Endogenous Retirement and Portfolio Choice

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Abstract
This paper studies the interaction between the portfolio and retirement decisions. We solve explicitly a standard life-cycle model with consumption and leisure. We find that human capital can be valued as a portfolio of European put options on stock. We propose that there is a natural definition of retirement in the life-cycle models and show that the state space can be partitioned into four regions: “Retire,” “Vacation,” “Work,” and “Work-forever.” We find that an agent with a constant wage will retire early (late) if his investment performs well (poorly) and will invest less in stocks than those who never retire but more than those who never work. We show that poor agents (those who have low initial financial wealth) should invest more of their wealth in stock. Somewhat surprisingly, if the wage is correlated with the stock return, an agent may retire only when his stock does not perform well.

PRELIMINARY-PLEASE DO NOT QUOTE

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1 Introduction

This paper studies the following problems: how does the performance of an agent’s portfolio affect the retirement decision? In turn, how does the retirement decision affect the portfolio choice of the agent?

We answer these questions using a standard life-cycle model. An agent has preferences over consumption and leisure and a bequest motive. The agent can invest in a riskless bank account and risky stock, where the stock price dynamics are described by a binomial tree. The wage is a function of the stock price. Therefore, the only state variable is the stock price. This simple setup allows us to easily characterize the solution and its implications.

We propose a definition of retirement in life-cycle models. We then show that the state space can be divided into four regions: “Retire,” “Vacation”, “Work,” and “Work-forever.” The agent will be in one of these four regions for a given level of initial wealth, wage, etc., depending on the value of two variables: his age, and how well his stock has done. The results are not so surprising for the cases of a constant wage or of a wage which is decreasing in stock price. If the agent’s stock performs well enough, then in a given period he may choose not to work (enter the “Vacation” region). If the agent’s stock performs exceptionally well, he may find it optimal to retire (enter the “Retire” region). That is, it may be optimal for the agent to never even think of working again, no matter how poorly his future stock returns are. Most of the time, the agent is working, and it is possible that the agent could decide to take a vacation in the future, retire in the future, or neither (here the agent is in the “Work” region). If the agent’s stock has performed poorly in the past, the agent is likely to work more, reflecting his need to use labor income to cushion the shock to financial wealth. If the performance of his stock has been sufficiently poor, the agent will work for the rest of his life (enter the “Work-forever” region). That is, no matter how well his stock performs in the future, the agent will never find it optimal to not work at least some positive amount.

When the wage is an increasing function of the stock price, the results can change drastically. If the relationship between the wage and the stock price is calibrated so that the volatility of the wage matches that observed empirically, the agent will continue to work when his stock portfolio has done the better, and retire when his stock portfolio has done the worse. The reason for this is that when the wage is an increasing function of the stock price, there is little value in working when the stock price is low because the wage
is low. Conversely, when the stock price is high, the wage is very high, and so the agent is best off continuing to work, even though the agent may be very wealthy.

The agent takes into account the value of his human capital, which is the present value of his future labor income, when making his portfolio decision. The retirement decision affects the human capital therefore the portfolio decision. We show that the agent’s human capital can be valued as a portfolio of European put options, where the underlying asset is an increasing function of the stock price and the strike price is proportional to the wage, with the proportional constant being the maximum number of feasible working hours.

When the wage is a constant or decreasing function of stock price, the human capital is negatively correlated with the stock return. Because of this, Bodie, Merton, and Samuelson (1992) (BMS) show that an agent will invest more of his financial wealth in the stock than those who do not have labor income. We show that, on the other hand, an agent facing retirement will invest less in stock than one who will never retire. When the wage is an increasing function of stock price, the human capital is positively correlated with the stock return. As argued by Bodie, Merton, and Samuelson (1992), the agent will invest less of his financial wealth in the stock than one who does not have labor income and may possibly short the stock. However, we show that an agent facing retirement will invest more in stock than one who will never retire.

Our paper also shows that agents with less financial wealth (poorer people) should participate more in the stock market. When their wages are constant or negatively correlated with the stock market, they should invest more in stock than agents with more financial wealth. Conversely, when their wages are positively correlated with the stock market, agents with less financial wealth should short more than those with more financial wealth.

Our work is closely related to Bodie, Merton, and Samuelson (1992). The life-cycle model in our paper and in BMS are almost identical. BMS solve the model only in the case where the agent is sufficiently poor so that he will work no matter how good stock returns are. We solve the model for all cases. This solution allows us to study endogenous retirement.

Basak (1999) considers the labor, consumption, and investment decision of a representative agent in a general equilibrium framework with a single-sector production firm. In that case, the agent will never retire because the marginal utility of the production firm goes to infinity as the labor supply goes to zero.
Sundaresan and Zapatero (1997) study optimal retirement with an emphasis on the valuation of pension funds. They abstract from modeling the disutility associated with labor. In most papers on endogenous retirement, for example, Kingston (2000) and Gustman and Steinmeier (2001), the time of retirement is independent of the state of the economy. Because of this, there is no interaction between the retirement and portfolio decision.

The paper is structured as follows. In Section 2 we set up the basic model. We derive the optimal consumption, labor and investment choices. In Section 3 we characterize the “Retire” and “Work-forever” regions in the state space and we then study the effect of retirement on portfolio choice. Section 4 examines the optimal portfolio choice. Section 5 presents numerical calculations. Section 6 concludes. In the appendix, we solve a continuous-time version of the model and we also leave technical details there.

2 Life-Cycle Model

2.1 Setup

The agent lives for $T$ periods, indexed from 0 to $T - 1$. The agent has a bequest in period $T$. At the beginning of period $t$, the agent has financial wealth $F_t$, will consume $c_t$ dollars of consumption, and work $k_t$ hours for a wage $p_t$.\footnote{Throughout this paper $F_t$ denotes financial wealth at time $t$. It is the total wealth minus the human capital of the agent.} The wage at any time $t$ is related to the stock price via

$$p_t = \alpha S^t_t.$$  

This is a discrete time version of the formulation used by Bodie, Merton, and Samuelson (1992). In this case, the wage is either perfectly positively correlated with the stock price ($\rho > 0$), perfectly negatively correlated with the stock price ($\rho < 0$), or uncorrelated with the stock price ($\rho = 0$). The agent is endowed with a total supply of $K$ hours which he can choose to allocate between labor and leisure. Therefore, within a given period $t$, the total amount of leisure the agent consumes is $l_t = K - k_t$. After consuming $c_t$ and earning his labor income $p_t k_t$, he invests the end-of-period wealth $F_t + p_t k_t - c_t$ in financial assets.

There are two financial securities available. The riskless asset, which we call a “bank account,” has a constant gross return of $R_f$. The risky asset, which we call “stock,” has
a gross return $R_f + c_t$ in period $t$. The financial wealth $F_{t+1}$ of the agent at the beginning of period $t + 1$ is

$$F_{t+1} = (F_t + p_t k_t - c_t) \left(R_f + \phi_t \epsilon_{t+1}\right).$$  \hspace{1cm} (2)

Here $\phi_t$ is the proportion of total wealth, after consumption but including labor income, which is invested in stock. Based on the form of the wage in equation (1) and the budget equation, markets are complete. While these assumptions reduce the generality of the model, they will allow us to exhibit the interaction between retirement and portfolio decision in a parsimonious fashion.

Equation (2) is written in a form which shows that financial wealth and labor income are used to invest and consume. An alternative formulation of the budget equation can be obtained by thinking of the labor income in a different way. One can consider the agent as having an endowment per period of $p_t K$ dollars (in addition to the financial wealth $F_t$ he already has coming into period $t$), out of which he consumes, invests, and “buys” leisure, with the total cost of leisure being $p_t l_t$. Using these variables, the budget equation can be written as

$$F_{t+1} = (F_t + p_t K - p_t l_t - c_t) \left(R_f + \phi_t \epsilon_{t+1}\right).$$  \hspace{1cm} (3)

Here $\phi_t (F_t + p_t K - p_t l_t - c_t)$ is the amount of investment wealth (wealth after consumption and leisure) invested in stock, with $\phi_t$ being the portfolio weight. We write the agent’s optimization problem as

$$\max_{\{c_t, \{l_t\}, \{\phi_t\}} \quad E_0 \left[ \sum_{t=0}^{T-1} e^{-\beta t} \left( \frac{c_t^{1-\gamma}}{1 - \gamma} + \frac{\alpha_l (p_t l_t)^{1-\gamma}}{1 - \gamma} \right) + e^{-\beta T \alpha_b} \frac{F_T^{1-\gamma}}{1 - \gamma} \right],$$  \hspace{1cm} (4)

s.t. \hspace{0.5cm} $$F_{t+1} = (F_t + p_t K - p_t l_t - c_t) \left(R_f + \phi_t \epsilon_{t+1}\right),$$  \hspace{1cm} (5)

$$l_t \leq K.$$

where

$$\beta = \text{Time discount rate},$$

$$\alpha_l = \text{Relative weight of leisure expenditure in utility},$$

$$\alpha_b = \text{Relative weight of bequest in utility},$$

$$\gamma = \text{Coefficient of relative risk aversion}.$$

We use an additive utility function over consumption ($c_t$), the leisure expenditure ($p_t l_t$), and the bequest ($F_T$), where the risk aversion parameter ($\gamma$) is identical for each variable.
This is done to keep the solution simple and tractable. The time discount factor \((\beta)\) is standard. \(\alpha_t\) and \(\alpha_b\) can be thought of as the relative weight of the leisure and the bequest motives in the utility function. For example, the larger \(\alpha_t\), the more leisure counts in the agent’s utility.

It is straightforward to allow for the more usual multiplicative Cobb-Douglas utility function over leisure and consumption with different risk aversion parameters over these variables. We adopted the above additive form because it is commonly used in finance literature and it allows for risk aversion coefficient \(\gamma\) to be greater than 1. The additive functional form is used in recent empirical studies, for example, French (2001).

For the sake of brevity, we introduce the variables

\[
\bar{e}_t = p_t K, \quad e_t = p_t, \tag{7}
\]

which represent the ”labor income endowment” and the “leisure expenditure,” respectively. Using these variables, the budget equation in (5) can be rewritten as

\[
F_{t+1} = (F_t + \bar{e}_t - c_t - e_t) \left( R_f + \phi_t \epsilon_{t+1} \right). \tag{9}
\]

The agent has total resources which include financial wealth \((F_t)\) and the labor income endowment \((\bar{e}_t)\) out of which to consume \((c_t)\) and ”buy leisure,” where the total cost of leisure is \((e_t)\).

Thus, the optimization problem becomes

\[
\max_{\{c_t\}, \{\bar{e}_t\}, \{\phi_t\}} \quad \mathbb{E}_0 \left[ \sum_{t=0}^{T-1} e^{-\beta t} \left( \frac{\epsilon_t^{1-\gamma}}{1-\gamma} + \frac{\alpha_t \epsilon_t^{1-\gamma}}{1-\gamma} \right) + e^{-\beta T} \frac{\alpha_b F_T^{1-\gamma}}{1-\gamma} \right], \tag{10}
\]

s.t.

\[
F_{t+1} = (F_t - c_t + \bar{e}_t - e_t) \left( R_f + \phi_t \epsilon_{t+1} \right), \tag{11}
\]

\[
e_t \leq \bar{e}_t. \tag{12}
\]

The reason for specifying preferences over the “leisure expenditure” \((e_t)\) rather than just “leisure” \((l_t)\) is so that both consumption and the leisure choice are measured in dollars.\(^2\)

\(^2\)Another way to see why this is appropriate is to consider the dimensions (or units) of \(\alpha_t\), which must be such that \((\alpha_t \epsilon_t^{1-\gamma} t)\) has units of dollars. A natural scale factor in this case would be the price of labor \(p\), so that

\[
\dim(\alpha_t) \sim p^{1-\gamma}. \tag{13}
\]

For \(\gamma = 5\) and \(p = \$50000\), this gives \(\alpha_t \sim 10^{-10}\). Rather than having to justify choosing \(\alpha_t\) to have a particular set value, we let the price scale the leisure choice appropriately.
2.2 Solution

We solve for the most general case including consumption ($c_t$), the leisure consumption decision ($e_t$), and bequests ($F_t$) in a framework where stock returns have a binomial distribution.

