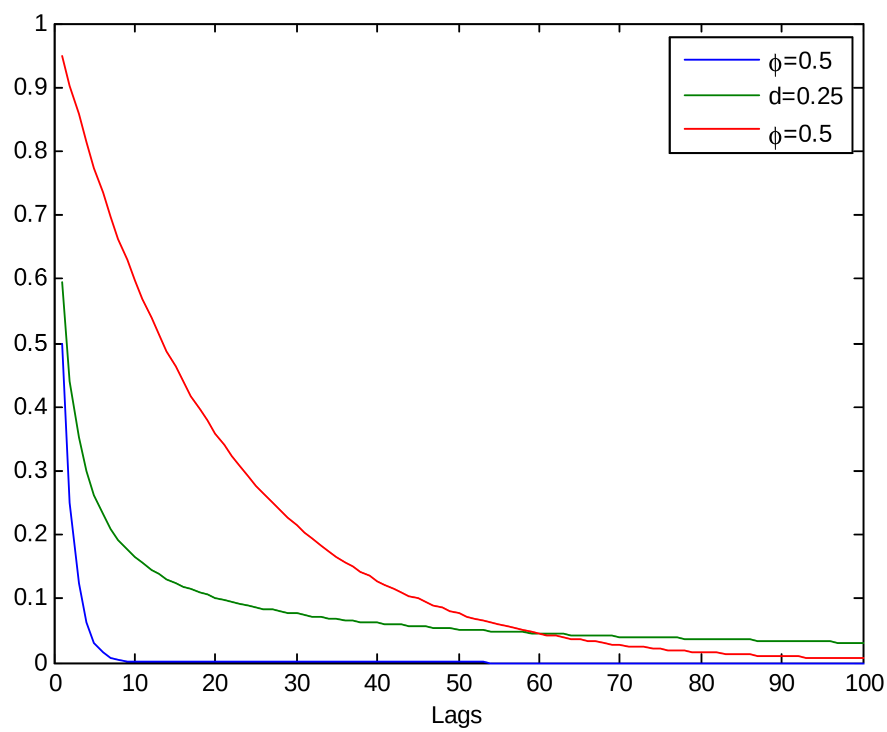
- Another interesting process: Fractionally differencing

$$Y_t = (1-L)^{-d}\,\varepsilon_t$$

- where $d$ is a number between $0$ and 0.5.
- It can be shown (Granger and Joyeux (1980), and Josking (1981)) that

$$\begin{aligned} Y_t &= \sum_{j=0}^{\infty} \eta_j \varepsilon_{t-j} \\ \eta_j &= (1/j!)\,(d+j-1)\,(d+j-2)\,(d+j-3)\ldots(d+1)\,d \\ \eta_j &\approx (j+1)^{d-1}\,,\ \text{for large j} \end{aligned}$$

- Plot of $\eta_j$ for $d = 0.25$ and $\phi^j$ for $\phi = 0.5$ and $\phi = 0.95$



- There is a similar representation in the spectral domaine

Spectral Representation Theorem [e.g., Cramer and Leadbetter (1967)]: Any covariance stationary process $Y_t$ with absolutely summable autocovariances can be represented as

$$Y_t = \mu + \int_0^{\pi} \left[\alpha\left(\omega\right)\cos\left(\omega t\right) + \delta\left(\omega\right)\sin\left(\omega t\right)\right] d\omega$$

where $\alpha\left(.\right)$ and $\delta(.)$ are zero-mean random variables for any fixed frequency $\omega \in \left[0, \pi\right]$. Also, for any frequencies $0 < \omega_1 < \omega_2 < \omega_3 < \omega_4 < \pi$, $\int_{\omega_1}^{\omega_2} \alpha(\omega) d\omega$ is uncorrelated with $\int_{\omega_3}^{\omega_4} \alpha(\omega) d\omega$ and the variable $\int_{\omega_1}^{\omega_2} \delta(\omega) d\omega$ is uncorrelated with $\int_{\omega_3}^{\omega_4} \delta(\omega) d\omega$.

- Different (but equivalent) way of looking at a time-series.