Abstract

I present a large competitive economy with rational strategic traders, in which prices are noisy due to a stochastic aggregation of idiosyncratic noises in the traders’ signals. This partial-revelation property is new, because the existing framework implies that ever-increasing competition has puzzling effects on price noise: the market impact of liquidity trades becomes negligible, and the price moves arbitrarily close to the dividend. The model is further distinguishable from the canonical framework in terms of trader welfare and cross-sectional demand variation. An intriguing property of the economy is that it can accommodate a small minority fraction of risk-seeking traders.

Keywords: Noise, Liquidity, Aggregation, Competition.

JEL Codes: D41, D53, D82, E19, G14.
1 Introduction

According to a vast literature in microstructure and rational expectations, price noise is an essential feature of well-posed theories of financial markets. It is well known that prices aggregate the private information of individual investors about fundamentals, but not in a perfectly revealing manner due to the presence of noise. But where does this noise come from? The existing literature points to liquidity trading, noise trading, and other effects as the source of price noise. As I show in this article, another possibility is that price noise emerges out of the aggregation of idiosyncratic noises of competitive rational traders. Prices, therefore, do not only aggregate private information, but they also aggregate noise.

My model uses a large-economy limit of a competitive version of Kyle (1985) with a risk-neutral market maker, strategic informed traders with mean-variance preferences, and pure noise traders. In addition, the model has one period of trade and one risky asset; this allows us to focus the discussion on a large cross-section of competitive traders. The only departure from the traditional framework is in the structure of information, in a manner that I explain below. As Holden and Subrahmanyam (1992) show, competition makes the market impact of noise trading negligible in a large economy. In the traditional framework, this has the puzzling side effect that prices fully reveal dividend information. While it is very intuitive that information aggregation improves as competition intensifies, it remains unclear why a large competitive market should also be a market with fully revealing prices.

My first main result is that it possible to have a large-market version of Kyle (1985) in which prices are noisy. To construct an economy with this feature I depart from the traditional framework by introducing a cross-sectional Itô process as a model of traders’ signals. The resulting economy is effectively an enlargement of the class of economies of Holden and Subrahmanyam (1992). The cross-sectional Itô structure, however, is not subject to the usual Laws of Large Numbers that underpin existing mechanisms of information aggregation. My information structure can therefore break the full-revelation property. Nevertheless, there is still a concept of information aggregation. This aggregation concept does not have the usual mathematical representation of a sample average of signals, but it uses a stochastic integral, instead. In fact, this particular feature of my model allows us to think of price noise as an aggregate quantity made up of a large number of idiosyncratic shocks. I therefore refer to this type of noise as “aggregate price noise.”

My second main result is that the market impact of noise trading does not vanish in a limiting large economy. This property is shared with the “group model” of Rostek and Weretka (2015), with the difference that the agents in Rostek and Weretka (2015) have different valuations of the traded asset, and that their equilibrium is Bayes-Nash in demand curves. In my economy the valuation of the traded asset is the same for every rational trader, and the equilibrium is a competitive version of
Kyle (1985), albeit with Itô-style signals. This results says that a rational trader’s demand function does not converge to that of an agent in a price-taking equilibrium. As Rostek and Weretka (2015) put it, there is price-making behavior in the large-limit market. The mechanism for this result in my model is that the contribution of each rational agent’s signal to the aggregate order seen by the market maker grows at a rate proportional to the size of the economy. Consequently, the market impact of the aggregate order flow does not decay to zero with increasing competition, as it does in Holden and Subrahmanyam (1992). This effect is reminiscent of the rational expectations models of García and Urošević (2013) and Kovalenkov and Vives (2014), where the amount of pure noise trading in the economy is explicitly scaled by the size of the economy. Herein, however, the effect rests on the variance properties of the Brownian increments contained in traders’ signals.¹

In later sections of the paper I compare my economy with Itô signals with existing competitive microstructure models based on Kyle (1985). I show that the market-impact parameter in Kyle (1985)-based models converges to zero at a rate inversely related to the square root of the size of the economy. This is an extension of results contained in Holden and Subrahmanyam (1992), and it is true irrespective of risk preferences, and of whether informed traders are informed perfectly or not. I explore additional differences between existing competitive microstructure models and my framework in more detail. As I show, the Itô-signal economy is qualitatively different not only in terms of revelation and market impact, but also in terms of trade and welfare.

An intriguing question arises as a consequence of the above results. Since the price contains aggregate noise in the limit of a large economy, is it possible to do away with pure noise traders and still have a noisy price? More concretely, if we start from a large economy as described above, and we set the variance of pure noise trading to zero, do we still have an equilibrium?

For the third main result of the paper I construct an example where this is indeed possible. A sufficient condition for it is that we have a small minority fraction of strategic traders with risk-seeking attitudes. It may initially appear that risk seeking is a troublesome ingredient of any well-founded economy. After all, intuition from price-taking frameworks dictates that a negative risk-aversion coefficient would imply that the demand-choice problem of a utility-maximizing trader is not well posed. There are, however, several reasons why risk seeking is not only internally consistent in my economy, but also one of its interesting features.

First, the demand-choice problem of risk-seeking traders is well posed indeed. In a similar fashion to results contained in Kyle (1989), the second-order condition of any rational trader is the sum of two terms: a risk-preference term and a market-impact term.² The second-order condition

¹In particular, the order of the signal-to-noise ratio coincides with the distance between traders in an increasingly refined grid of agents. In the limit this grid converges to a continuum of unit mass, and the Itô signals converge to a cross-sectional Brownian Motion.

²Note that the non-risk-preference term that appears in the second-order condition has a different interpretation in Kyle (1989) than it does here. Algebraically, however, the optimal demands are very similar.
is satisfied with a small amount of negative risk aversion exactly because in a large economy the market-impact parameter converges to a positive constant.

Second, the function of the risk-seeking minority is to ensure there is enough noise in the economy. The variance of the price noise generated out of aggregate demand is larger when the rational agents’ trading intensities are larger in the aggregate. It can be shown that, in order to maintain equilibrium, as we reduce the variance of liquidity traders we must compensate by increasing the trading intensity of some part of the population of rational agents. The boundary of when agents are risk-neutral is not enough to guarantee that we can set the variance of liquidity traders to zero. To achieve zero variance of liquidity trading we must thus cross into a domain where some rational agents are risk-seeking.

Third, that a minority fraction of people are risk seekers is a common finding in the empirical and experimental literature. Coombs and Pruitt (1960) carry out experiments using simple gambles with a population of 99 American subjects. They find that about one third of the participants prefer gambles with higher variance. Ali (1977) estimates the risk preferences of a representative, rational, expected-utility-maximizing agent who has objective knowledge of risky outcomes. Using data from more than 22,000 horse races in New York state, he estimates that the representative race bettor is a risk lover. Kachelmeier and Shehata (1992) elicit certainty equivalents for lotteries in experiments under high monetary incentives with a group of 80 Chinese subjects. Their evidence suggests that risk-seeking behavior is present in a fraction of individuals. Holt and Laury (2002) provide measures of relative risk aversion from experiments with 175 American subjects, also with significant monetary incentives. Their estimated risk-aversion coefficients are negative for a small fraction of individuals. Dohmen et al. (2011) use a German survey of 22,000 respondents combined with a field experiment of 450 subjects. They provide quantitatively similar estimates of fractions of risk-seekers in the German population to what Holt and Laury (2002) find for their American subjects.

The most popular approach of modeling noisy markets in the existing literature assumes that the supply of the traded asset varies exogenously. This idea first appears in Grossman (1976), who introduces the concept of stochastic supply as a device that obscures the fundamental information otherwise contained in prices. The idea is further developed in Grossman and Stiglitz (1980) and Hellwig (1980), who established what have since become workhorse models in the information economics of financial markets. As Dow and Gorton (2008) point out, there are two economic interpretations of stochastic supply. One interpretation portrays variation in supply as liquidity shocks to investors’ personal circumstances, whereas another interpretation argues that some individuals’ trading decisions are driven by irrational, and therefore “noisy,” dispositions.
Since Grossman’s original paper, the literature that uses noisy prices has grown tremendously.\textsuperscript{3} In addition, a number of other ways of building noise into prices have been proposed. One such approach is contained in Diamond and Verrecchia (1981) and Verrecchia (1982), who introduce random endowments in order to both generate trade and to avoid full revelation of the asset’s fundamental value. As discussed more recently in the treatise of rational expectations of García and Urošević (2013), these endowment models belong to a class of economies in which the size of noise grows with the number of agents. The scaling properties of noise are further discussed in Kovalenkov and Vives (2014), who allow the variance of noise trading to grow with the number of competitive agents within the framework of Kyle (1989).

Another approach is to relax specific assumptions of the rational expectations, demand-curve Bayes-Nash, and microstructure paradigms. One such example is Banerjee and Green (2015), where uninformed investors are uncertain about whether the individuals they trade against are informed traders or noise traders. Kyle, Obizhaeva, and Wang (2016) introduce noise in a finite economy by modeling traders as overconfident informed individuals who agree to disagree about the precision of each other’s information. Vives (2011), Rostek and Weretka (2012), and Rostek and Weretka (2015) have traders with correlated asset valuations in privately-revealing equilibria. Vives and Yang (2017) propose an equilibrium in which informed agents are unable to perfectly comprehend the information contained in prices because they are subject to receiver’s noise à la Myatt and Wallace (2012).

The rest of the paper proceeds in the following manner. In Section 2 I present the finite version of the economy. I then derive the perfectly competitive limit by making the number of traders very large in Section 3. In Section 4 I also compare the properties of my model with those of the standard microstructure paradigm. I present a brief example of other possibilities of perfect competition with cross-sectional Itô processes in 5. I draw up suggestions for further work and I conclude in Section 6. The proofs of all the results are in the Appendix.

2 The Model

The economy comprises a dividend-paying asset, and several traders who trade the asset. There are $N$ informed competitive rational traders and one uninformed risk-neutral market maker. There are also liquidity traders who provide a shock to the supply of the asset. The liquidity traders are not necessary for the main result of the paper, but I include them for the purpose of comparing my model with existing benchmark models.