For the stock in this binomial model, we define the excess return ($\epsilon_t$) as the return over the gross riskless rate ($R_f$), so that the stock’s gross return in period $t$ is $R_f + \epsilon_t$. The probability distribution of the excess stock return in period $t+1$ conditional on period $t$ information is shown in figure 1. Note that we must have $ud < 0$ to exclude arbitrage opportunities. We will only assume that the expected excess stock return is strictly positive ($\mu = \zeta_u u + \zeta_d d > 0$). The stock-price dynamics, can be represented in the form of the well-known binomial tree (Cox, Ross, and Rubinstein (1979)). Over several periods, the stock price dynamics follow a multiperiod binomial tree, where several of the single period binomial tress are connected together as in figure 2.
\[ S_{t+1} = (R_f + u)S_t \]
\[ S_{t+2} = (R_f + u)(R_f + u)S_t \]

Figure 2: Multiperiod Binomial Tree Model for the Stock Price $S$. 

\[ S_{t+1} = (R_f + d)S_t \]
\[ S_{t+2} = (R_f + d)(R_f + d)S_t \]
We define the single-period pricing kernel\footnote{No arbitrage conditions guarantee the existence of a pricing kernel. See Duffie (1996) or Harrison and Kreps (1979).} $\xi_t$ as

$$\xi^a_t = \frac{-d}{\zeta u R_f(u - d)}$$

$$\xi^b_t = \frac{u}{\zeta d R_f(u - d)}$$

(14)

Then it can be readily verified that

$$E_{t-1} [\xi_t] = \frac{1}{R_f}, \quad E_{t-1} [\xi_t S_t] = S_{t-1},$$

(15)

or equivalently

$$E_{t-1} [\xi_t R_f] = 1, \quad E_{t-1} [\xi_t (R_f + \epsilon_t)] = 1.$$  

(16)

We define the multi-period pricing kernel $(\pi_t)$ as

$$\pi_t = \prod_{s=0}^{t} \xi_s, \quad t = 1, 2, ..., T,$$

(17)

where we use the convention that $\pi_0 = 1$. It should be easy to see that while $\xi_t$ will price payoffs in period $t$ condition on the information set of $t - 1$, $\pi_t$ will price payoffs in period $t$ condition on the information of period 0. Using the pricing kernel $\pi_t$, we can reduce the multiple budget constraints (equation (11)) to a single wealth constraint (equation (20)), as shown in the following lemma.

**Lemma 2.1** The financial wealth process $\{F_t\}_{t=0}^{T}$, consumption process $\{c_t\}_{t=0}^{T-1}$, and leisure expenditure process $\{\bar{e}_t\}_{t=0}^{T-1}$ satisfy the budget constraint if and only if

$$E_0 [\pi_T F_T] + \sum_{t=0}^{T-1} E_0 [\pi_t c_t] = F_0 + \sum_{t=0}^{T-1} E_0 [\pi_t (\bar{e}_t - e_t)].$$  

(18)

The proof is given in appendix B. This lemma is very intuitive. It states that the sum of the present value of the bequest ($F_T$) and the future consumption stream ($\{c_t\}_{t=0}^{T-1}$) is equal to the sum of the initial financial wealth ($F_0$) and the present value of the labor income stream ($\{\bar{e}_t - e_t\}_{t=0}^{T-1}$).

We now use this simplified budget constraint to rewrite the original optimization problem:
Proposition 2.1: The optimization problem (10)-(12) is equivalent to the following problem

\[
\max_{\{\alpha_t, \{e_t\}_{F_T}} \quad \mathbb{E}_0 \left[ \sum_{t=0}^{T-1} e^{-\beta t} \left( \frac{c_t^{1-\gamma}}{1-\gamma} + \frac{\alpha_t e_t^{1-\gamma}}{1-\gamma} \right) + e^{-\beta T} \frac{\alpha_b F_T^{1-\gamma}}{1-\gamma} \right], \tag{19}
\]

s.t. \[
\mathbb{E}_0 [\pi_T F_T] + \sum_{t=0}^{T-1} \mathbb{E}_0 [\pi_t e_t] = F_0 + \sum_{t=0}^{T-1} \mathbb{E}_0 [\pi_t (\bar{e}_t - e_t)] , \tag{20}
\]

\[e_t \leq \bar{e}_t \quad \text{for all states and } t = 0, 1, ..., T - 1. \tag{21}\]

This is a recasting of the problem given in equations (10)-(12) and is a direct consequence of lemma 2.1. Using this proposition, we can solve directly for the optimal consumption, leisure and bequest. After this is done, we can solve for the portfolio choice that implements the optimal consumption, leisure and bequest. The problem as formulated in equations (19-21) is the discrete-time analog of the martingale approach to solving the portfolio choice problem first proposed by Cox and Huang (1989).4

The Lagrangian for the optimization problem in (19) can be written as

\[
L = \mathbb{E}_0 \left[ \sum_{t=0}^{T-1} e^{-\beta t} \left( \frac{c_t^{1-\gamma}}{1-\gamma} + \frac{\alpha_t e_t^{1-\gamma}}{1-\gamma} \right) + e^{-\beta T} \frac{\alpha_b F_T^{1-\gamma}}{1-\gamma} \right] \\
- \lambda \left( \mathbb{E}_0 [\pi_T F_T] + \sum_{t=0}^{T-1} \mathbb{E}_0 [\pi_t e_t] - \sum_{t=0}^{T-1} \mathbb{E}_0 [\pi_t (\bar{e}_t - e_t)] - F_0 \right) \\
+ \sum_{t=0}^{T-1} \mathbb{E}_0 [\eta_t (\bar{e}_t - e_t)] , \tag{22}\]

where \(\lambda\) and \(\eta_t\) are the Lagrange multipliers associated with the constraints in (20) and (21), respectively. The first order conditions are obtained by taking the derivatives of the Lagrangian with respect to the choice variables \(c_t, e_t,\) and \(F_T:\)

\[
e^{-\beta t} c_t^{\gamma-\gamma} = \lambda \pi_t, \quad \text{for all states and } t = 0, 1, ..., T - 1 ,
\]

\[
\alpha_t e^{-\beta t} e_t^{\gamma-\gamma} = \lambda \pi_t + \eta_t, \quad \text{for all states and } t = 0, 1, ..., T - 1 , \tag{23}
\]

\[
\alpha_b e^{-\beta t} F_T^{\gamma-\gamma} = \lambda \pi_T, \quad \text{for all states and } t = 0, 1, ..., T - 1 .
\]

Since the pricing kernel \(\pi_t\) is low when the stock price at time \(t\) is high, the above equations implies that the agent will have a higher level of consumption \((c_t^*)\), a higher level of leisure consumption \((e_t^*)\), and leave a larger bequest \((F_T^*)\) when the stock price is high.

Note that the lagrange multiplier (of the resource constraint) \(\lambda\), which is also the shadow price of the resource constraint (20), is independent of state and time. Because

\footnote{For a binomial version of the martingale approach, see He and Pearson (1991) and Pliska (1997).}
the resource constraint is also binding, we know that $\lambda > 0$. The lagrange multiplier (of the leisure constraint) $\eta_t$, which is the shadow price of the leisure constraint (21), is dependent on state and time and it is zero in many states when the leisure constraint is not binding. In fact, this is only case studied by Bodie, Merton, and Samuelson (1992).

When the leisure constraint is binding, that is, when the agent takes the maximal leisure $\bar{e}_t$, the lagrange multiplier $\eta_t \neq 0$. When this happens, the agent would like to take the leisure at the level given by

$$\hat{e}_t = \left( \frac{\alpha_t}{\lambda \pi_t e^{\beta t}} \right)^{\frac{1}{\gamma}},$$

(24)

or equivalently enjoy

$$\hat{t}_t = \left( \frac{\alpha_t}{\lambda \pi_t e^{\beta t}} \right)^{\frac{1}{\gamma}}$$

(25)

hours of leisure. However, the maximum number of hours that he can consume is $K$ hours of leisure. In this case, the agent can only consume this maximum possible number of hours of leisure. The agent will not work at all this period. We can also obtain this result the following way. The shadow price $\eta_t$, for the leisure expenditure, is given by

$$\eta_t = \alpha_t e^{-\beta_t} e_t^{s-\gamma} - \lambda \pi_t$$

(26)

which is the difference between the attainable marginal utility of leisure ($\alpha_t e^{-\beta_t} e_t^{s-\gamma}$) and the optimal marginal utility of leisure ($\lambda \pi_t$). It is clear from this equation that the shadow price $\eta_t$ is zero when the constraint on leisure is not binding and the agent exercises his endogenous put option on labor.

It is often convenient to express the optimal solution without $\eta_t$, as we do in the follows.

**Proposition 2.2** The optimal consumption, leisure expenditure, and bequest are given by

$$e_t^* = \left( \frac{1}{\lambda \pi_t e^{\beta_t}} \right)^{\frac{1}{\gamma}}, \quad t = 0, 1, 2, ..., T - 1,$$

$$e_t^* = \min \left( \left( \frac{\alpha_t}{\lambda \pi_t e^{\beta_t}} \right)^{\frac{1}{\gamma}}, \bar{e}_t \right), \quad t = 0, 1, 2, ..., T - 1,$$

(27)

$$F_T^* = \left( \frac{\alpha_0}{\lambda \pi_T e^{\beta T}} \right)^{\frac{1}{\gamma}}.$$
Note that

\[ e_t^* = \bar{e}_t \]

only if \( \eta_t > 0 \), that is, only if the leisure expenditure resource constraint in (21) is binding. Otherwise,

\[ e_t^* = \left( \frac{\alpha_t}{\lambda \pi_t e^{\beta t}} \right)^{\frac{1}{7}}. \]

This is why the optimal solution for the leisure expenditure in equation (27) involves the \( \min \) function.

Also, it should be clear from the definition of \( \pi_t \) in (17) and the form of the solutions in (27) that the optimal values which the choice variables take on at time \( t \) only depend on the stock price at time \( t \) (or equivalently on the value of the pricing kernel \( \pi_t \)) and not on the history of stock prices. That is, if \( S_t \) is related to \( S_0 \) by

\[ S_t = S_0 (R_f + u)^{n_u} (R_f + d)^{t-n_u}, \]  

then the optimal quantities do not depend on which order the \( n_u \) up-ticks and \( t-n_u \) down-ticks in the stock price occurred. Therefore, the value of the consumption \( (c_t^*) \), leisure expenditure \( (e_t^*) \), or bequest \( (F_t^*) \) can be determined by knowing two things: the time \( (t) \), and the number of up-ticks \( (n_u) \) that there have been in the stock price up to and including that period.

According to Proposition 2.2, we only need to solve the Lagrange multiplier \( \lambda \) to obtain the optimal consumption, leisure expenditure, and bequest. The expression for \( \lambda \) is given in the next proposition.

**Proposition 2.3** The equation for the Lagrange multiplier is given by

\[
\lambda^{-\frac{1}{7}} \left( \alpha_0^\frac{1}{7} e^{-\frac{\theta_T}{\pi_T}} E_0 \left[ \pi_T^{1-\frac{1}{7}} \right] + \sum_{t=0}^{T-1} e^{-\frac{\theta_T}{\pi_T}} E_0 \left[ \pi_t^{1-\frac{1}{7}} \right] \right) = \\
F_0 + \sum_{t=0}^{T-1} E_0 \left[ \pi_t \max \left( \bar{e}_t - \left( \frac{\alpha_t}{\lambda \pi_t e^{\beta t}} \right)^{\frac{1}{7}}, 0 \right) \right].
\]

A proof that a solution to this equation always exists is given in appendix C. It is the simplicity of this equation which motivated the specification of the model. The fact that
there is only this algebraic equation to solve tremendously simplifies the problem. We should remark that the assumption of complete markets is crucial for this simplification.

One can show that the Lagrange multiplier \( \lambda \) is decreasing in \( F_0 \) (a proof is given in appendix C). Since the Lagrange multiplier \( \lambda \) is the shadow price of wealth, and thus the marginal utility with respect to wealth, this says that the marginal utility of wealth is declining with wealth. Using this property and the relations for the optimal consumption, leisure, and bequest choices in (27), one can easily show that as the agent’s initial financial wealth \( (F_0) \) increases, he tends to consume more, work less, and leave a larger bequest.

Having solved for optimal consumption, optimal leisure, and optimal bequest, we now derive the optimal financial wealth \( F^*_t \).