\textsuperscript{3}Dow and Gorton (2008) conduct a survey of the literature as of 2008. For a more recent comprehensive list of papers, the interested reader can consult the introduction of García and Urošević (2013), and of Han, Tang, and Yang (2016).
Prices are set by competition for every informed investor, in a manner akin to Holden and Subrahmanyam (1992), who generalize Kyle (1985) to allow for an arbitrary number of perfectly informed rational investors. I use a limiting version of their structure, but with two differences. First, I allow the investors to have mean-variance preferences. These preferences specialize to the risk-neutral setup by setting the risk aversion parameter to zero. Second, there is one important difference in the structure of information, which I explain in more detail below. Other than that, the model is a standard economy with one trading period. Investors select their demand so as to maximize expected utility from profits, conditional on their private signals. They independently submit their demands to a market maker, who observes the total order flow from all investors, including that coming from liquidity-motivated trades. The market maker sets the price equal to the expected fundamental value of the asset, conditional on the total order flow.

I now provide the technical details of the economy. For each \( n = 1, \ldots, N \), agent \( n \) corresponds to point \( a_{n-1} \), where \( a_{n-1}, n = 1, \ldots, N \), are points that partition the interval \([0, 1]\) such that \( 0 = a_0 < a_1 < a_2 < \ldots < a_{N-1} < a_N = 1 \), with \( \Delta a_n = a_n - a_{n-1} \). Note that I have assigned the edges of the partition to be zero and one for convenience. In this manner, as the partition becomes finer and finer by increasing \( N \), it will eventually converge to a continuum of agents in the interval \([0, 1]\). To simplify the exposition I adopt the following conventions. I make the points equidistant, that is, \( \Delta a_n = a_n - a_{n-1} = \Delta a = 1/N \) for each \( n \). In addition, because each partition point \( a_{n-1} \) is one-to-one with \( n = 1, \ldots, N \), I use \( a_{n-1} \) as an index instead of \( n \), where appropriate.

I assume that the distribution of the dividend \( D \) is \( \mathcal{N}(0, \tau^{-1}D) \). Each investor \( n \) is endowed with a signal \( \Delta z_{a_n} \) about the liquidating dividend, where

\[
\Delta z_{a_n} = D\Delta a + \sqrt{\tau_{a_{n-1}}^{-1}} \Delta B_{a_n}. \tag{1}
\]

Here, \( B \) is a standard Brownian motion in the interval \([0, 1]\), independent of \( D \), and \( \Delta B_{a_n} \) is the \( n \)th Brownian increment,

\[
\Delta B_{a_n} = B_{a_n} - B_{a_{n-1}}. \tag{2}
\]

One may recognize the structure in (1) as a discrete version of an Itô process, where I have replaced the usual “time” argument with a discrete set of values corresponding to the cross section of agents, \( a_{n-1}, n = 1, \ldots, N \). This is deliberate, because it is the Itô integral indeed that allows us to construct aggregate noise in the competitive limit. For this reason I refer to the signal structure in (1) as an “Itô signal.”

I would like to stress that by independence of Brownian increments, the noise \( \Delta B_{a_n} \) is independent across traders, and that its distribution is \( \mathcal{N}(0, \Delta a) \). The existing literature treats trader-specific noise as an independent random variable whose distribution is normal, but its vari-
ance is a constant that does not scale with respect to $\Delta a$. This distinction is important, because in a large competitive economy such independent noises aggregate away to zero via Kolmogorov’s Law of Large Numbers. In contrast, in my setup the aggregation of trader-specific noises does not converge to zero; the noises instead aggregate together to form a stochastic integral.

The demand of agent $n$ is $X_n$. Each agent submits their demand of the asset to the market maker, without knowing what demand quantities other agents will submit. The demand of the liquidity-motivated traders is modeled by the random random variable $\theta$, with distribution $\mathcal{N}(0, \tau_{\theta}^{-1})$, independently of $D$ and the Brownian increments $\Delta B_{a_n}$ for all $n$.

The market maker sees only the aggregate order flow, and cannot infer which agent demanded which quantity. The market maker sets the price $P$ equal to what he expects the asset value to be given the aggregate order flow,

$$P = \mathbb{E} \left[ D \sum_{i=1}^{N} X_i + \theta \right].$$

As in Kyle (1985) and Holden and Subrahmanyam (1992), I consider demand strategies linear in signals

$$X_n = \beta_{a_{n-1}} \Delta z_{a_n},$$

and prices linear in aggregate order flow,

$$P = \lambda \left( \sum_{i=1}^{N} X_i + \theta \right).$$

The profit for agent $n$ is $\pi_n = X_n(D - P)$. Given the signal $\Delta z_{a_n}$, the utility of agent $n$ is

$$u(\pi_n; \Delta z_{a_n}) = \mathbb{E} \left[ \pi_n \right| \Delta z_{a_n} \right] - \frac{\delta_{a_{n-1}}}{2} \text{Var} \left( \pi_n \right| \Delta z_{a_n} \right),$$

where $\delta_{a_{n-1}}$ measures the risk aversion of agent $n$. Risk neutrality corresponds to $\delta_{a_{n-1}} = 0$. This utility function is akin to the Constant Absolute Risk Aversion (CARA) utility used in Subrahmanyam (1991).

### 3 The large-limit economy

Writing out the price function with the Itô signal I get

$$P = D \sum_{i=1}^{N} \lambda \beta_{a_{n-1}} \Delta a + \sum_{i=1}^{N} \lambda \beta_{a_{n-1}} \sqrt{\tau_{a_{n-1}}^{-1}} \Delta B_{a_n} + \lambda \theta.$$
As we can see on the right-hand side of (7), the price contains two different kinds of summation. These two kinds of summation have very different limiting behaviors. Recall that as the number of agents becomes infinite, the partition \(a_{n-1}, n = 1, \ldots, N\) of the interval \([0, 1]\) converges to a continuum. Therefore, the sequence \(\tau_{a_{n-1}}, n = 1, \ldots, N\) converges to a continuous function \(\tau(a)\) on \([0, 1]\), and similarly, the sequence \(\beta_{a_{n-1}}, n = 1, \ldots, N\) converges to a continuous function \(\beta(a)\) on \([0, 1]\). These facts imply that the summation in the coefficient of the dividend \(D\) in (7) converges to

\[
\int_0^1 \lambda \beta(a) da,
\]

in the sense of Riemann. The second summation in (7) is the aggregate price noise, and as we make the number of agents infinite it converges to

\[
\int_0^1 \lambda \beta(a) \sqrt{\tau(a)^{-1}} dB_a,
\]

in the sense of Itô. I give more details on the function \(\beta(a)\) and the constant \(\lambda\) in Theorem 1 below, in which I show the economy of the large limit. To derive this limit, I first use the Brownian properties of the Itô signal to provide an explicit solution of the finite economy in Proposition A.2 of the Appendix. The price is as in (7). I then send the number of investors \(N\) to infinity, thereby obtaining the following result.

**Theorem 1** As \(N \to \infty\), the market-impact parameter \(\lambda\) converges to

\[
\lambda = \frac{\int_0^1 \beta(s) ds}{\left(\int_0^1 \beta(s) ds\right)^2 + \tau_D \left(\int_0^1 \beta^2(s) ds + \frac{1}{\tau} \right) + 2 \lambda D},
\]

the demand coefficient sequence converges to a function \(\beta\) on \([0, 1]\) that satisfies

\[
\beta(a) = \frac{\tau(a)}{\delta(a) + 2 \frac{\lambda D}{1 - \lambda \int_0^1 \beta(s) ds}},
\]

and the price converges to

\[
P = \int_0^1 \lambda \beta(a) dz_a + \lambda \theta = \lambda \left( D \int_0^1 \beta(a) da + \int_0^1 \beta(a) \sqrt{\tau(a)^{-1}} dB_a + \theta \right).
\]

A few important comments are in order. First, as is standard in the literature, and as we can see in Equation (10c), we can think of the price as a noisy version of an aggregate signal of all the traders. The signal of each trader \(a\) is \(dz_a\), the infinitesimal limit of the signal assumption in (1),
with mathematical representation

\[ dz_a = Dda + \sqrt{\tau(a)^{-1}}dB_a. \]  

(11)

Each signal carries a coefficient that equals the market-impact parameter times the trading intensity of each agent, \( \lambda \beta(a) \). Thus, signals with higher precision contribute more to the dividend information contained in the price, whereas signals of more risk averse traders contribute less to the dividend information contained in the price. These facts conform to standard intuition that traders with higher precision trade more aggressively, and traders with higher risk aversion trade less aggressively.

Second, the limiting economy features a continuum of infinitesimally small strategic investors, whose decision problems remain well-defined in the infinitesimal limit. As we can see from Theorem 1, optimal agent demands are characterized as a Normal random variable with mean zero and variance \( [\beta^2(a)/\tau(a)] da \).

Third, a novel feature of information aggregation in this model is that the price contains an amount of noise beyond that contributed by noise traders. Due to the Brownian structure of idiosyncratic signal noises, the aggregate order flow observed by the market maker contains a stochastic integral of signal noises in the form of \( \int_0^1 \beta(a)\sqrt{\tau(a)^{-1}}dB_t \). This ensures that in my model there is always price noise, even in a large economy. As I explain in more detail below, this feature is not possible in existing models due to the mechanics of aggregation via the Law of Large Numbers.

Fourth, the limiting economy of Theorem 1 has very different implications for market impact than existing microstructure models with competition. It is well known that the information structure employed by extant models of competition in the Kyle (1985) framework implies that in the large limit the market-impact parameter converges to zero, and the price converges to the dividend.\(^4\) This has the consequence that the noise traders do not influence the price at all in a large economy. In contrast, as we can see in (10a), the market-impact parameter in my model converges to a positive constant. I further explore how my model compares to existing microstructure models in this, and other aspects, in Section 4 below.

Let

\[ \rho = \frac{\lambda \tau_D}{1 - \lambda \int_0^1 \beta(s)ds}. \]  

(12)

\(^4\)It may seem paradoxical that in existing canonical models \( \lambda \) converges to zero and the price \( P \) converges to the dividend \( D \), because \( \lambda \) appears as a coefficient of the aggregate demand of rational traders. Nevertheless, even though \( \lambda \to 0 \) in the canonical framework, it is also true that \( \lambda \sum_{i=1}^N X_i \to D \) as \( N \to \infty \) because \( \lambda \) and \( \lambda \sum_{i=1}^N X_i \) converge at completely different rates of \( N \). See Sections 4.1 and 4.2 and Lemma A.7 of the Appendix for more details.
which allows me to rewrite the inverse of the market-impact parameter as

\[
\frac{1}{\lambda} = \int_0^1 \beta(s)ds + \frac{\tau_D}{\rho}. \tag{13}
\]

This shows that every quantity above is a function of \(\rho\) and \(\int_0^1 \beta(s)ds\) alone, and it allows us to completely characterize the equilibrium by the following proposition.

**Proposition 2** Given a precision function \(\tau\), the equilibrium in the financial market satisfies

\[
\rho \int_0^1 \frac{\tau(a)}{[\delta(a) + 2\rho]^2} da + \frac{\rho}{\tau_0} = \int_0^1 \frac{\tau(a)}{\delta(a) + 2\rho} da, \tag{14a}
\]

and

\[
\int_0^1 \beta(s)ds = \int_0^1 \frac{\tau(a)}{\delta(a) + 2\rho} da, \tag{14b}
\]

where \(\rho\) and \(\int_0^1 \beta(s)ds\) have the same sign.

We can think of the two conditions in (14a) and (14b) as two equations in two unknowns, \(\rho\) and \(\int_0^1 \beta(s)ds\). Proposition 2 therefore completely characterizes the equilibrium, provided that we can solve the system of equations in (14), and provided that we have an exogenous precision function \(\tau\) in hand. I endogenize the precision function in the following section, but before I do so it is important to highlight an interesting property of the model.

The model can accommodate a mild degree of risk seeking behavior, at least for a small fraction of the trader population. Risk-seeking behavior may appear to be counterfactual, and what is more, mathematically troublesome of an assumption. Nevertheless, as I discuss in the introduction, several experimental and empirical studies provide support to the idea that a small fraction of the population may in fact be risk seeking rather than risk averse. Moreover, a mild degree of negative risk aversion can, in fact, be supported by the model. This would definitely not be so in a model with price-taking agents, because negative risk aversion would violate the second-order condition of these agents. This works very differently in my model due to an interesting consequence of that the market-impact parameter \(\lambda\) does not vanish in the limit of a large economy. It is straightforward to show that

\[
X_n = \frac{\mathbb{E} \left[ D - P_{-n} \right| \Delta z_{a_n}]}{2\lambda + \delta_{a_{n-1}} \text{Var} \left( D - P_{-n} \right| \Delta z_{a_n})}, \tag{15}
\]

where \(P_{-n} = \lambda \left( \sum_{i=n}^N X_i + \theta \right)\) is the price excluding the demand of trader \(n\). This demand function follows the intuition that strategic traders set their demand by responding to the aggregation of demand strategies of other traders. Note that as \(N\) becomes large, \(P_{-n}\) converges to the price \(P\).
If, in addition, $\lambda$ did converge to zero as $N$ became large, the optimal demand of each agent would approach that of a price-taking agent.\(^5\) In such a case, negative risk aversion would indeed be troublesome because the second-order condition with respect to demand would be positive rather than negative, implying that the demand would not correspond to an maximum. In my model, however, since $\lambda$ does not converge to zero as $N$ becomes large, it is possible to have traders with small degrees of risk seeking and still have the denominator of the demand function remain positive.

In particular, as soon as we rewrite the demand in (53a) in the form of the linear demand strategy in (4), we can see that due to the additive term $2\rho$ in the denominator of the demand function, the second-order condition of trader $t$ is satisfied if and only if

$$\delta(a) + 2\rho > 0. \quad (17)$$

This implies that mildly negative risk aversion is allowed, as long as it does not exceed $-2\rho$.

### 3.1 Information Acquisition

Up to this point I have treated the precision function $\tau$ as exogenous. In this section I allow agents to choose their precision level $\tau_{a_{n-1}}$. I assume that agent $n$ faces a quadratic cost function in precision, equal to

$$c_{a_{n-1}}(\tau_{a_{n-1}}) = \frac{\tau_{a_{n-1}}^2}{4\psi_{a_{n-1}}} \Delta a, \quad (18)$$

where $\psi_{a_{n-1}}$ is the inverse of agent $n$’s marginal cost of precision. Each agent chooses $\tau_{a_{n-1}}$ that is optimal ex-ante; that is, $\tau_{a_{n-1}}$ maximizes

$$\mathbb{E}[u(\pi_n; \Delta z_{a_n})] - c_{a_{n-1}}(\tau_{a_{n-1}}). \quad (19)$$

I define surplus to be the limiting aggregate expected utility minus the costs of information acquisition.

$$S = \lim_{N \to \infty} \sum_{t=1}^{N} \{ \mathbb{E}[u(\pi_n; \Delta z_{a_n})] - c_{a_{n-1}}(\tau_{a_{n-1}}) \} \quad (20)$$

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\(^5\)The optimal demand $X$ of a price-taking trader with CARA utility, risk-aversion coefficient $\delta$ and information set $\mathcal{F}$ is

$$X = \frac{\mathbb{E}[D - P|\mathcal{F}]}{\delta \text{Var}(D - P|\mathcal{F})}. \quad (16)$$
**Proposition 3** The endogenous precision of agent $t$ is

$$
\tau(a) = \frac{(1 - \lambda \int_0^1 \beta(s) ds)}{\tau_D} \frac{\psi(a)}{\delta(a) + 2\rho},
$$

and the demand coefficient is

$$
\beta(a) = \frac{(1 - \lambda \int_0^1 \beta(s) ds)}{\tau_D} \frac{\psi(a)}{[\delta(a) + 2\rho]^2},
$$

Moreover, the surplus is

$$
S = \frac{1}{4} \int_0^1 \frac{\tau^2(a)}{\psi(a)} da = \left(1 - \frac{\rho}{\tau_D \int_0^1 \beta(s) ds}\right) \int_0^1 \frac{\psi(a)}{[\delta(a) + 2\rho]^2} da = \left(\frac{\lambda}{2\rho}\right)^2 \int_0^1 \frac{\psi(a)}{[\delta(a) + 2\rho]^2} da.
$$

**Proposition 4** The equilibrium with information acquisition satisfies

$$
\rho \int_0^1 \frac{\psi(a)}{[\delta(a) + 2\rho]^2} da = \left(1 - \frac{\rho}{\tau_D \int_0^1 \beta(s) ds}\right) \int_0^1 \frac{\psi(a)}{[\delta(a) + 2\rho]^2} da
$$

and

$$
\rho \left(\int_0^1 \beta(s) ds\right)^2 + \tau_D \left(\int_0^1 \beta(s) ds\right) - \int_0^1 \frac{\psi(a)}{[\delta(a) + 2\rho]^2} da = 0,
$$

under the restriction that $\rho$ and $\int_0^1 \beta(s) ds$ have the same sign.

As for Proposition 2, we can think of the two conditions of Proposition 4 as two equations in two unknowns, $\rho$ and $\int_0^1 \beta(s) ds$. These equations are quite general, and they depend on the risk-aversion function $\delta$ and the inverse marginal-cost function $\psi$. I derive the equilibrium for three cases below.

### 3.1.1 Homogeneous risk aversion

The first case for which we can easily derive the equilibrium of Proposition 4 is when every agent has the same risk aversion coefficient, $\delta(a) = \delta$, for all $t$. Here I assume that $\delta > 0$, and I deal with the risk-neutral case separately as a special case below. In this case (22a) gives

$$
\int_0^1 \beta(s) ds = \frac{\rho \delta + 2\rho}{\tau_D \delta + \rho},
$$

(23)
and using this in (22b) gives
\[ \rho^3 (\delta + 2\rho)^4 + \rho (\delta + \rho) (\delta + 2\rho)^3 \tau_D \tau_\theta - (\delta + \rho)^2 \tau_\theta^2 \int_0^1 \psi(a) da = 0. \] (24)

This equation is a seventh-order polynomial in \( \rho \), which makes it hard to solve analytically. It is nevertheless straightforward to prove that it has a unique positive solution in \( \rho \).\(^6\)

### 3.1.2 Risk neutrality

The solution with risk neutrality is a special case of the solution with common risk aversion above, with \( \delta = 0 \). In this case (23) gives
\[ \int_0^1 \beta(s) ds = \frac{2}{\tau_\theta^3} \rho, \] (27)
and (24) gives
\[ 16\rho^5 + 8\tau_D \tau_\theta \rho^4 - \tau_\theta^2 \int_0^1 \psi(a) da = 0, \] (28)
which has a unique positive solution.\(^7\)

### 3.1.3 Heterogeneous risk preferences without liquidity traders

Let the precision of the liquidity trades be infinite, \( \tau_\theta = \infty \). In this case the random variable \( \theta \) degenerates to the constant zero. Moreover, Equations (22a) and (22b) can be solved independently of each other, because when \( \tau_\theta^{-1} = 0 \) Equation (22a) no longer depends on \( \int_0^1 \beta(s) ds \). Futhermore, as long as we can obtain a positive solution for \( \rho \) from (22a), Equation (22b) yields a unique positive solution for \( \int_0^1 \beta(s) ds \).

I proceed with constructing an illustrative example for which we can analyze the equilibrium.

---

\(^6\)By Descartes’s rule of signs, if the coefficients of \( \rho \) in the polynomial (24) have one sign change, then there is a unique positive solution for \( \rho \). After writing out the polynomial, it follows by inspection that the coefficients of \( \rho \) in (24) of order seven, six, five, four, and three are all positive. The coefficient of order two is
\[ \tau_\theta \left(7\delta^3 \tau_D - \tau_\theta \int_0^1 \psi(a) da \right), \] (25)
the coefficient of the linear term is
\[ \delta \tau_\theta \left(\delta^3 \tau_D - \tau_\theta \int_0^1 \psi(a) da \right) \] (26)
and the constant term is negative by inspection. It is straightforward to show that there is only one sign change in the coefficients of \( \rho \). The only thing that depends on the model parameters is in which coefficient the sign changes. If \( \delta^3 \tau_D > \tau_\theta \int_0^1 \psi(a) da \) then the sign changes in the constant term, if \( \tau_\theta \int_0^1 \psi(a) da / 7 < \delta^3 \tau_D < \tau_\theta \int_0^1 \psi(a) da \) then the sign changes in the linear term, and if \( \delta^3 \tau_D < \tau_\theta \int_0^1 \psi(a) da / 7 \) the sign changes in the quadratic term.