**Proposition 2.4** For time \( 0 \leq t < T \), the financial wealth \( F^*_t \) attained by optimal strategies is given by

\[
F^*_t = \mathbb{E}_t \left[ \frac{\pi_T}{\pi_t} F^*_T \right] + \sum_{s=t}^{T-1} \mathbb{E}_t \left[ \frac{\pi_s}{\pi_t} c^*_s \right] - \sum_{s=t}^{T-1} \mathbb{E}_t \left[ \frac{\pi_s}{\pi_t} \left( \tilde{c}_s - c^*_s \right) \right],
\]

\[
= \mathbb{E}_t \left[ \frac{\pi_T}{\pi_t} F^*_T \right] + \sum_{s=t}^{T-1} \mathbb{E}_t \left[ \frac{\pi_s}{\pi_t} c^*_s \right] - \sum_{s=t}^{T-1} \mathbb{E}_t \left[ \frac{\pi_s}{\pi_t} \left( \max \left( \tilde{c}_s - \left( \frac{\alpha \lambda}{\alpha \pi_t e^{\beta s}} \right)^{\frac{1}{\beta}} \right), 0 \right) \right]. \tag{30}
\]

Equation (30), shows that the optimal wealth at time \( t \) consists of the present value of the bequest plus the present value of the consumption minus the value of the future labor income.

Here again it should be clear that since the quantities in equation (30) only depend on the value of \( \pi_t \) in a given state of the economy, the optimal financial wealth only depends on the current state (i.e., \( \pi_t \)) as well. Only the time and the number of up-ticks that have occurred in the stock price to date are required to figure out the optimal financial wealth.

Note that, from (30), there is the possibility of optimal financial wealth becoming negative (i.e., borrowing), which is well known to happen in the life-cycle models with labor income. The agent will optimally choose borrowing to smooth consumption if there are large future income. Because our agent has a power utility function in consumption and bequest, the agent’s consumption and financial wealth in the last period have to be strictly positive and the total wealth is positive in all states and time.

We will study the optimal portfolio weight after next section.
3 Vacation and Retirement

In this section, we propose a definition of retirement and study the various classes of agent’s leisure choice.

The constraints on the leisure choice are $l_t \leq K$ and $e_t \leq \bar{e}_t$ for the problems in (4)-(6) and (10)-(12), respectively. Consider the formulation of the leisure choice constraint in terms of hours ($l_t \leq K$). The agent is endowed with a maximum of $K$ hours of leisure. When $l_t^* = K$, the agent will not work at all. For some states with $l_t^* = K$, the agent may optimally choose to work later in life, that is $l_s^* < K$ for $s > t$. We define such a state as vacation state when this occurs (the agent is in the “Vacation” region). For other states, $l_s^* = K$ for all $s \geq t$. We define such a state as retire state—the agent is retired when this occurs (the agent is in the “Retire” region). If $l_s^* < K$ for all $s \geq t$, then the agent will not consume the maximum amount of feasible leisure in this period and all subsequent periods. That is, the agent will work for this period and all subsequent periods, we define such states as “Work forever” states, (the agent is in the “Work-forever” region). If none of these cases obtains, then we define such states as “Working” states—the agent is working (in the “Work” region).

In the literature, the retirement age or decision has been defined by the following characteristics:

- The retirement age is exogenously determined in a random fashion,
- The retirement age is exogenously specified due to some policy,
- The retirement decision is independent of the state of the economy,
- The decision to retire is possibly due to declining health.

Here we abstract from the issues outlined above. We wish to study, from an efficiency point of view, the joint decision over consumption, labor, and investment. Our definition of retirement is an alternative to the one often used in studies involving labor, consumption and the retirement decision. Often it is defined as the time when an agent decides that the marginal utility of more time in the workforce outweighs the marginal benefit (Benitez-Silva (2000), Kingston (2000), Stock and Wise (1990), Sundaresan and Zapatero (1997)). However, some studies suggest that studying what an agent would do in cases where the above characteristics are not in force may be interesting.
Sevak (2001) tests whether the timing of retirement responds to unexpected changes in wealth. His motivation stems from two recently observed phenomena. First is that the percentage of defined contribution pension plans held in the U.S. has risen from 39% in 1975 to 52% in 1993. Second, the number of households owning stock has risen from approximately one-third in 1989 to one-half in 1999. In order to disentangle the effect of wealth on leisure consumption (in particular, whether leisure is a normal good) from other effects (in particular the effects of retirement plans on wealth, or the disutility associated with labor), he studies a subsample from the Health and Retirement Study. He uses the asset holdings section of the data to isolate individuals who experienced large windfall gains due to stock holdings. By decomposing capital gains into expected and unexpected components (and taking account of other effects, such as the subject’s health), he is able to estimate the effect of a wealth increase on the probability of retirement. The number he uses to measure the effect is the elasticity of the probability of retirement with respect to wealth. He finds that for a wealth shock which would allow consumption to double, this number (the elasticity of the probability of retirement) is approximately 39-52% for men aged 55-65. For women, the same figure is between 0 and 35%.

In a related article, Khitatrakun (2001) also uses the Health and Retirement Study to try to ascertain the effect of a wealth shock on the retirement decision. First, he establishes that subjects are able to form reasonably accurate projections of their retirement ages. By using the subjects’ own estimates of when they think they will retire, he establishes that those in the 90th percentile of wealth gains in the 1990s (times of large stock market returns) will typically retire 5-6 months earlier than they originally thought.

Gustman and Steinmeier (1986) do not use our definition directly, but point out that one drawback of their model is that it “...is not appropriate for analyzing short term labor supply responses to unanticipated changes in incentives.”

Since the “Vacation” and “Work” regions need no characterization beyond that given in section 2.2, in the next two sections we obtain expressions for the level of wealth such that the “Work-Forever” and “Retire” regions are entered.

3.1 Work-Forever Region

We will denote $F_{wf}(t)$ as the level of wealth at time $t$ that separates the “Work forever” region from “Working” region. Consider the value of $F_{wf}(0)$. If $F_0 \leq F_{wf}(0)$, the agent will never choose to go on vacation before he leaves the economy. Clearly from equation
(27), we must have that
\[
\left( \frac{\alpha_t}{\lambda \alpha t e^{\beta t}} \right)^{\frac{1}{2}} \leq \bar{e}_t \quad \text{for all states and } t = 0, 1, ..., T - 1. \tag{31}
\]
This implies that
\[
\lambda \geq \lambda_{\text{max}} = \frac{\alpha_t}{\min_{t, \text{state}} \left( (\alpha t e^{\beta t})^{\frac{1}{2}} \bar{e}_t \right)^{\frac{1}{2}}}. \tag{32}
\]
Combining (32), (27) and (18) gives the condition on the initial wealth \(F_0\) such that the agent will work forever as
\[
F_0 \leq F_{w_f}(0), \tag{33}
\]
where
\[
F_{w_f}(0) = \left( \frac{1}{\lambda_{\text{max}}} \right)^{\frac{1}{2}} \left( \alpha_0^{\frac{1}{2}} \Omega^T + (1 + \alpha_q^{\frac{1}{2}}) \frac{\Omega^T - 1}{\Omega - 1} \right) - H_{\text{max}}^c(0), \tag{34}
\]
\(H_{\text{max}}^c(t)\) is defined as
\[
H_{\text{max}}^c(t) = \sum_{s = t}^{T-1} E_t \left( \frac{\pi_s}{\pi_t} \bar{f}_s \right). \tag{35}
\]
\(\Omega\) is a positive constant which is defined in appendix C. It is clear from this last term that \(F_{w_f}(0)\) could be negative, which simply means that there is some probability that the agent will go on vacation in some period if initial financial wealth \((F_0)\) is positive. This will usually be an increasing function of time or age, or equivalently a decreasing function of horizon; the longer the horizon of the agent, the more likely it is that he will possibly have great stock performance in the future.

Clearly, the value \(F_{w_f}(t)\) for \(t > 0\) could also be calculated. It is given by
\[
F_{w_f}(t) = \left( \frac{1}{\lambda_{\text{max}}} \right)^{\frac{1}{2}} \left( \alpha_0^{\frac{1}{2}} \Omega^{T-t} + (1 + \alpha_q^{\frac{1}{2}}) \frac{\Omega^{T-t} - 1}{\Omega - 1} \right) - H_{\text{max}}^c(t), \tag{36}
\]
where we have labeled \(\lambda_{\text{max}}\) with a time superscript.\(^5\) If \(F_t^* \leq F_{w_f}(t)\), then the agent will always find it optimal at time \(t\), and every period after time \(t\), to work some strictly positive amount.

### 3.2 Retire Region

In the previous section, we calculated an expression for how low the agent’s wealth would have to fall for him to work for the rest of his life. Here we calculate an expression for

\(^5\lambda_{\text{max}}\) corresponds to the Lagrange multiplier one would find by solving (31) at time \(t\)
the wealth level which the agent must attain so that he will find it optimal to retire.\textsuperscript{6} We will denote this level of wealth at time 0 as $F_{\text{ret}}(0)$. Clearly from equation (27), we must have that
\[
\left( \frac{\alpha_t \lambda_{t+1}}{\lambda_t e^{\beta t}} \right)^{\frac{1}{\gamma}} \leq \bar{\epsilon}_t \quad \text{for all states and } t = 0, 1, ..., T - 1. \tag{37}
\]
This implies that
\[
\lambda \leq \lambda_{\text{min}} = \frac{\alpha_t}{\max_{t, \text{state}} \left( \left( \frac{\lambda_t e^{\beta t}}{\lambda_{t+1}} \right)^{\frac{1}{\gamma}} \bar{\epsilon}_t \right)^{\gamma}}. \tag{38}
\]
Combining (32), (27) and (18) gives the condition on the initial wealth such that the agent will retire as
\[
F_0 \geq F_{\text{ret}}(0), \tag{39}
\]
where
\[
F_{\text{ret}}(0) = \left( \frac{1}{\lambda_{\text{min}}} \right)^{\frac{1}{\gamma}} \left( \frac{1}{\alpha_0} \Omega^T + \frac{\Omega^T - 1}{\Omega - 1} \right). \tag{40}
\]
$\Omega$ is a positive constant which is defined in appendix C. Here it is clear that this is always positive. This will be a decreasing function of horizon. The longer the horizon of the agent, the more likely that stock returns will be sufficiently bad so that he will find it optimal to work at least some small amount in the future.

Clearly, we can also calculate this value for any time $t > 0$. It is evident from (40) that the expression for $F_{\text{ret}}(t)$ is given by
\[
F_{\text{ret}}(t) = \left( \frac{1}{\lambda_{\text{min}}} \right)^{\frac{1}{\gamma}} \left( \frac{1}{\alpha_0} \Omega^{T-t} + \frac{\Omega^{T-t} - 1}{\Omega - 1} \right), \tag{41}
\]
where we have labeled $\lambda_{\text{min}}$ with a time superscript.\textsuperscript{7} If $F_t^* \geq F_{\text{ret}}(t)$, then the agent will always find it optimal at time $t$, and every period after time $t$, to stay retired.

Our results differ from those of Bodie, Merton, and Samuelson (1992). In our solution, we explicitly take into account the leisure constraint (21) so that we are able to obtain the retirement region. BMS were primarily interested in how the possibility of labor might affect the portfolio choice of the agent, and ignored the effects of retirement. In particular, they used a solution for the value function which is interior with respect to the leisure

\textsuperscript{6}It should be noted that in our simple model of stock returns, the level to which the stock price can fall with any positive probability is strictly limited above zero, and that the higher the current stock price, the higher this lower limit is.

\textsuperscript{7}$\lambda_{\text{min}}^*$ corresponds to the Lagrange multiplier one would find by solving (37) at time $t$. 

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constraint.\textsuperscript{8} This is equivalent, in our model, to setting $F_0 \leq F_{wf}(0)$ (see section 3.1). That is, the initial wealth level is so low that the agent would find it optimal to work every period regardless of how well his stock performed. In other words, BMS only study the "work forever" region.