\(^7\)Note that Proposition 4 excludes the solution \( \rho = 0 \) when \( \delta(a) = 0 \) for all \( t \).
I assume that a fraction \( f_G \) of the agents has a risk-aversion coefficient \( \delta_G > 0 \), and a fraction \( f_B \) of the agents has a risk-aversion coefficient \( \delta_B < 0 \). Notice that the assumption \( \delta_B < 0 \) implies risk-seeking behavior, rather than risk aversion. As I discuss above, a small degree of risk seeking is economically well founded as long as \( \delta_B + 2\rho > 0 \). I therefore require that this condition is met in equilibrium.

For simplicity, I assume that every agent has the same marginal cost of acquiring information, so that \( \psi(a) = \psi \) for all \( t \). In this case (22a) gives

\[
f_B (\delta_B + \rho)(\delta_G + 2\rho)^3 + f_G (\delta_G + \rho)(\delta_B + 2\rho)^3 = 0. \tag{29a}
\]

Given a positive solution \( \rho \) of (29a), (22b) gives a solution for \( \int_0^1 \beta(s)ds \) as the unique positive solution of

\[
\rho \left( \int_0^1 \beta(s)ds \right)^2 + \tau_D \left( \int_0^1 \beta(s)ds \right) - \psi \left[ \frac{f_B}{(\delta_B + 2\rho)^2} + \frac{f_G}{(\delta_G + 2\rho)^2} \right] = 0, \tag{29b}
\]

Notice that the polynomial in (29a) cannot have a positive solution in \( \rho \) if \( \delta_B \) is not negative, or if the fraction \( f_B \) of agents with risk-seeking behavior is not positive. If \( \delta_B < 0 \) and \( f_B > 0 \) indeed, then the equilibrium has the following property.

**Proposition 5** If \( \delta_B < 0 \) and \( f_B > 0 \), then there exists a positive root \( \rho^* \) of (29a) for which \( \delta_B + 2\rho > 0 \), and thus the risk-seeking agents’ second order condition is satisfied.

### 4 Comparison with standard microstructure models

The argument I have presented thus far raises the question of whether aggregate price noise can be created out of a signal structure that is already employed by leading microstructure models. The answer to this question is no. To make the exposition concrete, I present limiting results for a broad unifying framework that subsumes the static versions of leading microstructure models. For reasons that will become clear in what follows, I refer to this unifying framework as an economy with “Kolmogorov signals.” I then compare the model with Kolmogorov signals with my model with Itô signals, and I highlight ways in which the two models are distinguishable. Note that the unified models to which I compare the Itô-signal economy do not typically feature information acquisition. For this reason, I use the equilibrium of the Itô-signal economy described in Proposition 2, which does not have information acquisition, either, for the comparison below.

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4.1 A unifying framework of existing literature

I construct an economy that is more general than the following models: the one-trading-period version of Kyle (1985), the one-trading-period version of Holden and Subrahmanyam (1992), and Subrahmanyam (1991) without the common noise component in the signal for the informed.

Suppose that $\varepsilon_n, n = 1, \ldots, N$ is a collection of independent random variables, where each $\varepsilon_n$ has distribution $\mathcal{N}(0, \tau_{n}^{-1})$ and is independent of $D$ and $\theta$. I do not assume that $\varepsilon_n$ are identically distributed, but as I explain further below, I instead assume that certain cross-sectional population moments of the precisions $\tau_n$ exist. I also assume that agents have mean-variance preferences, but in order to simplify the exposition I assume that all agents have the same risk-aversion coefficient.

The traders’ signal is

$$\Delta \tilde{z}_{a_n} = (D + \varepsilon_n) \Delta a.$$  

(30a)

This modeling choice may appear at odds with the established literature, which usually models traders’ signals as

$$s_n = D + \varepsilon_n.$$  

(30b)

The two signals in (30a) and (30b) appear to be quite different, because the former uses the cross-sectional partition size $\Delta a$, whereas the latter is the standard way of representing a signal in the existing literature. It turns out, however, that both signals represent exactly the same economy. A summary intuition for this fact is that the two signal representations have the same signal-to-noise ratio, which is free of $\Delta a$. Before I explain why in more detail, I would like to note that representation (30a) has the advantage of being similar to the Itô signal in (1). It is a version of the Itô signal, but with the noise in the “drift” part of the Itô process rather than in the “volatility” part. This fact allows us to cast (30b), the usual way of representing signals, in the form of (30a). We can therefore think of the economy with Itô signals as in the same large class of economies as Holden and Subrahmanyam (1992), with the one-period version of Holden and Subrahmanyam (1992) being a special case of the Itô-signal economy for which every trader is risk-neutral and has infinite signal precision.

To show that the signals in (30a) and (30b) represent the same economy, I maintain assumptions (3) and (4), and I adopt the notation

$$X_n = \beta_{a_{n-1}} \Delta \tilde{z}_{a_n}$$  

(31a)

for the signal in (30a), and the notation

$$X_n = \hat{\beta}_n s_n$$  

(31b)
for the signal in (30b).

**Lemma 6** The economies represented by the signals in (30a) and (30b) are equivalent. For either signal representation, agents’ demands, the aggregate signal, the price, and the market-impact parameter are identical. In particular, the relationship of the coefficients in agent demands in the signal representations (30a) and (30b) is

\[
\hat{\beta}_n = \tilde{\beta}_{a_{n-1}} \Delta a.
\] (32)

Detailed expressions for the market-impact parameter \(\lambda\) and the demand coefficients \(\tilde{\beta}_{a_{n-1}}, n = 1, \ldots, N\), are given in Lemma A.6 of the Appendix.

As Lemma 6 shows, the signals in (30a) and (30b) are equivalent. This happens because the demand representations (31a) and (31b) are different ways of writing the same thing:

\[
\hat{\beta}_n \left( D + \epsilon_n \right) = X_n = \tilde{\beta}_{a_{n-1}} \left( D + \epsilon_n \right) \Delta a.
\] (33)

To summarize, no matter which representation we choose for the agent signal, the demand is always as in (33), where we can see that there is a \(\Delta a\) incorporated into the demand coefficient. This, of course, is no accident. After all, \(\Delta a = 1/N\), which says that the demand coefficient in (33) is inversely related to the number of agents in the economy. This is a straightforward effect of competition. As the number of agents increases, each agent has to trade less aggressively so as not to reveal too much to the market maker, which naturally decreases the magnitude of his demand. Another way to frame this effect comes from the standard intuition that competition erodes profits. Because profits are proportional to squared demands, increasing the number of traders decreases the magnitude of demand of any given trader.

In this light, what the signal representation of (30a) achieves is to bake the scale of competition, \(1/N\), directly into the agent signal. This now provides the mathematical representation of signals with \(\Delta a\) in them with a solid economic justification. As I have proved above, in order to have aggregate noise in the large-economy limit, it suffices that the scale of the variance of individuals’ noises is of order \(\Delta a\). This is in fact the defining characteristic of Brownian motion. The economic takeaway is that in order to have aggregate price noise in a large market, the variance of the individuals’ noises must scale directly with the scale of competition.

### 4.2 Comparison in terms of the revelation property

Let me now answer the question of whether the signal structure of (30a) –or equivalently, (30b)– involves noise in the large limit. As I establish in the next theorem, the answer is no.
Theorem 7 Suppose that the traders’ signals are as in (30a) or (30b), and assume that

\[ \mathbb{E} \left[ \frac{1}{\tau_n} \right] = \lim_{N \to \infty} \frac{1}{N} \sum_{n=1}^{N} \frac{1}{\tau_n} < \infty, \]  

(34a)

and

\[ \lim_{N \to \infty} \sum_{n=1}^{N} \frac{1}{n^2 \tau_n} < \infty. \]  

(34b)

As \( N \to \infty \) the market-impact parameter \( \lambda \) converges to zero and the price converges to \( P = D \).

The assumptions in (34a) and (34b) allow for heterogeneity in the precisions of traders.\(^8\) These assumptions are sufficient to maintain some generality for the convergence results below, and they are fairly weak. For example, they are easy to satisfy for homogeneous signal precisions.\(^9\) In the limit two things happen. First, the impact of the liquidity traders, measured by \( \lambda \), vanishes. Second, the price \( P \) is identical to the liquidating value of the asset, \( D \). In other words, the price is fully revealing, as it contains no noise whatsoever. This result is consistent with Holden and Subrahmanyam (1992), who similarly show that as the number of strategic traders becomes large the market impact of noise traders vanishes and the price becomes fully revealing.\(^10\) Holden and Subrahmanyam (1992) derive this result for an economy of risk-neutral traders who know the liquidating value of the dividend with infinite precision. One may therefore wonder if the price becoming fully revealing in the large limit is an artifact of risk neutrality or of infinite precision. As the argument would go, risk aversion makes traders trade less aggressively, and therefore the aggregation of their signals through the order flow would be weaker than with risk neutrality. What is more, if the signal of each individual agent contained some noise, the aggregation of these noises would also show up in the price. As Theorem 7 shows, however, the result of Holden and Subrahmanyam (1992) extends, for a one-period economy, to risk-averse traders who have a signal with finite precision about the liquidating value of the dividend. This result obtains due to two reasons.

First, it is indeed true that risk aversion weakens the aggregation of traders’ information through

---

\(^8\)I use the expectation operator with a bar over it to denote cross-sectional moments of diversely distributed precisions, to distinguish it from the “regular” expectation operator which applies on the distribution of \( D \) and \( \epsilon_n \).

\(^9\)If the precision \( \tau_n \) is the same for every \( n \), \( \tau_n = \tau \), the left-hand side of (34a) equals \( 1/\tau \), and the left-hand side of (34b) equals

\[ \frac{1}{\tau} \sum_{n=1}^{\infty} \frac{1}{n^2} < \infty. \]  

(35)

The quantity in (35) is finite because the infinite sum equals the Riemann zeta function \( \zeta(s) \) evaluated at \( s = 2 \), which is a finite number.

\(^10\)See the “first auction approaching perfect competition” result of Proposition 3 in Holden and Subrahmanyam (1992)
order flow, because it reduces the demand coefficient of each trader. The demand coefficient of each trader \( n \) converges, up to an appropriate inverse order of \( N \), to a trader-specific coefficient, which I call \( b_n \). Risk aversion does reduce this trader-specific coefficient in a finite economy, but not strongly enough to matter asymptotically.