4 Portfolio Choice

Proposition 4.1 The optimal portfolio weight $\phi_{t-1}^*$ is given by

$$
\phi_t^* = \frac{1}{\mu} \left( \frac{E_t \left[ F_{t+1}^* \right]}{F_t^* - c_t^* + \bar{e}_t - e_t^*} - R_f \right),
$$

$$
= \phi_M \left( 1 + \frac{\sum_{s=t}^{T-1} E_t \left[ \frac{\varphi_s}{\pi_t} \right]}{F_t^* - c_t^* + e_t^* - e_t^*} \right)
+ \frac{\sum_{s=t}^{T-1} E_t \left[ \frac{\varphi_s}{\pi_t} (\hat{e}_s - e_t^*) \right]}{F_t^* - c_t^* + e_t^* - e_t^*} \left( \phi_M - \psi_t \right)
$$

where the risk premium $\mu$ is given by

$$
\mu = \zeta_u u + \zeta_d d,
$$

$\phi_M$ is given by

$$
\phi_M = \left( \left( -\frac{\zeta_u}{\zeta_d} \frac{1}{d} \right)^{\frac{1}{2}} - 1 \right) R_f
\frac{u - \left( -\frac{\zeta_u}{\zeta_d} \frac{1}{d} \right)^{\frac{1}{2}} d}{R_f}
$$

and $\psi_t$ is given by

$$
\psi_t = \frac{1}{\bar{e}_t} \left( \frac{\sum_{s=t}^{T-1} E_t \left[ \frac{\varphi_s}{\pi_t} (e_s^* - \hat{e}_s) \right]}{\sum_{s=t}^{T-1} E_t \left[ \frac{\varphi_s}{\pi_t} (e_s^* - \hat{e}_s) \right]} - R_f \right).
$$

$\phi_M$ is the portfolio weight which obtains in the case where there is no labor/leisure choice.

The first term in equation (43) is exactly the portfolio weight found in Bodie, Merton, and Samuelson (1992), which is the human capital which the agent would have if he were to never take any leisure time now or in the future.

Bodie et. al., neglect the constraint imposed on leisure, by assuming an interior solution. Using our notation, their result is that the portfolio weight on total wealth

\textsuperscript{8}This is pointed out in footnote 6 of their paper.
(financial wealth $F_t$, plus maximum potential human capital $H^c_{\text{max}}(t)$) is $\phi^*_M$ and is related to the portfolio weight on financial wealth $\phi^*$ by

$$\phi^*_M (H^c_{\text{max}}(t) + F_t) = \phi^*_M F_t,$$

or

$$\phi^*_M = \left(1 + \frac{H^c_{\text{max}}(t)}{F_t}\right).$$

(47)

(48)

Returning to the expression for the more general case being studied in this paper (equation 42), it is an unfortunate fact that a simpler expression for the portfolio weight in terms of fundamental parameters of the model is not possible. There are only two realistic limiting scenarios which could be used to simplify the portfolio weight expression in equation (42). One is that the agent is so rich that he never chooses to work again. In this case, the problem is trivial as it is really just the standard portfolio choice problem in discrete time, where the agent consumes out of financial wealth. The other extreme is that the agent is sufficiently poor so that he always works. In this case the constraint on leisure is never binding. However, this is exactly the case studied in (Bodie, Merton, and Samuelson (1992)). The contribution of our analysis to this problem lies primarily in what happens when the wealth level is between these extremes.

Some simplification can be achieved by rewriting the portfolio weight as follows:

$$\phi^*_M = \phi_M \left(1 + \frac{E_t \frac{\pi_{t+1}}{\pi_t} H^c(t + 1)}{F^*_t} \right) - \frac{R_f}{\mu} \left(\frac{E_t \frac{1}{R_f} H^c(t + 1) - E_t \frac{\pi_{t+1}}{\pi_t} H^c(t + 1)}{F^*_t}\right),$$

where $H^c_t$ is defined by

$$H^c_t = \sum_{s=t}^{T-1} E_t \left(\frac{\pi_{t}}{\pi_t} (\bar{e}_s - e^*_s)\right).$$

(50)

We see that the first term:

$$\phi_M \left(1 + \frac{E_t \frac{\pi_{t+1}}{\pi_t} H^c(t + 1)}{F^*_t}\right),$$

consists of the portfolio choice in the absence of labor times a number which becomes larger than one as the value of human capital in the next period increases. Clearly, this

$\phi^*_M$ is the portfolio weight obtained in continuous time where the agent allocates wealth between consumption and investment, and where the agent can choose to invest in riskless and risky assets. When the risky asset has an expected excess return of $\mu$, an instantaneous volatility of returns $\sigma$, and the agent has constant relative risk aversion parameter $\gamma$, then $\phi^*_M = \frac{\mu}{\gamma \sigma}$ (Merton (1969)).
shows that as the agent anticipates working in the future, he invests more in the risky asset. The second term:

\[-\frac{R_f}{\mu} \left( \frac{E_t \left[ \frac{1}{\bar{R}_f} H^c(t + 1) \right] - E_t \left[ \frac{\bar{R}_f}{\bar{R}_f} H^c(t + 1) \right]}{F_t^{t^*}} \right) \]  

(52)

is proportional to the difference in the present value of the next period human capital when discounted at the risk-free rate of discount (under the physical probability measure) minus its value at the risky rate of discount. Another way to say this is as follows: it is proportional to how much the value of human capital is discounted from its risk-free value. As the value of next-period human capital is discounted more from its "risk-free" value, i.e. as human capital gets riskier, a smaller proportion of wealth is invested in stock.

5 Calibration Exercises

In this section we show the results of some numerical exercises to show what the model predicts regarding the consumption, labor, and investment choices of an agent under a set of reasonably realistic circumstances.

5.1 Model Parameters

In table 1, we show the parameters which we used in the calculation exercise. The parameters are chosen to be in line with those used by Bodie, Merton, and Samuelson (1992). The values chosen for some of the parameters warrant some comment. The rate of time preference is chosen so that expected consumption in each period is equal in the log utility case. We carry out the calculations for low risk-aversion (log utility) and high risk-aversion (γ = 5) cases. Most importantly, the wage is assumed to be constant for all states and times.

To numerically solve the model, we simply specify the initial parameters as in table 1, which include the investor’s horizon (or age), initial financial wealth, initial wage, the
return process for stock, the risk-free rate, etc.\(^{10}\) Then equation (29),

\[
\lambda^{-\frac{1}{\gamma}} \left( \alpha_0 e^{-\frac{\beta}{\gamma} T} E_0 \left[ \frac{1}{\lambda T} \right] + \sum_{t=0}^{T-1} e^{-\frac{\beta}{\gamma} t} E_0 \left[ \frac{1}{\lambda t} \right] \right) =
\]

\[
F_0 + \sum_{t=0}^{T-1} E_0 \left[ \lambda t \max \left( \bar{e}_t - \left( \frac{\alpha_t}{\lambda t \pi^t e^{\beta t}} \right)^{\frac{1}{\gamma}}, 0 \right) \right],
\]

is solved numerically for the value of the Lagrange multiplier \(\lambda\). This is done for every horizon. Once the value of \(\lambda\) is known, then the agent’s consumption and leisure choices follow from equations (23),

\[
e^{-\beta t} e^{s-\gamma} = \lambda \pi, \text{ for all states and } t = 0, 1, \ldots, T - 1,
\]

\[
\alpha_t e^{-\beta t} e^{s-\gamma} = \lambda \pi_t + \eta_t, \text{ for all states and } t = 0, 1, \ldots, T - 1,
\]

\[
\alpha_0 e^{-\beta t} F_T^{s-\gamma} = \lambda \pi_T, \text{ for all states and } t = 0, 1, \ldots, T - 1.
\]

The agent’s financial wealth can be determined from equation (30)

\[
F_t^* = E_t \left[ \frac{\pi T}{\pi t} F_T^* \right] + \sum_{s=t}^{T-1} E_t \left[ \frac{\pi s}{\pi t} c_s^* \right] - \sum_{s=t}^{T-1} E_t \left[ \frac{\pi s}{\pi t} \left( \bar{e}_s - e_s^* \right) \right],
\]

\[
= E_t \left[ \frac{\pi T}{\pi t} F_T^* \right] + \sum_{s=t}^{T-1} E_t \left[ \frac{\pi s}{\pi t} c_s^* \right] - \sum_{s=t}^{T-1} E_t \left[ \frac{\pi s}{\pi t} \left( \max \left( \bar{e}_s - \left( \frac{\alpha_t}{\lambda \pi^t e^{\beta s}} \right)^{\frac{1}{\gamma}}, 0 \right) \right) \right].
\]

The portfolio weight is then determined from (43),

\[
\phi_t^* = \frac{1}{\mu} \left( \frac{E_t \left[ F_{t+1}^* \right]}{F_t^* - c_t^* + \bar{e}_t - e_t^*} - R_f \right).
\]

\(^{10}\)From table 1, we have that \(\zeta(u) = \frac{1}{2}, \mu = 0.6, \sigma^2 = 0.12, R_f = 1.03, \Delta t = \frac{1}{2}\). Then

\[
\begin{align*}
r &= \log (R_f), \\
u &= \exp \left( \left( r + \mu - \frac{\sigma^2}{2} \right) \Delta t + \sigma \sqrt{\Delta t} \right) - (R_f)^{\Delta t} = 0.282, \\
d &= \exp \left( \left( r + \mu - \frac{\sigma^2}{2} \right) \Delta t - \sigma \sqrt{\Delta t} \right) - (R_f)^{\Delta t} = -0.221, \\
\mu &= \zeta(u) u + \zeta(d) d = 0.031.
\end{align*}
\]

In this model, we need the binomial tree to represent the physical process for the stock price. Therefore, we use a binomial tree with a drift that is equal to the physical drift on the stock.

Once the \(u\) and \(d\) factors are known, then the value of the single-period pricing kernel \(\xi_t\) can be determined from equation (14). The multi-period pricing kernel \(\pi_t\) is determined from equation (17).
5.2 Portfolio weights versus horizon

In table 2 we show the results of the calculation summarized for the log utility and risk-averse ($\gamma = 5$) cases, respectively, where we use the parameters shown in table 1. We calculate the initial portfolio weight for an individual at ages of 31,41,51, and 61. The agent's terminal age is 71. The portfolio weight shown at each age is the value which the portfolio weight takes on at the mean financial wealth level for that age.

We see that the agent's portfolio share of stock is initially much higher than 1. This reflects the fact that the maximum human capital $H_{max}(t)$ is like a riskless asset. This view does not contradict our earlier claim about the value of endogenous labor choice being a portfolio of put options. When the agent's actual choice of labor earnings in the future is examined, it is seen that the choice of labor, valued now, can be valued as a portfolio of put options. However, the sum total of all human capital that could be called upon is like bank account. It is the sum total of human capital that can be called upon, in combination with the actual choice of leisure (leisure choice acts like shorted stock, moving inversely with the stock price) which leads to the expected present value of human capital being valued like a portfolio of put options.

As expected, the portfolio weights for the more risk-averse agent are significantly lower than for the log utility agent. However, even for the risk-averse agent, these weights indicate that the agent optimally invests a large fraction of his financial wealth in stock. These results are very similar to those of Bodie, Merton, and Samuelson (1992).

In figure 3, a plot of optimal portfolio weight versus age is shown for various levels of initial wealth. Younger agents will invest more, and agents with lower wealth levels will invest higher proportions of their wealth in stock.

5.3 Comparison of Low and High Risk Aversion

Here we show the results of calculations for two types of agents. One has low risk aversion ($\gamma = 1$ or log utility). The other is highly risk-averse ($\gamma = 5$).

Figures 4-9 show sample paths of wealth, consumption, and labor choice versus age (only a limited number of paths are shown). Each path shown is a possible realization, for a given history of stock returns, of each of these quantities. These plots emphasize the rich dynamic structure of the choice variables which can be solved for in this model. Generally, better stock returns imply that agents consume more, work less, and attain a
higher level of financial wealth. The degree to which these choice variables vary changes significantly with changing risk aversion, as can be seen by noting the scales on each of these plots. For example, we can see in plots 6 and 7 that consumption at age 41 under the best possible stock performance is over ten times higher for the log utility agent than the risk-averse agent.

Figure 10 presents various regions of interest in regards to the agent’s labor choices over the life cycle depending on state. The “Retire” region shows where the wealth in that state at that time is above the “Retire” wealth threshold (derived in section 3.2). Here the agent will never find it optimal to work again. The “Vacation” region shows states where the agent will not work at least for that period, but will consider returning to work at some later time. The “Work” region shows where the agent neither retires nor goes on vacation but possibly has some chance to do either of these in the future. The “Work-Forever” region shows where the wealth will be such that the agent always chooses to work at least some amount for all states and periods in the future. Not surprisingly, each of these four regions tends to be characterized by lower and lower levels of financial wealth. Also not surprising is the fact that more risk-averse agents retire later and less often than less risk-averse agents. Put another way, the wealth threshold for retirement is higher for more risk-averse agents. This reflects the fact that more risk-averse agents wish to cushion financial shocks to consumption and leisure more than less risk-averse agents. If the agent retires, his leisure consumption will not be subject to dealing with financial shocks. However, his consumption and bequest will absorb all of the financial wealth shocks. Figure 10 prefaces the phenomena we see in figures 11-14.

In figure 11, the labor choice fraction (that is, the fraction of his time the agent chooses to work) is shown. We see that for either value of risk aversion, the agent uses labor to cushion financial wealth shocks more when stock performance is poor. However, the more risk-averse agent does not always work more; he is also risk-averse to fluctuations in leisure consumption. When stock performance is bad, the log utility agent actually works more. However the more risk-averse agent often works more when stock performance is not so bad.

In figure 12, the conditional standard deviation of labor earnings is shown. We see that the conditional standard deviation of labor earnings is typically higher when stock returns are good for the log utility agent, though it is higher when stock returns are bad for the more risk-averse agent. The less risk-averse agent absorbs financial shocks by
adjusting both consumption and labor to a higher degree for a few bad financial shocks (number of up-ticks in stock price not too different from the number of down-ticks) but has less “room” to adjust if stock returns are really bad (lots of down-ticks). The more risk-averse agent adjusts more slowly for stock returns which are moderately bad (a few down-ticks), and thus if stock returns are later worse (lots of down-ticks), then he has more “room” to adjust.

This suggests what we see in figure 13. Here we plot the human capital $H^c(t)$, which is defined by

$$H^c(t) = \sum_{s=t}^{T-1} E_t \left( \frac{\pi_s}{\pi_t} \max \left( \hat{e}_s - \left( \frac{\alpha_t}{\lambda \pi_s e^{\beta_s}} \right)^{\frac{1}{\beta}}, 0 \right) \right).$$  (53)

Human capital “plateaus” more quickly for the less risk-averse agent, reflecting the fact that he adjusts quickly as the path of stock returns deviates some small amount from average, but has less room to adjust when stock returns are really bad. The more risk-averse agent adjusts to a lower degree.

This is also evident in figure 14, where we see that the conditional volatility of human capital for the less risk-averse (log utility) agent is higher near the center of the tree, where he does most of his adjustment to good and bad times. The more risk-averse agent does less adjustment for slightly less than average stock returns, but continues to adjust more evenly as the stock performance gets very bad.

These results for the conditional standard deviation of human capital also have one implication that is counter to empirical studies on human capital: the conditional volatility of human capital goes down in recessions. In this model this is because the agent works more and more as stock performance get worse. However he can only work so much. For lower stock prices, if he is already working at or near the limit of time he can spend, he will not be able to improve his situation much by working more (because he has no additional time). Of course there is also a constraint on the other side as well; the agent can only enjoy so much leisure. This is why the conditional volatility of human capital (as well as labor earnings) is hump-shaped, and not monotonically decreasing with stock price (see e.g. Storesletten, Telmer, and Yaron (2001)). One could also understand this as an effect of endogenous labor put options. The conditional volatility of a put option is hump-shaped and going to zero as the underlying value (the interior leisure expenditure choice $\hat{e}_t$ in equation (24)) goes to zero.

Of course, one shortcoming of our model is that it contains no prospect for the individual to be laid off, lose his job, or have his wages reduced. In these calculations, wages are
constant for all states of the world. Thus, it is not surprising that given the availability to an agent of the option to work to “cushion” financial shocks that he takes advantage of it when he needs it most (in recessions).

In figure 15, we see that consumption also is much more variable across states for the log utility agent than the more risk-averse agent. The values of consumption for the risk-averse agent are not completely unrealistic, as they are typically on the order of a few tens of thousands of dollars. The variability of consumption across states for a given time is smaller for the more risk-averse agent.

Figure 16 shows the optimal financial wealth. Of particular interest are the large negative values. This somewhat surprising result is possible in this discrete time model (and will not lead to negative bequest). There are two reasons for this. First is the fact that the stock price one period into the future is always bounded at some value strictly above zero. Second, the agent has endogenous labor put options, essentially on stock, which are always worth something even if stock performance is poor. Put another way, the agent always has the option to work if his stock performs poorly. Thus, it is often optimal for the agent to borrow large amounts when stock performance is bad. Figure 17 shows the ratio of debt (absolute value of investment wealth when it is negative) to human capital. We see that even in the risk-averse ($\gamma = 5$) case, it will be optimal for the agent to borrow the equivalent of more than three-fourths of his human capital to consume, take leisure time, and invest when his stock performs poorly. We also see that in the last period shown (the period before the agent leaves the economy), the agent never takes on any debt. There is no issue with the agent borrowing more and more, since his bequest motive prevents him from taking on more debt than he can repay before leaving the economy.

Figure 18 shows the portfolio weight $\phi^*$. We see that in the “Retire” region, since there is no labor choice, the portfolio weight in stocks is the Merton value ($\phi_M$), appropriately adjusted to the discrete-time case. Also, in the second to last period, since the agent is simply investing to satisfy his bequest motive, with no possibility of future labor income, the value is also the Merton value (even for poor stock returns). We see negative values for poor stock returns. It is not the case, however, that the agent is shorting stock when stock returns are bad. Recall that the total dollar value invested in stock is

$$I_t^S \equiv \phi_t F_t^i,$$  \hspace{1cm} (54)
where the investment wealth \((F_t^i)\) is given by

\[
F_t^i \equiv F_t - c_t + \bar{e}_t - e_t,
\]

(55)

The investment wealth is the wealth the agent has left over after buying consumption and leisure. This is the wealth which will be divided between stock and bank account. In figure 19, both the portfolio weight \((\phi^*)\) and the investment wealth \((F^{i*})\) are plotted separately. Of particular interest is the fact that the portfolio weight seems to be negative somewhere below the mid-line of the tree, and is at its largest value somewhere near or below the center of the tree. We see the behavior of the portfolio weight as a function of stock price in figure 20, when the agent is 68 years old. The more risk-averse agent must experience significantly worse stock performance than the log utility agent to have a negative portfolio weight.

The behavior of \(\phi^*\) in figure 20 is intuitive. Consider the behavior of \(\phi^*\) in the limiting cases of the stock price going to \(+\infty\). As the stock price goes to \(+\infty\), if the agent has been investing optimally, his wealth is also going to \(+\infty\). Therefore it must be that his investment wealth \((F^{i*})\) is going to \(+\infty\) as well. Clearly, the agent will retire. In this case, there is no labor/leisure decision to be made, so that the agent simply invests \(\phi_M\) in stock. As the stock price gets very low, if the agent has been investing optimally, he will have negative financial wealth (see figure 16). If the agent borrows too heavily to invest in the stock market, he stands some chance of ending up with a negative bequest in the last period. Therefore as the stock price goes to 0, the agent’s total investment in stock will go to zero, which dictates that the portfolio weight be negative but approaching 0.

The agent’s behavior around where the investment wealth \((F^{i*})\) switches sign is a bit more complicated. Note that the optimal investment wealth is strictly increasing in stock price. This is shown explicitly in figure 21, where investment wealth is plotted versus stock price for the risk-averse individual at age 68. We see from this plot that the optimal investment wealth crosses from positive to negative as the stock price decreases. The most important point is that the agent will always invest a positive amount in stock since the expected excess stock return is strictly positive. As the investment wealth goes to zero from either slightly above or slightly below zero, the agent has less and less wealth to lose, as well as the option to work, which he can exercise if his stock decreases in value in the future. Therefore the agent will invest a higher and higher fraction of his wealth in stock, even borrowing heavily if need be, since he can easily work and make up whatever is lost for very small wealth levels.
Figure 22 shows the wealth the agent invests in stock in the log utility case at age 68, versus stock price. Here we see that this amount is monotonically increasing with stock price (note that this is on a log-log scale). This should be compared to figure 23, which is the same plot but for the case of the more risk-averse ($\gamma = 5$) agent. Here we see that the amount invested in stock is generally monotonic with stock price until the stock price is such that the agent is close to where in the future he may retire. Once the agent is likely enough to retire in the future, he may actually invest slightly less in the stock market if he has slightly better stock returns.

5.4 Increasing aversion to work with age

While the numerical exercises of the previous sections have shed considerable light on the interaction of uncertainty, aversion to labor, and consumption, they are lacking in one respect: the agent’s preference over leisure is constant over his entire life. Indeed in the case of the risk-averse ($\gamma = 5$) investor, the probability that the agent retires during his life with the parameters as in table 1 is only .073% (for the log utility investor it is 13%).

Anecdotal inference would suggest that people’s preference for leisure increases with age. Here we try to see what the effects of this are on the consumption, labor, and portfolio choice. We do this by making $\alpha_t$ vary over time. Specifically, $\alpha_t$ has a constant value of $1/2$ before age 62, and increases linearly to a final value of 10 when the agent turns $70^{\frac{1}{2}}$. A plot of $\alpha_t$ as a function time is shown in figure 25. The effects of the increase in $\alpha_t$ are clear in figure 26, where the size of the “Retire” region starts to increase significantly faster just when $\alpha_t$ starts to increase at age 62. Particularly interesting is the behavior of $I_t^S$ (the dollar amount the agent invests in stock) versus stock price. In figure 27, we see this plotted at age 68 for the risk-averse ($\gamma = 5$) investor. This can be compared to figure 23, which is the same figure plotted where $\alpha_t$ has a constant value of $1/2$. We see that the stock price at which the agent invests less in stock for better stock returns is significantly lower for the case of $\alpha_t$ increasing with age after age 62, than with $\alpha_t$ constant over the lifespan of the agent. The geometric annual mean return associated with the agent investing less in stock for better past stock performance was 22.4% for $\alpha_t$ constant at a value of $1/2$ for the entire life of the agent (figure 23). With $\alpha_t$ increasing after age 62, it only need be 4.4% (figure 27). The point of this is that for reasonable parameterizations of the agent becoming more averse to work as he gets older, the range of stock prices in which an agent with better past stock performance actually invests less
in the stock market is much lower, and thus much more likely.

In table 3, we see the probability of retirement on or before a given age, and the required wealth level to retire at a given age, for ages at which the agent has some nonzero probability of retiring. The log utility agent has a 94% chance of retiring before leaving the economy. For the risk-averse agent, the probability is 33%.

5.5 Results for wage positively correlated with stock price

Here we consider what the model says for wages which are perfectly positively correlated with the stock price. Recall equation (1), which specifies the assumed relationship between the wage and the stock price:

\[ p_t = \alpha S_t^\rho. \]  

(56)

Up until now, all of the numerical exercises we have shown were done with \( \rho = 0 \). Now we will examine what occurs when \( \rho > 0 \). The specific value chosen is \( \rho = 0.203 \). This value of \( \rho \) turns out to imply a conditional volatility of the wage process, \( \sigma_w = 0.05 \). This value for the conditional volatility of wages is relatively conservative when empirical studies are taken into account.\(^{11}\)

In figure 28, we see a plot depicting the agent’s labor choices over the course of the life cycle. Compared to figure 10, we see that the ”Work-Forever” region for the highly risk-averse agent now occurs when stock returns are high. Of course this occurs because the wage increases sufficiently with stock price to induce the agent to work even more. The agent now retires when stock returns are poor. When stock returns are poor, the agent’s wages are low enough so that it is not worth it for him to work.

6 Conclusion

In this paper we have examined endogenous labor-leisure choice and endogenous retirement in the context of a life-cycle model. Under the assumption of complete asset markets, an elegant and simple solution is found which completes the analysis done originally in Bodie, Merton, and Samuelson (1992). Abstracting from regulatory definitions of retire-

\(^{11}\)Gottschalk and Moffitt (1994) decompose the earnings process into permanent and transitory components. They find that the volatility of the transitory component is 10.4% for the period 1970-78, and 14.8% for the period 1979-1987.
ment, we study, from the point of view of efficient resource allocation, the labor and retirement choices.

We find the following results when the agent’s wage is constant. The agent works harder and retires later (if at all) when stock returns are poor. More risk-averse agents adjust their consumption and labor choices more gradually to financial wealth shocks. Agents will always invest a positive amount in stock, provided that its expected excess return is strictly positive. The agent facing retirement will invest less in stock than those who never retire but more than those who never work. The agent may optimally borrow to consume and invest when stock performance is poor; for reasonable parameterizations, highly risk-averse agents ($\gamma = 5$) are found to borrow up to $3/4$ of the value of their human capital. While an agent’s total dollar investment in stock is generally increasing with stock price (or equivalently with financial wealth), it may be decreasing for a range of stock prices where the agent’s wealth is such that he is likely to retire in the near future. If the agent’s aversion to work is increasing with age, the agent predictably retires more often, and the range of stock prices where stock investment is decreasing with stock price occurs at a significantly lower stock price.