Second, the inverse relationship of demand with the number of traders allows a classical type of aggregation to operate on the signals in (30a) and (30b) via laws of large numbers. The composition of signal noises in the price component that is driven by aggregate order flow, \( \lambda \sum_{n=1}^{N} X_n \), converges to zero via Kolmogorov’s Law of Large Numbers.\(^{11}\) For this reason I refer to the signals in (30a) and (30b) as “Kolmogorov signals.” In particular, as I show in detail in Lemma A.7 of the Appendix, the market-impact parameter is of order \( 1/\sqrt{N} \), and the demand coefficient of each trader is also of order \( 1/\sqrt{N} \). The order-flow component of the price is therefore

\[
\lambda \sum_{n=1}^{N} X_n \approx \text{constant} \cdot \frac{1}{N} \sum_{n=1}^{N} b_n (D + \varepsilon_n). \tag{36}
\]

For example, if all the traders in the economy have identical signal precisions, the aggregate noise is proportional to

\[
\frac{1}{N} \sum_{i=n}^{N} \varepsilon_n, \tag{37}
\]

which converges to zero as \( N \) tends to infinity via the strong version of the Law of Large Numbers.

### 4.3 Comparison in terms of trade

Here I compare my model with Itô signals to the model with Kolmogorov signals in terms of their implications for trade. Motivated by that the impact of noise trading vanishes in the large limit in the Kolmogorov-signal economies, I explore what happens to measures of trading activity coming only from rational traders.

Because the model is static, it is not possible to measure “trading” in the traditional sense of the word without taking a stance on the ex-ante position of the agents. Since this is not typically explicitly modeled in static Kyle (1985)-type models, I adopt a cross-sectional measure of trade using the realized demands of the traders.

I define rational variation to be the cross-sectional sum of squared demands of rational traders,

\(^{11}\) The Kolmogorov version of the Law of Large Numbers allows for random variables that are independent, but not identically distributed. This feature allows us to have heterogeneity in signal precisions, as long as we conform to (34a) and (34b).
as in
\[ V_R = \sum_{i=1}^{N} X_i^2. \] (38)

I now explore differences between my model and the standard microstructure framework with respect to the above measure of trade. As we see below, the two models have different implications not only in terms of full and non-full revelation, but also in terms of rational variation in the large-economy limit.

**Proposition 8** For the economy with Itô signals,
\[ \lim_{N \to \infty} V_R = \int_0^1 \frac{\beta^2(a)}{\tau(a)} da. \] (39)

For the economy with Kolmogorov signals, if, in addition to Assumptions (34a) and (34b) we also have
\[ \lim_{N \to \infty} \sum_{n=1}^{N} \frac{1}{n^2 \tau_n^2} < \infty, \] (40a)
then
\[ \lim_{N \to \infty} V_R = \frac{\tau_D}{\tau_b \mathbb{E} \left[ \frac{\tau_n (\tau_D + \tau_n)}{(2\tau_D + \tau_n)^2} \right]} \left( \mathbb{E} \left[ \left( \frac{\tau_n}{2\tau_D + \tau_n} \right)^2 \right] D^2 + \mathbb{E} \left[ \frac{\tau_n}{(\tau_n + \tau_D)^2} \right] \right). \] (40b)

The two models are distinguishable empirically, as long as we have data that can be used to calculate rational variation and the number of rational traders. On the one hand, the model with Itô signals predicts that the volume \( V \) asymptotes to a constant, that we can immediately recognize as the variance of the aggregate price noise. On the other hand, the model with Kolmogorov signals predicts that the volume \( V \) asymptotes to a random variable with a very specific form as the number of traders becomes large, in that the volume reveals the square of the dividend.

### 4.4 Comparison in terms of welfare

The last comparison I make between the economy with Itô signals and the economy with Kolmogorov signals is in terms of implications for welfare. I define welfare \( W \) as the aggregate expected utility of the agents,
\[ W = \lim_{N \to \infty} \sum_{i=1}^{N} \mathbb{E} [u(\pi_n; \Delta z_{a_n})]. \] (41)

We have the following result.
Proposition 9  For the economy with Itô signals,

\[
\lim_{N \to \infty} W = \frac{\lambda}{2\rho} \int_0^1 \beta(a) da = \frac{\lambda}{2} \left( \int_0^1 \frac{\beta^2(a)}{\tau(a)} da + \frac{1}{\tau_{\theta}} \right).
\]  \hfill (42)

For the economy with Kolmogorov signals, under Assumptions (34a) and (34b),

\[
\lim_{N \to \infty} W = 0.
\]  \hfill (43)

5  Are there other possibilities with Itô signals?

As I have shown in the previous section, it is impossible to generate price noise in a large economy with Kolmogorov signals with independent noises. But is the signal structure of (1) the only structure that can generate price noise out of the aggregation of idiosyncratic noises? Put differently, does the non-revelation property of prices come from the particular structural form of (1), or from the Brownian structure of noise at the individual level? As it turns out, it is the latter that is responsible for non-revelation. In fact, the structure of (1) is one particular member of a larger class of models with non-fully revealing prices. I provide one more such example below, and I leave the development of further examples for future work.

5.1  An example with distributed information

Here I represent the dividend as the stochastic integral

\[
D = \sqrt{\tau_D^{-1}} \int_0^1 dB^D_s
\]  \hfill (44)

where \(B^D\) is a standard Brownian motion in the interval \([0, 1]\). As above, the dividend \(D\) has the distribution \(N(0, \tau_D^{-1})\), so the assumption in (44) is merely for modeling convenience. For simplicity I assume that all the agents in the economy are risk neutral. The signal of agent \(n\) is

\[
\Delta z_{an} = \sqrt{\tau_D^{-1}} \Delta B^D_{an} + \sqrt{\tau_{\theta}^{-1}} \Delta B^z_{an}.
\]  \hfill (45)

where \(B^z\) is a standard Brownian motion in the interval \([0, 1]\) independent of \(B^D\). The signal in (45) is informative about the dividend because \(D\) can be written as

\[
D = \sqrt{\tau_D^{-1}} \left( B^D_{an-1} - B^D_{a0} + \Delta B^D_{an} + B^D_{aN} - B^D_{an} \right),
\]  \hfill (46)
which says that trader $n$ observes a noisy version of the component $\Delta B_a^n$ of the dividend, but not about the components $B_{a_{n-1}}^D - B_{a_0}^D$ and $B_{a_N}^D - B_{a_n}^D$. Moreover, by independence of Brownian increments, it follows that the dividend information that trader $n$ receives is not shared with any other agents in the economy. I refer to this type of information structure as “distributed” information.

The large limit of this economy can be obtained in a manner similar to that in Section 3. I present an informal derivation of the main points below, using the infinitesimal version of the information structure directly, leaving the full-blown derivation of the limit for the interested reader.\(12\) Adapting Lemma A.1 to the signal structure of (45) we have that the limiting demand of trader $a$ is

$$X(a) = \frac{1}{2\lambda} \mathbb{E}\left[D \mid d\tilde{z}_a\right] = \frac{\tau(a)}{2\lambda(\tau(a) + \tau_D)} d\tilde{z}_a,$$

where the first equality follows from independence of Brownian increments and risk neutrality, and the second equality follows from the projection theorem for Normal random variables. The quantity $d\tilde{z}_a$ is the infinitesimal limit of (45), with mathematical representation

$$d\tilde{z}_a = \sqrt{\tau_D^{-1}} dB_a^D + \sqrt{[\tau(a)]^{-1}} dB_a^\ast.$$  

We immediately obtain that

$$\beta(a) = \frac{\tau(a)}{2\lambda[\tau(a) + \tau_D]}$$

for each trader $t$, and that the limiting aggregate demand from rational traders is

$$\int_0^1 \beta(a) d\tilde{z}_a.$$  

Using this expression we can develop an equation for $\lambda$, with solution

$$\lambda = \frac{1}{2} \sqrt{\frac{\tau_D}{\tau_D} \int_0^1 \frac{\tau(a)}{\tau(a) + \tau_D} da}.$$  

It now follows that the limiting price is

$$P = \frac{1}{2\sqrt{\tau_D}} \int_0^1 \frac{\tau(a)}{\tau(a) + \tau_D} dB_a^D + \frac{1}{2} \int_0^1 \sqrt{\tau(a)} dB^\ast_a + \lambda \theta.$$  

\(12\)The derivation I present here is admittedly a shortcut for the full derivation of the limit. It is, however, instructive for those readers that may be interested in the large-economy price without going through a detailed derivation of the limit.
6 Conclusion

I have presented a model of price noise that does not originate from pure noise trading or liquidity trading. The price noise instead arises as a combination of the noises in the signals of rational individuals in a large competitive economy à la Kyle (1985). Standard economic intuition dictates that such noises should, by the Law of Large Numbers, average away to zero. This intuition, however, rests on the particular way the existing literature models traders’ signals about the dividend. In an environment where the signal structure is a cross-sectional Itô process, it is possible to generate price noise that is a non-degenerate aggregation of individual noises. This aggregate price noise is well-defined, and it has the mathematical representation of a stochastic integral.

In addition, in a comparison to existing microstructure models, I show that the impact of liquidity trading on the price vanishes in a large economy, but only under extant assumptions about the structure of noise. In a standard framework, this would imply that the price would fully reveal the dividend to the market maker. In my model, however, due to the presence of aggregate noise, the price remains a noisy version of the dividend in a large economy.

I have modeled trader signals as a cross-sectional Itô process of the simplest form. It may be fruitful to pursue more general versions of the signal structure, as the class of Itô processes is admittedly much larger than what I use here. Moreover, the model is static, but its extension to an intertemporal framework may yield insightful implications. In addition, it may be interesting to extend the model to cover the case of correlated errors in the traders’ signals, as in the information structures employed by Foster and Viswanathan (1996), to give an example from microstructure, and Manzano and Vives (2011), to give an example from the rational expectations literature. Finally, the model I present here is within the tradition of the microstructure framework of Kyle (1985). It would also be interesting to explore whether we can provide micro-foundations of aggregate price noise in other frameworks. One notable such framework would be that with Bayes-Nash equilibria in demand curves, such as in the common-value model of Kyle (1989), the correlated-values model of Vives (2011), and the equi-commonal model of Rostek and Weretka (2012). I leave such questions for future work.