Interestingly, when wages are (perfectly) positively correlated with the stock price, and (conservatively) calibrated to the empirically-observed conditional volatility of wages, the agent’s labor choice follows exactly the opposite of the pattern observed when the wage is constant. The agent retires when his stock returns are poor, and continues to work when his stock returns are good.

In most of the calculations done in this work, the analysis was limited to the constant-wage case, with a slight examination of what occurs when the wage is correlated with the stock price. A full analysis of the second case is sure to bring many interesting insights, and will form the basis of future research.

Appendices

A Continuous-Time Model

In this section, we recast the model in continuous time. This will allow us to do two things which are not so easy with the discrete time model. First, we will be able to obtain an
expression for the portfolio weight which is easier to interpret, particularly for the case of a varying wage. Second, we will be able to demonstrate in a simplified case how one can understand negative financial wealth.

A.1 Continuous-Time Version

The discrete version of the model can be readily cast in continuous time. The state of the economy is specified by the price \( S_t \) of a stock which satisfies

\[
dS_t = S_t(r + \mu)dt + S_t\sigma dZ_t.
\]

Here, \( r = R_f - 1 \) is the one-period interest rate. The utility of the agent is

\[
\max_{\{c_t, l_t, \phi_t\}} \mathbb{E}_0 \left[ \int_0^T e^{-\beta t} \left( \frac{c_t^{1-\gamma} + \alpha_t (p_t l_t)^{1-\gamma}}{1 - \gamma} \right) dt + e^{-\beta T} \alpha_t F_T^{1-\gamma} \right],
\]

s.t. \( dF_t = F_t (r + \phi_t \mu) dt - c_t dt + p_t (K - l_t) dt + F_t \sigma dZ_t, \)

\( l_t \leq K, \)

where the consumption stream \( \{c_t\} \) and the bequest \( F_T \) are financed by income streams from trading in the financial markets and the endogenous labor income stream \( \{p_t (K - l_t)\} \), and \( \{\phi_t\} \) is the stock portfolio weight. We define the wage process as

\[
p_t = \alpha S_t^\phi
\]

If we recast this equation in terms of

\[
\tilde{e}_t = p_t K, \tilde{e}_t = p_t l_t,
\]

we have

\[
\max_{\{c_t, \tilde{e}_t, \phi_t\}} \mathbb{E}_0 \left[ \int_0^T e^{-\beta t} \left( \frac{\tilde{e}_t^{1-\gamma} + \alpha_t \tilde{e}_t^{1-\gamma}}{1 - \gamma} \right) dt + e^{-\beta T} \alpha_t F_T^{1-\gamma} \right],
\]

s.t. \( dF_t = F_t (r + \phi_t \mu) dt - c_t dt + (\tilde{e}_t - e_t) dt + F_t \sigma dZ_t, \)

\( e_t \leq \tilde{e}_t. \)

We define the pricing kernel as

\[
\pi_t = \exp \left( \int_0^t - \left( r + \frac{\mu^2}{2\sigma^2} \right) ds - \frac{\mu}{\sigma} dZ_s \right).
\]

We can use the martingale approach to solve the problem in (62-64). The following proposition is completely parallel to the discrete time version.
Proposition A.1  The maximization problem (62-64) is equivalent to

$$\max_{\{c_t^1, c_t^2\}, F_T} E_0 \left[ \int_0^T e^{-\beta t} \left( c^1_{t-1} - \frac{\alpha_t}{\lambda t} c^2_t \right) dt + e^{-\beta T} \alpha_0 F_T^{1-\gamma} \right],$$  \hspace{0.5cm} (65)

s.t. \hspace{0.5cm} \begin{align*}
E_0 & \left[ \pi_T F_T + \int_0^T \pi_t c_t dt \right] = F_0 + E_0 \left[ \int_0^T \pi_t (\bar{e}_t - e_t) dt \right], \hspace{0.5cm} (66) \\
e_t & \leq \bar{e}_t. \hspace{0.5cm} (67)
\end{align*}

Proposition A.2  The optimal consumption, leisure, and wealth are given by:

$$c^*_t = \left( \frac{1}{\lambda \pi_t e^{\beta t}} \right)^{\frac{1}{\gamma}}, \hspace{0.5cm} t = 0, 1, 2, ..., T - 1, \hspace{0.5cm} (68)$$

$$e^*_t = \min \left( \left( \frac{\alpha_t}{\lambda \pi_t e^{\beta t}} \right)^{\frac{1}{\gamma}}, \bar{e}_t \right), \hspace{0.5cm} t = 0, 1, 2, ..., T - 1, \hspace{0.5cm} (69)$$

$$F_T^* = \left( \frac{\alpha_0}{\lambda \pi_T e^{\beta T}} \right)^{\frac{1}{\gamma}}, \hspace{0.5cm} (70)$$

with $\lambda$ satisfying the equation:

$$\lambda^{-\frac{1}{\gamma}} \left( \frac{1}{\alpha_0} e^{\left( \frac{r}{\gamma}-1 \right)T} + \left( \frac{1}{\gamma}-1 \right) \left( \lambda \pi_T e^{\beta T} \right) - 1 \right) = \lambda^{-\frac{1}{\gamma}} \left( \frac{1}{\alpha_0} e^{rT} + \frac{1}{\gamma} \left( \frac{1}{\gamma}-1 \right) \left( \lambda \pi_T e^{\beta T} \right) - 1 \right) + \int_0^T E_0 \left[ \pi_t \min \left( \left( \frac{\alpha_t}{\lambda \pi_t e^{\beta t}} \right)^{\frac{1}{\gamma}}, \bar{e}_t \right) \right] dt = F_0 + \int_0^T E[\pi_t \bar{e}_t] dt. \hspace{0.5cm} (71)$$

The left hand side of equation (71) decreases monotonically from $+\infty$ to 0 when $\lambda$ varies from 0 to $+\infty$, while the right hand side is fixed. Therefore, there is always a solution to equation (71) for all $F_0 > 0$. It is easy to prove that the Lagrange multiplier is decreasing in $F_0$.

Proposition A.3  For time $0 \leq t < T$, the wealth $F_t^*$ attained by optimal strategies is given by

$$F_t^* = E_t \left[ \frac{\pi_T}{\pi_t} F_T^* \right] + \int_t^T \left( E_t \left[ \frac{\pi_s}{\pi_t} c_t^* \right] - E_t \left[ \frac{\pi_s}{\pi_t} (\bar{e}_s - e_s^*) \right] \right) ds, \hspace{0.5cm} (72)$$

where

$$E_t \left[ \frac{\pi_s}{\pi_t} (\bar{e}_s - e_s^*) \right] = E_t \left[ \frac{\pi_s}{\pi_t} \max \left( \bar{e}_s - \left( \frac{\alpha_t}{\lambda \pi_s e^{\beta s}} \right)^{\frac{1}{\gamma}}, 0 \right) \right]. \hspace{0.5cm} (73)$$

The budget constraint says that the sum of the present value of the optimal bequest $F_t^*$ and the optimal consumption stream $c_t^*$ equals to the sum of the initial wealth $F_0$ and the present value of a portfolio of put options.
Proposition A.4 For time $0 \leq t < T$, the optimal portfolio weight $\phi_t^*$ is given by

$$
\phi_t^* = \phi_{M}^t \left( 1 + \frac{1}{F_{t}} \int_{t}^{T} E_t \left[ \frac{\pi^s}{\pi_t} \varepsilon_s (e_s^* < \bar{\varepsilon}_s) \right] ds \right) - \frac{\rho}{F_{t}} \int_{t}^{T} E_t \left[ \frac{\pi^s}{\pi_t} \left( \varepsilon_s - \left( 1 - \frac{1}{\gamma} \right) e_s^* \right) \chi(e_s^* < \bar{\varepsilon}_s) \right] ds.
$$

where $\phi_{M}^t$ is as defined in footnote 9. $\chi$ is the indicator function.

The first term is the standard Merton term, which is the portfolio weight which would obtain if there were no labor/leisure choice (as in Merton (1969)). The second term arises from the agent’s labor income flexibility. If the agent is likely to work in the future anyway, he is willing to invest more of his financial wealth in stock. The third term is due to the time-varying wage. Note that since $l_s^* \leq K$, the factor $K - \left( 1 - \frac{1}{\gamma} \right) l_s^*$ is always positive. Therefore, the component of $\phi_t^*$ due to time-varying wage always has the opposite sign of $\rho$.

A.2 An Example: Constant Wage

We consider a special case of (67) where $\alpha_t = 0$; the agent has no preference for leisure. We show that negative financial wealth can be an optimal choice in this simple case.

Specifically, the utility of the agent is

$$
\max_{\{c_t\}, \{\phi_t\}} \mathbb{E}_0 \left[ \int_0^T e^{-\beta t} \frac{c_t^{1-\gamma}}{1-\gamma} dt + e^{-\beta T} a_{F_T} \frac{F_T^{1-\gamma}}{1-\gamma} \right], \quad (74)
$$

s.t. $dF_t = F_t (r + \phi_t \mu) dt - c_t dt + K dt + F_t \sigma dZ_t. \quad (75)$

The income $K$ is a constant and $\{\phi_t\}$ is the stock portfolio weight. Using techniques similar to those in section (A.1), we find that

$$
c_t^* = \left( \frac{1}{\lambda t e^{\beta t}} \right)^{\frac{1}{\gamma}}, \quad t = 0, 1, 2, ..., T - 1, \quad (76)
$$

$$
F_T^* = \left( \frac{a_{F_T}}{\lambda T e^{\beta T}} \right)^{\frac{1}{\gamma}} \quad (77)
$$

For time $0 \leq t < T$, the wealth $F_t^*$ attained by optimal strategies is given by

$$
F_t^* = E_t \left[ \frac{\pi_T}{\pi_t} F_T^* \right] + \int_t^T E_t \left[ \frac{\pi_s}{\pi_t} c_s^* \right] ds - K \frac{1 - e^{-r(T-t)}}{r}.
$$

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These expressions imply that

\[ F^*_T = \pi_t^{-\frac{1}{\gamma}} f(t) - \frac{K(1 - e^{-r(T-t)})}{r}, \]

where

\[ f(t) = \left( \alpha_0 \delta e^{-\Delta(T-t)} + \frac{1}{\Delta} \left(1 - e^{-\Delta(T-t)} \right) \right) \left( \frac{1}{\lambda e^{\beta t}} \right)^{-\frac{1}{\gamma}}, \]  

(78)

and

\[ \Delta = \left( r + \frac{\mu^2}{2\gamma \sigma^2} \right) \left( 1 - \frac{1}{\gamma} \right). \]  

(79)

Therefore, for any time \( 0 < t < T \), when the stock price is high, the optimal financial wealth \( F^*_t \) will be high and positive. When the stock price is close to zero, the optimal financial wealth \( F^*_t \) will be

\[ -\frac{K(1 - e^{-r(T-t)})}{r}. \]  

(80)

In the binomial model, because the lower bounds (for different \( t \)) on the stock price are greater than zero, only at some time \( t < T \) can the financial wealth \( F^*_t \) become negative.

Intuitively, the investor optimally chooses total wealth, which is

\[ E_t \left[ \frac{\pi_T}{\pi_t} E^*_T \right] + \int_t^T E_t \left[ \frac{\pi_s c^*_s}{\pi_t} \right] ds = \pi_t^{-\frac{1}{\gamma}} f(t). \]

The optimal total wealth is zero when the stock price is zero and infinity when the stock price is infinity. However, the human capital is strictly positive for \( 0 < t < T \) and is independent of the stock price. Therefore, when the stock price is low, the financial wealth, which is the difference between the total wealth and financial wealth, will be negative.

Denote the stock price at which the financial wealth is zero as \( S^*_t \). When \( S_t \) decreases from \( S^*_t \) to 0, the financial wealth changes from 0 to \( -\frac{K(1 - e^{-r(T-t)})}{r} \) and the portfolio weight changes from \( -\infty \) to 0.

### A.3 Endogenous Labor Income as a Put Option on Stock

Equation (91) shows that the value which underlies the put option is

\[ S^*_{t,M} \]  

(81)

where

\[ \phi^*_M = \frac{\mu}{\gamma \sigma^2} \]  

(82)
is the portfolio weight obtained in the case with no labor choice (Merton (1969)). This is quite intuitive: the more the agent would invest when there is no labor choice, the higher the response of the agent’s labor choice to financial shocks. Interestingly, a more risk-averse investor has a put option which is less responsive to stock price changes than a less risk-averse one. This is because preferences are such that the agent is risk-averse over both consumption and leisure. The agent will not adjust his leisure expenditure as much in response to a financial shock as a less risk-averse investor would.

We now calculate the value of the labor put option explicitly for the case of a constant wage. The process for stock is

$$\frac{dS}{S} = (r + \mu) dt + \sigma dz,$$

or

$$S_t = S_0 e^{(r + \mu - \frac{\sigma^2}{2})t + \sigma \Delta z(t)}.$$

Here \( r = R_f - 1 \) is the instantaneous interest rate on risk free loans. The process for the pricing kernel is given by

$$\frac{d\pi_t}{\pi_t} = -r dt - \frac{\mu}{\sigma} dz,$$

or

$$\pi_t = e^{-\frac{\mu^t}{r} - \frac{\mu}{\sigma} \Delta z(t)},$$

thus

$$\pi_t = e^{-\frac{\mu^t}{r}} S_t^{\frac{\mu}{\sigma^2}},$$

where

$$\Gamma = \frac{\mu r}{\sigma^2} + \frac{\mu^2}{2\sigma^2} - \frac{\mu}{2} - r.$$

The leisure choice \( e_t \) is related to the stock price by

$$e_t = \left( \frac{\alpha_t}{\lambda} \right)^{\frac{1}{2}} e^{\frac{\beta + x_t}{\gamma}} S_t^{\frac{\mu}{\sigma^2}}.$$

Using the process for this, and evaluating the expression

$$E_0(\max(\tilde{c} - e_t, 0)),$$

(here the wage is constant across state and time, so \( \tilde{c} = pK \), the maximum possible leisure expenditure is also constant across states and time). We obtain as the value of the put option:

$$V_t^\text{put} = \tilde{c} e^{-rt} N(d_1) - e^{-rt} \left( \frac{\alpha_t}{\lambda} \right)^{\frac{1}{2}} S_0^{\frac{\mu}{\sigma^2}} N(d_2),$$
where

\[
\begin{align*}
    d_1 &= \frac{\log(\bar{v}) - \bar{q}}{\sigma_q}, \\
    d_2 &= d_1 - \sigma_q, \\
    \sigma_q^2 &= \frac{\mu}{\gamma \sigma}, \\
    \bar{q} &= \log \left( \left( \frac{\alpha_t}{\lambda} \right)^{\frac{1}{2}} S_0^{\frac{\mu}{\gamma \sigma^2}} \right) + \left( \psi - \frac{\eta^2}{2} \right) t, \\
    \psi &= (r + \mu) \frac{\mu}{\gamma \sigma^2} + \frac{\mu}{\gamma \sigma^2} \frac{(\frac{\mu}{\gamma \sigma^2} - 1) \sigma^2}{2} - \frac{\beta + \Gamma}{\gamma}, \\
    \beta &= -\frac{\mu}{\sigma^2}, \\
    \Gamma &= \frac{\mu \sigma^2}{\sigma^2} + \frac{\mu^2}{2\sigma^2} - \frac{\mu}{2} - R_f, \\
    \bar{v} &= pK.
\end{align*}
\]

\[\text{B} \quad \text{Discrete-Time Multiperiod Budget Equation}\]

Start with

\[
E_{t-1}[\xi_t R_t] = 1.
\]

This must be true for all gross returns. It is also true for the gross risk-free return, \( R_f \), and the gross risky return \( R_f + \epsilon_t \),

\[
E_{t-1}[\xi_t R_f] = 1, \quad E_t[\xi_t (R_f + \epsilon_t)] = 1.
\]

Let \( \zeta_u \) be the probability of an up-ticket in stock price occurring next period, and \( \zeta_d = 1 - \zeta_u \) the probability of a down-ticket. Dropping time subscripts, it must be that

\[
0 = \zeta_u \xi_u u + \zeta_d \xi_d d.
\]

and

\[
\frac{1}{R_f} = \zeta_u \xi_u + \zeta_d \xi_d.
\]

Solving these gives

\[
\begin{align*}
    \xi_t(\epsilon_{t+1} = u) &= \frac{-d}{\zeta_u R_f (u - d)}, \\
    \xi_t(\epsilon_{t+1} = d) &= \frac{u}{\zeta_d R_f (u - d)}.
\end{align*}
\]
Recall equation (11). From this equation we see that

$$\pi_{t+1} F_{t+1} = \pi_t \xi_{t+1} (R_f + \phi_t c_{t+1}) (F_t - c_t + \bar{e}_t - e_t), \quad (105)$$

so

$$E_t [\pi_{t+1} F_{t+1}] = \pi_t F_t - \pi_t c_t + \pi_t (\bar{e}_t - e_t). \quad (106)$$

Continuing in this way, we can see that

$$E_0 [\pi_{t+1} F_{t+1}] = \pi_0 F_0 - \sum_{s=0}^{t} E_0 [\pi_s c_s] + \sum_{s=0}^{t} E_0 [\pi_s (\bar{e}_s - e_s)]. \quad (107)$$

Thus we see that the budget constraint is equivalent to (18).

C The Lagrange Multiplier

The equation for the Lagrange Multiplier on wealth ($\lambda$) is given by

$$\lambda^{-\frac{1}{\gamma}} \left( \alpha_t e^{-\frac{\bar{e}_T}{\gamma}} E_0 \left[ \pi_t^{1-\frac{1}{\gamma}} \right] + \sum_{t=0}^{T-1} e^{-\frac{\bar{e}_t}{\gamma}} E_0 \left[ \pi_t^{1-\frac{1}{\gamma}} \right] \right) = F_0 + \sum_{t=0}^{T-1} E_0 \left[ \pi_t \max \left( \bar{e}_t - \left( \frac{\alpha_t}{\lambda^{\gamma} \pi_t e^{\theta t}} \right)^{\frac{1}{\gamma}}, 0 \right) \right]. \quad (108)$$

To evaluate this equation, we use the results of the following lemma:

**Lemma C.1** The conditional expectation of $\pi_t^{1-\frac{1}{\gamma}}$ is given by

$$E_0 \left[ \pi_t^{1-\frac{1}{\gamma}} \right] = \Xi_t, \quad (109)$$

Using this, we can obtain the following expression for the equation for $\lambda$:

$$\lambda^{-\frac{1}{\gamma}} \alpha_t e^{-\frac{\bar{e}_T}{\gamma}} \Omega^T + \frac{1 - \Omega^T}{1 - \Omega} = F_0 + \sum_{t=0}^{T-1} E_0 \left[ \pi_t \max \left( \bar{e}_t - \left( \frac{\alpha_t}{\lambda^{\gamma} \pi_t e^{\theta t}} \right)^{\frac{1}{\gamma}}, 0 \right) \right], \quad (110)$$

where straightforward calculations show that the constants $\Upsilon$, $\Xi$, and $\Omega$ are given by

$$\Upsilon = \left( -\frac{\zeta_{u}}{\zeta_{d}} \right)^{\frac{1}{\gamma}}, \quad \Xi = \frac{u^{\frac{1}{\gamma}} \zeta_{d} \left( u - \Upsilon d \right)}{(R_f (u - d))^{1-\frac{1}{\gamma}}}, \quad \Omega = e^{-\frac{\bar{e}}{\gamma} \Xi}. \quad (111)$$

The left hand side of equation (110) decreases monotonically from $+\infty$ to 0 when $\lambda$, varies from 0 to $+\infty$, while the right hand side increases from $F_0$ to $F_0 + \sum_{s=t}^{T-1} E_0 \left[ \pi_s \bar{e}_s \right]$. Therefore, there is always a solution to equation (29) for all $F_0 > 0$. 

35
We now show that \( \lambda \) is decreasing in initial wealth. Begin with equation (29). Consider two different values of initial wealth, \( F_{01} \) and \( F_{02} \) with \( F_{02} > F_{01} \). Suppose that the corresponding values of \( \lambda \) are \( \lambda_1 \) and \( \lambda_2 \), respectively, and suppose that \( \lambda_2 \geq \lambda_1 \). Then taking the difference of equation (29) with the values corresponding to \( F_{02} \) and \( F_{01} \), we obtain

\[
F_{02} - F_{01} = \left( \frac{1}{\lambda_2^2} - \frac{1}{\lambda_1^2} \right) \left( \frac{1}{\gamma} \Omega^T + \frac{\Omega^T - 1}{\Omega - 1} \right) + \sum_{t=0}^{T-1} \text{E}_0 \left[ \pi_t (e^*_{t2} - e^*_{t1}) \right],
\]

where

\[
e^*_{t1} = \min \left( \left( \frac{\alpha_t}{\lambda_1^2} \right)^{\frac{1}{\gamma}}, \, \bar{e}_t \right), \quad e^*_{t2} = \min \left( \left( \frac{\alpha_t}{\lambda_2^2} \right)^{\frac{1}{\gamma}}, \, \bar{e}_t \right).
\]

Based on the assumptions made, the left hand side is strictly positive, while the right hand side is weakly negative, which is a contradiction. Therefore we must have that \( \lambda_2 < \lambda_1 \).

Given the forms for the optimal consumption, leisure consumption, and terminal wealth in (27) this means that consumption and terminal wealth are strictly increasing with initial financial wealth (\( F_0 \)), and leisure consumption is weakly increasing with initial financial wealth.

D   Stock Investment is positive every period if the risk premium is strictly positive

Note that

\[
I^S_{t-1} \equiv \phi_{t-1} F^i_{t-1} = - \left( \frac{R_t}{\mu} \right) \text{cov}_{t-1} [\xi_t, F^i_t].
\]

Let \( \zeta_u \) be the probability of going to the up state in the next period and \( \zeta_d = 1 - \zeta_u \) be the probability of going to the down state. Let \( \xi_u, \xi_d \) be the next period Arrow-Debreu prices, and let \( F^u_t \) be the wealth in the up state and \( F^d_t \) the wealth in the down state at time \( t \).

The conditional covariance of \( \xi_t \) and \( F_t \) can be written as follows:

\[
\text{cov}_{t-1} [\xi_t, F_t] = \zeta_u \zeta_d (\xi_u - \xi_d) \left( F^u_t - F^d_t \right).
\]
Note that

\[
(\xi_u - \xi_d) = \frac{-\zeta_u u + \zeta_d d}{\zeta_u \zeta_d},
\]

\[
< 0.
\]

Denote \(F_t^{n_t, n_{t-1}}\) as the wealth at time \(t\) given that the state at \(t - 1\) is \(n_{t-1}^{l-1}\). Then we have that

\[
F_t^{n_t, n_{t-1}} - F_t^{d, n_{t-1}} = \left( E_t \left[ \pi_T(n_{t-1}^{l-1}) F_T^s(n_{t-1}^{l-1}) \right] \right) + \sum_{s=t}^{T-1} E_t \left[ \frac{\pi_s(n_{t-1}^{l-1})}{\pi_{t-1}(n_{t-1}^{l-1})} \left( e_{s+}^* - e_s^* \right) \right],
\]

\[
= \left( E_t \left[ \pi_T(n_{t-1}^{l-1}) F_T^s(n_{t-1}^{l-1}) \right] \right) (T - 1)
\]

where

\[
F_T^s(n_{t-1}^{l-1}) = \left( \frac{\alpha_f}{\lambda \pi_T(n_{t-1}^{l-1}) e^{\beta_T}} \right)^{\frac{1}{\gamma}},
\]

\[
e_s^* = \left( \frac{1}{\lambda \pi_s(n_{t-1}^{l-1}) e^{\beta_s}} \right)^{\frac{1}{\gamma}},
\]

\[
\gamma = \min \left( \left( \frac{\alpha_f}{\lambda \pi_T(n_{t-1}^{l-1}) e^{\beta_T}} \right)^{\frac{1}{\gamma}}, \bar{t} \right),
\]

\[
\epsilon_s^* = \min \left( \left( \frac{\alpha_f}{\lambda \pi_s(n_{t-1}^{l-1}) e^{\beta_s}} \right)^{\frac{1}{\gamma}}, \bar{t} \right).
\]

and \(\gamma\) is defined as in (111). Clearly the first term is strictly positive, and the second term is weakly positive in every state. Thus \(F_t^{n_t, n_{t-1}} - F_t^{d, n_{t-1}}\) is strictly positive. Therefore the covariance in equation (114) is strictly negative. Thus, (114) must be strictly positive.