References


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A Appendix

Lemma A.1 For each finite $N$, the optimal demand is

$$X_n = \frac{\mathbb{E}\left[D - P_{-n} \mid \Delta z_{an}\right]}{2\lambda + \delta_{an-1} \text{Var}\left(D - P_{-n} \mid \Delta z_{an}\right)},$$

and the second-order condition of trader $n$ is satisfied if and only if

$$\delta_{an-1} + \frac{2\lambda}{\text{Var}\left(D - P_{-n} \mid \Delta z_{an}\right)} > 0.$$  \(53b\)

Moreover, the utility of the optimal profit is

$$u(\pi_n; \Delta z_{an}) = \frac{1}{2} X_n \mathbb{E}\left[D - P_{-n} \mid \Delta z_{an}\right],$$

and the market-impact parameter is

$$\lambda = \frac{\text{Cov}\left(D, \sum_{i=1}^N X_i\right)}{\text{Var}\left(\sum_{i=1}^N X_i + \theta\right)} = \frac{\text{Cov}\left(D, \sum_{i=1}^N X_i\right)}{\text{Var}\left(\sum_{i=1}^N X_i\right) + \tau^{-1}}.$$  \(53d\)

Proof. Due to assumptions (4) and (5) I can write the utility function as

$$u(\pi_n; \Delta z_{an}) = X_n \left\{ \mathbb{E}\left[D - P_{-n} \mid \Delta z_{an}\right] - \lambda X_n \right\} - \frac{\delta_{an-1}}{2} X_n^2 \text{Var}\left(D - P_{-n} \mid \Delta z_{an}\right).$$

The first-order condition with respect to demand proves (53a). The second-order condition with respect to demand is

$$- \left[ 2\lambda + \delta_{an-1} \text{Var}\left(D - P_{-n} \mid \Delta z_{an}\right) \right].$$

Because $\text{Var}\left(D - P_{-n} \mid \Delta z_{an}\right) > 0$, the second-order condition is negative if and only if (53b) holds. Combining (53a) with (54) proves (53c). Finally, from (3) we get

$$P = \mathbb{E}\left[D \mid \sum_{i=1}^N X_i + \theta\right] = \frac{\text{Cov}\left(D, \sum_{i=1}^N X_i + \theta\right)}{\text{Var}\left(\sum_{i=1}^N X_i + \theta\right)} \left(\sum_{i=1}^N X_i + \theta\right)$$

$$= \frac{\text{Cov}\left(D, \sum_{i=1}^N X_i\right)}{\text{Var}\left(\sum_{i=1}^N X_i + \theta\right)} \left(\sum_{i=1}^N X_i + \theta\right),$$ \(56\)
where the third equality follows because $\theta$ is independent of $D$. Matching the coefficient of aggregate demand in (56) with that in conjecture (5) we obtain

$$
\lambda = \frac{\text{Cov}(D, \sum_{i=1}^{N} X_i)}{\text{Var}(\sum_{i=1}^{N} X_i + \theta)} = \frac{\text{Cov}(D, \sum_{i=1}^{N} X_i)}{\text{Var}(\sum_{i=1}^{N} X_i) + \tau_{\theta}^{-1}},
$$

(57)

where the second equality is because $\theta$ is independent of agent demands. ■

**Proposition A.2** For the economy of Section 2 and for each finite $N$, (4) and (5) hold with

$$
\lambda = \frac{1}{\tau_D} \sum_{i=1}^{N} \beta_{a_{i-1}} \Delta a
$$

and

$$
\beta_{a_{n-1}} = \frac{\tau_{a_{n-1}}}{\delta_{a_{n-1}} + \frac{2\lambda \tau_D}{1 - \lambda \sum_{i=1}^{N} \beta_{a_{i-1}} \Delta a}} + A(\lambda, \tau_{a_{n-1}}, \beta_{t_0}, \ldots, \beta_{a_{n-1}}) \lambda \Delta a
$$

(58a)

where

$$
A(\lambda, \tau_{a_{n-1}}, \beta_{t_0}, \ldots, \beta_{a_{n-1}}) = \delta_{a_{n-1}} \left(2\beta_{a_{n-1}} + \frac{\tau_{a_{n-1}}}{\tau_D} \sum_{i=1}^{N} \beta_{a_{i-1}} \Delta a \right) + \frac{\tau_{a_{n-1}} - \delta_{a_{n-1}} \lambda \beta_{a_{n-1}}^2 \tau_{a_{n-1}}}{1 - \lambda \sum_{i=1}^{N} \beta_{a_{i-1}} \Delta a}
$$

(58b)

(58c)

**Proof.**

By Lemma A.1 and due to that

$$
\mathbb{E} \left[ \theta \Big| \Delta z_{a_n} \right] = 0
$$

(59)

because $\theta$ is independent of $\Delta z_{a_n}$, and that

$$
\mathbb{E} \left[ X_i \big| \Delta z_{a_n} \right] = \mathbb{E} \left[ D \big| \Delta z_{a_n} \right] \beta_{a_{i-1}} \Delta a
$$

(60)

because $\Delta z_{a_n}$ is independent of $\Delta B_{t_i}$ for $i \neq n$, it follows that

$$
X_n = \frac{1 - \lambda \sum_{i=1}^{N} \beta_{a_{i-1}} \Delta a}{2\lambda + \delta_{a_{n-1}}} \mathbb{E} \left[ D \big| \Delta z_{a_n} \right] + \mathbb{E} \left[ D \big| \Delta z_{a_n} \right] \beta_{a_{i-1}} \Delta a
$$

(61)

The projection theorem implies

$$
\mathbb{E} \left[ D \big| \Delta z_{a_n} \right] = \frac{\text{Cov}(D, \Delta z_{a_n})}{\text{Var}(\Delta z_{a_n})} \Delta z_{a_n} = \frac{\tau_{a_{n-1}}}{\tau_{a_{n-1}} \Delta a + \tau_D} \Delta z_{a_n},
$$

(62)
Using Equation (67) we can rewrite the term multiplying $\delta$, which establishes (58a). From this equation we also get

$$a_1 \beta_{b-1} \Delta a = \frac{1 - \lambda \sum_{i=1}^{N} \beta_{a_{i-1}} \Delta a}{2\lambda + \delta_{a_{n-1}} \left[ \left(1 - \lambda \sum_{i=1}^{N} \beta_{a_{i-1}} \Delta a \right)^2 + \lambda^2 \left( \sum_{i=1}^{N} \frac{\beta_{a_{i-1}}^2}{\tau_{a_{i-1}}} \Delta a + \frac{1}{\tau_\theta} \right) \right]}$$

and rearranging yields

$$\beta_{a_{n-1}} = \frac{\tau_{a_{n-1}} \left( 1 - \lambda \sum_{i=1}^{N} \beta_{a_{i-1}} \Delta a \right)}{\lambda \left( \tau_{a_{n-1}} \Delta a + 2\tau_D \right) + \delta_{a_{n-1}} \left[ \left(1 - \lambda \sum_{i=1}^{N} \beta_{a_{i-1}} \Delta a \right)^2 + \lambda^2 \left( \sum_{i=1}^{N} \frac{\beta_{a_{i-1}}^2}{\tau_{a_{i-1}}} \Delta a + \frac{1}{\tau_\theta} \right) \right]}$$

Substituting (4) into (57) gives

$$\lambda = \frac{\text{Cov} \left( D, \sum_{i=1}^{N} X_i \right)}{\text{Var} \left( \sum_{i=1}^{N} X_j + \theta \right)} = \frac{\frac{1}{\tau_D} \sum_{i=1}^{N} \beta_{a_{i-1}} \Delta a}{\frac{1}{\tau_D} \left( \sum_{i=1}^{N} \beta_{a_{i-1}} \Delta a \right)^2 + \sum_{i=1}^{N} \frac{\beta_{a_{i-1}}^2}{\tau_{a_{i-1}}} \Delta a + \frac{1}{\tau_\theta}}$$

which establishes (58a). From this equation we also get

$$\lambda^2 \left( \sum_{i=1}^{N} \frac{\beta_{a_{i-1}}^2}{\tau_{a_{i-1}}} \Delta a + \frac{1}{\tau_\theta} \right) = \lambda^2 \frac{1}{\tau_D} \sum_{i=1}^{N} \beta_{a_{i-1}} \Delta a - \lambda^2 \frac{1}{\tau_D} \left( \sum_{i=1}^{N} \beta_{a_{i-1}} \Delta a \right)^2$$

Using Equation (67) we can rewrite the term multiplying $\delta_{a_{n-1}}$ in the denominator of (65) as

$$\left(1 - \lambda \sum_{i=1}^{N} \beta_{a_{i-1}} \Delta a + \lambda \beta_{a_{n-1}} \Delta a \right)^2 + \lambda^2 \left( -\frac{\beta_{a_{n-1}} \Delta a}{\tau_{a_{n-1}}} + \sum_{i=1}^{N} \frac{\beta_{a_{i-1}}^2}{\tau_{a_{i-1}}} \Delta a + \frac{1}{\tau_\theta} \right) \left( \tau_{a_{n-1}} \Delta a + \tau_D \right)$$

$$= \left(1 - \lambda \sum_{i=1}^{N} \beta_{a_{i-1}} \Delta a \right)^2 + 2 \left(1 - \lambda \sum_{i=1}^{N} \beta_{a_{i-1}} \Delta a \right) \left( \lambda \beta_{a_{n-1}} \Delta a \right) + \left( \lambda \beta_{a_{n-1}} \Delta a \right)^2$$
\[-\lambda^2 \frac{\beta^2 a_{n-1}}{\tau a_{n-1}} \Delta a \big( \tau a_{n-1} \Delta a + \tau D \big) + \lambda^2 \left( \sum_{i=1}^{N} \frac{\beta^2 a_{i-1}}{\tau a_{i-1}} \Delta a + \frac{1}{\tau \theta} \right) \left( \tau a_{n-1} \Delta a + \tau D \right) \]

\[= 1 - \lambda \sum_{i=1}^{N} \beta a_{i-1} \Delta a \]

\[+ \left[ \lambda \left( 1 - \lambda \sum_{i=1}^{N} \beta a_{i-1} \Delta a \right) \left( 2\beta a_{n-1} + \frac{\tau a_{n-1}}{\tau D} \sum_{i=1}^{N} \beta a_{i-1} \Delta a \right) - \lambda^2 \beta^2 a_{n-1} \frac{\tau D}{\tau a_{n-1}} \right] \Delta a, \quad (68)\]

and therefore (65) becomes

\[\beta a_{n-1} = \tau a_{n-1} \left( 1 - \lambda \sum_{i=1}^{N} \beta a_{i-1} \Delta a \right) \]

\[\left\{ 2\lambda \tau D + \delta a_{n-1} \left( 1 - \lambda \sum_{i=1}^{N} \beta a_{i-1} \Delta a \right) \right\} \]

\[+ \left[ \lambda \tau a_{n-1} + \delta a_{n-1} \lambda \left( 1 - \lambda \sum_{i=1}^{N} \beta a_{i-1} \Delta a \right) \left( 2\beta a_{n-1} + \frac{\tau a_{n-1}}{\tau D} \sum_{i=1}^{N} \beta a_{i-1} \Delta a \right) - \delta a_{n-1} \lambda^2 \beta^2 a_{n-1} \frac{\tau D}{\tau a_{n-1}} \right] \Delta a \], \quad (69)

This proves (58b) and (58c).

**Proof of Theorem 1.** By inspection of (58a) of Proposition A.2, as \( N \to \infty \) we obtain

\[\lambda = \frac{\int_{0}^{1} \beta(s) ds}{\left( \int_{0}^{1} \beta(s) ds \right)^2 + \tau D \left( \int_{0}^{1} \frac{\beta^2(s)}{\tau(s)} ds + \frac{1}{\tau \theta} \right)}, \quad (70)\]

which proves (10a). Taking \( N \to \infty \) in (58b) of Proposition A.2 we also obtain

\[\beta(a) = \frac{\tau(a)}{\delta(a) + 2 \frac{\lambda \tau D}{1 - \lambda \int_{0}^{1} \beta(s) ds}}, \quad (71)\]

which proves (10b). Equation (10c) now follows by standard results in stochastic calculus, after substituting (4) into (5) and taking the limit.

**Lemma A.3**

\[\rho = \frac{\int_{0}^{1} \beta(s) ds}{\int_{0}^{1} \frac{\beta^2(s)}{\tau(s)} ds + \frac{1}{\tau \theta}}, \quad (72a)\]
which implies that in equilibrium $\rho$ and $\int_0^1 \beta(s)ds$ have the same sign. In addition,

$$\frac{\tau_D}{1 - \lambda \int_0^1 \beta(s)ds} = \frac{\rho}{\lambda} = \rho \int_0^1 \beta(s)ds + \tau_D.$$  \hspace{1cm} (72b)

**Proof.** Equation (10a) implies that

$$1 - \lambda \int_0^1 \beta(s)ds = \frac{\tau_D \left( \int_0^1 \frac{\beta^2(s)}{\tau(s)} ds + \frac{1}{\tau_0} \right)}{\left( \int_0^1 \beta(s)ds \right)^2 + \tau_D \left( \int_0^1 \frac{\beta^2(s)}{\tau(s)} ds + \frac{1}{\tau_0} \right)}. \hspace{1cm} (73)$$

Dividing (10a) by (73) establishes (72a). It now follows that $\rho$ and $\int_0^1 \beta(s)ds$ have the same sign, because the denominator on the right-hand side of (72a) is always positive. Equation (72b) follows from the definition of $\rho$ in (12). \Box

**Proof of Proposition 2.** From (72a) we get

$$\rho \int_0^1 \frac{\beta^2(a)}{\tau(a)} da + \frac{1}{\tau_0} = \int_0^1 \beta(a) da. \hspace{1cm} (74)$$

By (10b) of Theorem 1 and definition (12), the integral term on the right-hand side of (74) is

$$\int_0^1 \beta(a) da = \int_0^1 \frac{\tau(a)}{\delta(a) + 2\rho} da, \hspace{1cm} (75)$$

which proves (14b). Similarly, the integral term on the left-hand side of (74) is

$$\int_0^1 \frac{\beta^2(a)}{\tau(a)} da = \int_0^1 \frac{\tau(a)}{[\delta(a) + 2\rho]^2} da. \hspace{1cm} (76)$$

Substituting (75) and (76) into (74) proves (14a). The restriction that $\rho$ and $\int_0^1 \beta(s)ds$ have the same sign follows from Lemma A.3. \Box

**Corollary A.4** For the economy of Section 2, the ex-ante expectation of the utility of agent $n$ is

$$\mathbb{E} \left[ u(\pi_n; \Delta z_{a_n}) \right] = \frac{\beta_{a_{n-1}}}{2\tau_D} \left( 1 - \lambda \sum_{i=1}^{N} \beta_{a_{i-1}} \Delta a \right) \Delta a. \hspace{1cm} (77)$$
Proof. By Lemma A.1, Equation (59), and Equation (60) for $i \neq n$, we obtain that the utility is

$$u(\pi_n; \Delta z_{a_n}) = \frac{1}{2} \left( 1 - \lambda \sum_{i=1}^{N} \beta_{a_i-1} \Delta a \right) X_n \mathbb{E} \left[ D \mid \Delta z_{a_n} \right].$$

(78)

This together with (4) implies that the ex-ante expectation of the utility is

$$\mathbb{E}[u(\pi_n; \Delta z_{a_n})] = \frac{1}{2} \left( 1 - \lambda \sum_{i=1, i \neq n}^{N} \beta_{a_i-1} \Delta a \right) \beta_{a_n-1} \mathbb{E} \left[ \Delta z_{a_n} \mathbb{E} \left[ D \mid \Delta z_{a_n} \right] \right]$$

$$= \frac{1}{2} \left( 1 - \lambda \sum_{i=1, i \neq n}^{N} \beta_{a_i-1} \Delta a \right) \beta_{a_n-1} \frac{\tau_{a_n-1}}{\tau_{a_n-1} \Delta a + \tau_D} \mathbb{E} \left[ \Delta z_{a_n}^2 \right] = \beta_{a_n-1} \left( 1 - \lambda \sum_{i=1, i \neq n}^{N} \beta_{a_i-1} \Delta a \right) \Delta a. \quad (79)$$

**Proposition A.5** The first-order condition of the information-acquisition decision of agent $n$ is

$$\tau_{a_n-1} = \psi_{a_n-1} \frac{\tau_{a_n-1}}{\tau_D} \delta_{a_n-1} + \frac{2 \lambda \tau_D}{1 - \lambda \sum_{i=1}^{N} \beta_{a_i-1} \Delta a} + A \left( \lambda, \tau_{a_n-1}, \beta_{a_0}, \ldots, \beta_{a_n-1} \right) \lambda \Delta a + O(\Delta a), \quad (80)$$

where $O(\cdot)$ is the “big O” notation.

**Proof.** By corollary A.4, the first-order condition for $\tau_{a_n-1}$ is

$$\frac{\tau_D}{\psi_{a_n-1}} \tau_{a_n-1} = \frac{d}{d\tau_{a_n-1}} \left[ \beta_{a_n-1} \left( 1 - \lambda \sum_{i=1, i \neq n}^{N} \beta_{a_i-1} \Delta a \right) \right]. \quad (81)$$

By the product rule we obtain

$$\frac{\tau_D}{\psi_{a_n-1}} \tau_{a_n-1} = \left( \frac{d}{d\tau_{a_n-1}} \beta_{a_n-1} \right) \left( 1 - \lambda \sum_{i=1, i \neq n}^{N} \beta_{a_i-1} \Delta a \right) + \beta_{a_n-1} \frac{d}{d\tau_{a_n-1}} \left( 1 - \lambda \sum_{i=1, i \neq n}^{N} \beta_{a_i-1} \Delta a \right). \quad (82)$$

By Proposition A.2, the contribution to $d\beta_{a_n-1}/d\tau_{a_n-1}$ from the denominator of (58b) is of order $\Delta a$. For the same reason, the derivative $d \left( 1 - \lambda \sum_{i=1, i \neq n}^{N} \beta_{a_i-1} \Delta a \right) /d\tau_{a_n-1}$ is also of order $\Delta a$. We
therefore have
\[
\frac{\tau_D}{\psi_{a_{n-1}}} = \frac{1 - \lambda \sum_{i=1}^{N} \beta_{a_{i-1}} \Delta a}{\delta_{a_{n-1}} + \frac{2 \lambda \tau_D}{1 - \lambda \sum_{i=1}^{N} \beta_{a_{i-1}} \Delta a}} + A\left(\lambda, \tau_{a_{n-1}}, \beta_{t_0}, \ldots, \beta_{a_{n-1}}\right) \lambda \Delta a + O(\Delta a), \quad (83)
\]
which proves (80). ■

**Proof of Proposition 3.** Taking the limit as \(N \to \infty\) in (80) of Proposition A.5 proves (21a). Equation (21b) follows from substituting (21a) into (10b). By corollary A.4 and Equation (18), the surplus is
\[
S = \lim_{N \to \infty} \sum_{i=1}^{N} \{ \mathbb{E}[u(\pi_n; \Delta z_{a_n})] - c_{a_{n-1}}(\tau_{a_{n-1}}) \}
\]
\[
= \lim_{N \to \infty} \sum_{i=1}^{N} \left[ \frac{\beta_{a_{n-1}}}{2 \tau_D} \left( 1 - \lambda \sum_{i=1}^{N} \beta_{a_{i-1}} \Delta a \right) - \frac{\tau_{a_{n-1}}^2}{4 \psi_{a_{n-1}}} \right] \Delta a
\]
\[
= \int_{0}^{1} \left[ \frac{\beta(a)}{2 \tau_D} \left( 1 - \lambda \int_{0}^{1} \beta(s) ds \right) - \frac{\tau^2(a)}{4 \psi(a)} \right] da
\]
\[
= \int_{0}^{1} \left[ \frac{\tau^2(a)}{2 \psi(a)} - \frac{\tau^2(a)}{4 \psi(a)} \right] da. \quad (84)
\]
where the last equality follows from (21a) and (21b). This proves Equation (21c). ■

**Proof of Proposition 4.** From (72a) we get
\[
\rho \int_{0}^{1} \frac{\beta^2(a)}{\tau(a)} da + \rho \frac{1}{\tau_\theta} = \int_{0}^{1} \beta(a) da. \quad (85)
\]
By Proposition 3 and (72b) the integral term on the right-hand side of (85) is
\[
\int_{0}^{1} \beta(a) da = \frac{\lambda}{\rho} \int_{0}^{1} \frac{\psi(a)}{[\delta(a) + 2\rho]^2} da, \quad (86)
\]
and the integral term on the left-hand side of (85) is
\[
\int_{0}^{1} \frac{\beta^2(a)}{\tau(a)} da = \frac{\lambda}{\rho} \int_{0}^{1} \frac{\psi(a)}{[\delta(a) + 2\rho]^3} da. \quad (87)
\]
Substituting (86) and (87) into (85), multiplying through by \(\rho/\lambda\), and using (72b) gives
\[
\rho \int_{0}^{1} \frac{\psi(a)}{[\delta(a) + 2\rho]^3} da + \rho \frac{1}{\tau_\theta} \left( \rho \int_{0}^{1} \beta(s) ds + \tau_D \right) = \int_{0}^{1} \frac{\psi(a)}{[\delta(a) + 2\rho]^2} da. \quad (88)
\]
From (86) and (72b) we get

\[
\left( \rho \int_0^1 \beta(s)ds + \tau_D \right) \int_0^1 \beta(a) da = \int_0^1 \frac{\psi(a)}{[\delta(a) + 2\rho]^2} da,
\]

which proves (22b). Equation (89) further implies that

\[
\left( \rho \int_0^1 \beta(s)ds + \tau_D \right) = \frac{1}{\int_0^1 \beta(a) da} \int_0^1 \frac{\psi(a)}{[\delta(a) + 2\rho]^2} da,
\]

and using this in (88) proves (22a). The restriction that \( \rho \) and \( \int_0^1 \beta(s)ds \) have the same sign follows from Lemma A.3.