**References**


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### Derived parameters

| $u$                                           | 0.282  |
| $d$                                           | -0.221 |
| $\zeta_u u + \zeta_d d$                       | 0.031  |

Table 1: Parameters used in the calculations in table 2 and figures 4-24. Quantities marked with (*) are annualized counterparts to those used in the calculations of the model. $\zeta_u$ is the unconditional probability of an up-tick in the stock price between periods. $\alpha_l$ is the relative weighting which the leisure term receives in the utility function. $\alpha_b$ is the relative weighting the bequest term receives in the utility function. $\beta$ is the time discount. $R_f$ is the gross risk-free rate, $\mu$ is the risk premium, $\sigma^2$ is the variance of stock returns, $S_0$ is the initial value of the stock, $p_t$ is the annual wage, $K$ is the maximum amount of time the agent may work in a given period (here normalized to 1) and $F_0$ is the initial financial wealth. $u$ is the excess return of the stock for an up-tick. $d$ is the excess return of the stock for a down-tick.
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Table 2: Results for log-utility and risk-averse ($\gamma = 5$) agents with parameters as in table 1. All quantities are annualized. All quantities except age and $\phi^*$ are averages taken from the calculation with an age of 31. $\bar{F}$ is the average financial wealth from the initial calculation. $\bar{F}^i$ is the average investment wealth (wealth after consumption and labor earnings), $\bar{c}$ is the expected average consumption, $\bar{t}$ is the expected average leisure (fraction of the total available time devoted to leisure). Portfolio weights are significantly larger than 1. This is because the agent’s maximum labor income in the future $H^c_{\text{max}}$ is like a bank account, so that total wealth (financial + human) has both the actual financial bank accounts that the agent holds, plus the riskless asset which are represented by the maximum labor income the agent can call upon in the future.
Optimal Portfolio Weight vs. Age $\gamma = 5$

Figure 3: Plot of $\phi^* \text{ versus age for fixed initial financial wealth levels. Each curve in the plot is obtained by fixing the financial wealth of the agent but varying his age, and solving for the portfolio weight (}\phi^*\text{) using equation (43). Parameters other than initial wealth level and age are as in table 1. } \phi_M \text{ is the portfolio weight which obtains in the case where the agent has no labor/leisure choice (}\phi_M \text{ for the discrete time case is defined in equation (45)).}
Figure 4: Wealth paths for a log utility agent with parameters as in table 1. The extreme variations in wealth are associated with very unlikely paths for stock returns.
Wealth Paths, $\gamma = 5$

- Expected Ex-ante Wealth
- No Risky Asset
- Maximal Stock Returns
- Minimal Stock Returns
- Other Paths of Stock Returns

Figure 5: Wealth paths for a risk-averse ($\gamma = 5$) agent with parameters as in table 1. The extreme variations in wealth are associated with very unlikely paths for stock returns.
Figure 6: Consumption paths for a log utility agent with parameters as in table 1. The extreme variations in consumption are associated with very unlikely paths for stock returns.
Figure 7: Consumption paths for a risk-averse ($\gamma = 5$) agent with parameters as in table 1. The extreme variations in consumption are associated with very unlikely paths for stock returns.
Figure 8: Labor choice paths for a log utility agent with parameters as in table 1. Labor choice is the fraction of available time \( ((K - l^*_t)/K) \) the agent chooses to work.
Figure 9: Labor choice paths for a risk-averse ($\gamma = 5$) agent with parameters as in table 1. Labor choice is the fraction of available time ($\left( K - l^*_t \right) / K$) the agent chooses to work.
Figure 10: Retire, vacation, work, and work-forever regions for parameters as in table 1. The x-axis represents time, and the y axis represents state, with dots being nodes on the binomial tree. Selected points are shown. Successively larger probabilities of retirement at any age are clearly higher for better stock returns. The more risk-averse agent retires less often.
Figure 11: Optimal labor percentage with parameters as in table 1. The x-axis represents time, and the y-axis represents state, with dots being nodes on the binomial tree. Selected points are shown. The labor fraction distribution at a particular age across states is strictly increasing as stock prices decrease, reflecting the “cushioning” of financial wealth shocks with labor earnings. Two other points are of particular interest in comparing the behavior of the two types of agents. The variability across state (stock price) is clearly higher for the less risk-averse agent (observe values for age 41). This is related to the second fact, which is that the less risk-averse agent adjusts his leisure more in response to slight financial wealth shocks (a few up- or down-ticks in stock price) than the more risk-averse agent. The more risk-averse agent is more risk-averse over both consumption and leisure.
Figure 12: Conditional standard deviation of labor income (in dollars) with parameters as in table 1. The x-axis represents time, and the y-axis represents state, with dots being nodes on the binomial tree. Selected points are shown. First, it is interesting to note that the conditional standard deviation of labor earnings is “hump-shaped.” At any given date, if stock returns have been particularly bad (near the lower edge of the tree) the agent is already working at or near the maximum (see figure 11), whereas when stock returns have been particularly good (near the bottom of the “Retire” region) the agent is often working very little and if stock returns are better, cannot work much less. As in figure 11, the less risk-averse agent adjusts his labor much more in response to slight financial wealth shocks (a few up- or down-ticks). Thus the conditional standard deviation of labor income for the less risk-averse agent is more variable.
Figure 13: Human capital (in dollars) with parameters as in table 1. (Human capital $H_c^e(t)$ is defined in (53)). The x-axis represents time, and the y axis represents state, with dots being nodes on the binomial tree. Selected points are shown. The less risk-averse agent adjusts more in response to financial shocks, so that at any given date, his human capital “plateaus” more quickly in response to stock price decreases.
Figure 14: Conditional standard deviation of human capital (in dollars) with parameters as in table 1. (Human capital $H^c(t)$ is defined in (53)). The x-axis represents time, and the y axis represents state, with dots being nodes on the binomial tree. Selected points are shown. As in the case of conditional standard deviation of labor earnings (figure 12), we see that the conditional volatility of human capital is “hump-shaped,” and that it is more variable for the less risk-averse agent, reflecting the fact that the less risk-averse agent adjusts labor more in response to financial shocks to wealth.
Figure 15: Annualized optimal consumption (in dollars) with parameters as in table 1. The x-axis represents time, and the y axis represents state, with dots being nodes on the binomial tree. Selected points are shown. As in the case of labor choice (figure 11), the less risk-averse agent adjusts consumption more in response to financial wealth shocks.
Figure 16: Optimal financial wealth with parameters as in table 1. The x-axis represents time, and the y axis represents state, with dots being nodes on the binomial tree. Selected points are shown. Unlike figures 4-15, the last date for which figures are presented is actually at age 71 (when the agent leaves a bequest), to emphasize the point that the bequest cannot be negative, even though the optimal financial wealth is often negative for states on the lower-half of the tree. The agent optimally borrows heavily to invest and consume, thus avoiding having to work so much.
Figure 17: Ratio of debt to human capital. (Human capital $H^c(t)$ is defined in (53)). Parameters are as in table 1. The x-axis represents time, and the y-axis represents state, with dots being nodes on the binomial tree. Selected points are shown. First, it is notable that both agents will take on debt levels worth more than three-fourths the value of his human capital when stock returns are really bad. Also, the less risk-averse agent has higher levels of debt to human capital. In the second to last period, the agent never takes on any debt, so that when the agent leaves the economy, his bequest will be strictly positive.
Figure 18: Optimal portfolio weight with parameters as in table 1. $\phi_M$ is the optimal portfolio weight which obtains in the case when there is no labor choice (defined in (45). The x-axis represents time, and the y axis represents state, with dots being nodes on the binomial tree. Selected points are shown. As pointed out in table 2 the portfolio weight at age 31 is significantly larger than 1. The portfolio weights are largest in magnitude near the center of the tree. This region is also where the investment wealth is typically the smallest. The portfolio weights are often negative, though in that case the investment wealth is negative, so that the amount invested in the stock market is strictly positive (see figure 19). In the “Retire” region (figure 10), we see that the portfolio weight is the Merton value, as it must be since the leisure/labor choice is irrelevant. This is also true for all states in the last period, because there are no future periods in which labor income can be earned.
Figure 19: Regions where the optimal portfolio weight and and Optimal Investment Wealth ($F_{t}^{i*} = F_{t}^{*} - c_{t}^{*} + \bar{c} - \epsilon_{t}^{*}$) are positive or negative. Parameters are as in table 1. This shows that the amount of wealth invested in the stock market $I_{t}^{S}$ is strictly positive even when the agent goes into debt to invest and consume (i.e. $F_{t}^{i*} < 0$).
Figure 20: Optimal portfolio weight at age 68 versus stock price, for parameters as in table 1. $\phi_M$ is the portfolio weight when there is no labor/leisure choice. The portfolio weight is always a decreasing function of stock price on either side of the boundary where investment wealth $F_i^*$ switches sign. The portfolio weight increases as investment wealth decreases, though not quite as fast as investment wealth, so that the majority of the time, agents who have had better stock returns invest more in the stock market (also see figures 22 and 23).
Figure 21: Optimal investment wealth ($F_t^{i*}$) versus stock price for the risk-averse ($\gamma = 5$) agent. It is monotonically increasing with stock price, and switches sign at the vertex of the dotted lines. The point at which it crosses sign corresponds to the asymptote in figure 20.
Figure 22: Wealth invested in stock age at 68 for log utility with parameters as in table 1. Also shown is the stock price where the agent will retire at age 68. The wealth invested in the stock market is monotonically increasing versus stock price (note that this is not true for the $\gamma = 5$ case; see figure 23). Agents who have had better stock returns will invest more in the stock market.
Figure 23: Wealth invested in stock at age 68 for CRRA Utility ($\gamma = 5$) with parameters as in Table 1. Also shown is the stock price where the agent will retire at age 68. The optimal investment in the stock market is mostly increasing in the stock price until the agent has had stock returns that are high enough so that in the future he might retire. In this case, it is possible that an agent who has had slightly better stock returns may invest slightly less in the stock market. This is because if the agent has had slightly better stock returns, and is more likely to retire in the future, he does not want to have to work too much if stock returns are bad. Because this effect is especially pronounced right around the “Retire” region, the amount invested in the stock market actually decreases with stock price for a small range of stock prices. Also see Figure 24.
Figure 24: The top plot shows total dollar holdings (\(I^S_t\)) in stock for the risk-averse agent (\(\gamma = 5\)) including human capital (with \(HCTM\)), and not including human capital (without \(HCTM\)). When the agent has less human capital which he can use to cushion financial wealth shocks in the future (e.g., if he retires), then he will hold fewer total dollars in stock. The bottom plot shows the number of additional dollars the agent will hold in stock because of human capital (\(HCTM\)).
Increasing Disutility of Work with Age: $\alpha_l$ increasing

Figure 25: $\alpha_l$ versus time for the numerical exercises explained in section 5.4, figures 26 and 27, and table 3.
Figure 26: Retire, vacation, work, and work-forever regions with $\alpha_t$ as in figure 25, constant at the value $1/2$ until age 62, and increasing linearly to 10 by age $70\frac{1}{2}$. All other parameters are as in table 1. This can be compared to figure 10, where $\alpha_t$ has a constant value of $1/2$ over the entire life span of the agent.
Figure 27: Wealth invested in stock at age 68 for CRRA Utility ($\gamma = 5$) with parameters as in table 1, except $\alpha_t$ is as in figure 25, constant with value 1/2 until age 62, and increasing linearly to 10 by age $70\frac{1}{2}$. This can be compared with figure 23, where $\alpha_t$ has a constant value of 1/2 over the entire life of the agent.
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Table 3: Probabilities of retirement on or before a given a given age, and wealth levels required for retirement at that age, for both log utility and risk-averse investors. All parameters used in the calculation are as in table 1 except α(t), which is constant until age 62, after which it increases linearly to a value of 10 by age 70\frac{1}{2}, as in figure 25.
Figure 28: Retire, vacation, work, and work-forever regions for parameters as in table 1, except for $\rho$, which has the value .203, corresponding to a conditional volatility of wages of 5%. The x-axis represents time, and the y axis represents state, with dots being nodes on the binomial tree. Selected points are shown. We see that for the risk-averse investor, the ”Retire” and ”Vacation” regions are now on the lower side of the tree (i.e. occur when stock returns are poor), while the ”Work-Forever” region occurs for good stock returns (compare figures 10 and 26).