Proof of Proposition 5. Let \( F(\rho) \) denote the left-hand side of (29a), that is,

\[
F(\rho) = f_B (\delta_B + \rho) (\delta_G + 2\rho)^3 + f_G (\delta_G + \rho) (\delta_B + 2\rho)^3.
\]

We have

\[
F(0) = \delta_B \delta_G (f_B \delta_G^2 + f_G \delta_B^2) < 0,
\]

because \( \delta_B < 0 \), and

\[
F(-\delta_B) = f_G \delta_B^3 (\delta_B - \delta_G) > 0,
\]

because \( \delta_B < 0 \) and \( \delta_G > 0 \). By continuity of \( F \), (92) and (93) prove that there exists a \( \rho_* \), with \( 0 < \rho_* < -\delta_B \), that is a root of \( F \). Next, from the equilibrium condition (29a) we have that \( \rho_* \) satisfies

\[
(\delta_B + 2\rho_*)^3 = \frac{-f_B (\delta_B + \rho_*)}{f_G (\delta_G + \rho_*)} (\delta_G + 2\rho_*)^3 > 0,
\]

where the inequality follows from that \( \rho_* < -\delta_B \). Because the cube preserves the sign, we obtain that \( \delta_B + 2\rho_* > 0 \).

Finally, observe that

\[
\lim_{N \to \infty} \text{Var} \left( D - P_{-n} \mid \Delta z_{a_n} \right) = \frac{1}{\tau_D} \left( 1 - \lambda \int_0^1 \beta(s)ds \right)^2 + \lambda^2 \left( \int_0^1 \frac{\beta^2(s)}{\tau(s)} ds + \frac{1}{\tau \theta} \right)
\]

\[
= \left( \frac{\lambda \tau_D}{\rho} \right)^2 \frac{1}{\tau_D} + \lambda^2 \frac{\int_0^1 \beta(s)ds}{\rho} = \frac{\lambda}{\rho},
\]

where the second equality follows by definition of \( \rho \) in (12) and (72a) of Lemma A.3. Equation (95) shows that

\[
\rho = \lim_{N \to \infty} \frac{\lambda}{\text{Var} \left( D - P_{-n} \mid \Delta z_{a_n} \right)}.
\]
By Lemma A.1 and because, as established above, \( \delta_B + 2\rho > 0 \), we obtain that the second-order condition of the risk-seeking agents is satisfied at the root \( \rho_\star \).

**Proof of Lemma 6.** The first-order condition of the agents gives that for the representation in (30a) the demand coefficient is

\[
\tilde{\beta}_{a_{n-1}} = \frac{\text{Cov} \left( D - \lambda \sum_{i \neq n}^N X_i, \Delta \tilde{z}_{a_{n}} \right)}{\text{Var} (\Delta \tilde{z}_{a_{n}}) \left( 2\lambda + \delta \left[ \text{Var} \left( D - \lambda \sum_{i \neq n}^N X_i | \Delta \tilde{z}_{a_{n}} \right) + \lambda^2 \tau_\theta^{-1} \right] \right)}
\]

\[
= \frac{\tau_n}{\tau_D + \tau_n} \left( 1 - \lambda \sum_{i = 1 \neq n}^N \tilde{\beta}_{a_{i-1}} \Delta a \right)
\]

\[
\Delta a \left( 2\lambda + \delta \left[ \frac{1}{\tau_D + \tau_n} \left( 1 - \lambda \sum_{i = 1 \neq n}^N \tilde{\beta}_{a_{i-1}} \Delta a \right)^2 + \lambda^2 \left( \sum_{i = 1 \neq n}^N \frac{\left( \tilde{\beta}_{a_{i-1}} \Delta a \right)^2}{\tau_{a_{i-1}}} + \frac{1}{\tau_\theta} \right) \right] \right),
\]

(97a)

whereas for the representation in (30b) the demand coefficient is

\[
\hat{\beta}_n = \frac{\text{Cov} \left( D - \lambda \sum_{i \neq n}^N X_i, s_n \right)}{\text{Var} (s_n) \left( 2\lambda + \delta \left[ \text{Var} \left( D - \lambda \sum_{i \neq n}^N X_i | s_n \right) + \lambda^2 \tau_\theta^{-1} \right] \right)}
\]

\[
= \frac{\tau_n}{\tau_D + \tau_n} \left( 1 - \lambda \sum_{i = 1 \neq n}^N \tilde{\beta}_i \right)
\]

\[
\left( 2\lambda + \delta \left[ \frac{1}{\tau_D + \tau_n} \left( 1 - \lambda \sum_{i = 1 \neq n}^N \tilde{\beta}_i \right)^2 + \lambda^2 \left( \sum_{i = 1 \neq n}^N \frac{\left( \tilde{\beta}_i \right)^2}{\tau_{a_{i-1}}} + \frac{1}{\tau_\theta} \right) \right] \right).
\]

(97b)

Comparing (97a) with (97b) proves that \( \hat{\beta}_n = \tilde{\beta}_{a_{n-1}} \Delta a \), as long as the expression for \( \lambda \) is the same for the two signal representations if we replace \( \hat{\beta}_n \) with \( \tilde{\beta}_{a_{n-1}} \Delta a \). Recall now that

\[
\lambda = \frac{\text{Cov} \left( D, \sum_{i = 1}^N X_i \right)}{\text{Var} \left( \sum_{i = 1}^N X_i \right) + \tau_\theta^{-1}}.
\]

(98)

By inspection of (98) and the demand representations (31a) and (31b) it follows that \( \hat{\beta}_n = \tilde{\beta}_{a_{n-1}} \Delta a \). Because \( \lambda \) is identical for the two signal representations it now also follows that the price is as in (5) for both signal representations.

**Lemma A.6** For the economy of Section 4.1, for either of the signals in (30a) and (30b), the
The market impact of liquidity trading is given by

\[
\lambda = \frac{1}{\tau D} \frac{\sum_{i=1}^{N} \tilde{\beta}_{a_{i-1}} \Delta a}{\left( \sum_{i=1}^{N} \tilde{\beta}_{a_{i-1}} \Delta a \right)^2 + \sum_{i=1}^{N} \frac{\left( \tilde{\beta}_{a_{i-1}} \Delta a \right)^2}{\tau_i} + \frac{1}{\tau_a}},
\]

(99a)

and the demand coefficients are

\[
\tilde{\beta}_{a_{n-1}} \Delta a = \frac{\tau_n}{\delta + \frac{\lambda \tau_D}{1 - \lambda \sum_{i=1}^{N} \tilde{\beta}_{a_{i-1}} \Delta a} \left[ 2 + \frac{\tau_n}{\tau_D} - \frac{\delta \lambda \left( \tilde{\beta}_{a_{n-1}} \Delta a \right)^2}{\tau_n} \right] + \frac{\lambda \tilde{A} \left( \lambda, \tau_n, \tilde{\beta}_{t_0}, \ldots, \tilde{\beta}_{a_{n-1}} \right)}{\tau_n \sum_{i=1}^{N} \tilde{\beta}_{a_{i-1}} \Delta a}},
\]

(99b)

where

\[
\tilde{A} \left( \lambda, \tau_n, \tilde{\beta}_{t_0}, \ldots, \tilde{\beta}_{a_{n-1}} \right) = \delta \left( 2 \tilde{\beta}_{a_{n-1}} \Delta a + \frac{\tau_n}{\tau_D} \sum_{i=1}^{N} \tilde{\beta}_{a_{i-1}} \Delta a \right).
\]

(99c)

**Proof.** One can prove this lemma by repeating the argument in the proof of Proposition A.2, adapting it for the signals in (30a) and (30b). A shorter proof, however, uses Proposition A.2 together with a standard property of Brownian Motion. Comparing the signal in (30a) with the signal in (1), and because \[\Delta B_{a_n} \sim \mathcal{N}(0, \Delta a),\] we can represent the signal in (30a) in the notation of Proposition A.2 if we set

\[\tau_{a_{n-1}} = \frac{\tau_n}{\Delta a}.
\]

(100)

Doing so proves Equation (99a) immediately. From Proposition A.2 and Equation (100) we obtain

\[
\tilde{\beta}_{a_{n-1}} = \frac{1}{\Delta a} \frac{\tau_n}{\delta + \frac{\lambda \tau_D}{1 - \lambda \sum_{i=1}^{N} \tilde{\beta}_{a_{i-1}} \Delta a} \left[ 2 + \frac{\tau_n}{\tau_D} - \frac{\delta \lambda \left( \tilde{\beta}_{a_{n-1}} \Delta a \right)^2}{\tau_n} \right] + \frac{\lambda \tilde{A} \left( \lambda, \tau_n, \tilde{\beta}_{t_0}, \ldots, \tilde{\beta}_{a_{n-1}} \right)}{\tau_n \sum_{i=1}^{N} \tilde{\beta}_{a_{i-1}} \Delta a}},
\]

(101)

where

\[
\tilde{A} \left( \lambda, \tau_n, \tilde{\beta}_{t_0}, \ldots, \tilde{\beta}_{a_{n-1}} \right) = \delta \left( 2 \tilde{\beta}_{a_{n-1}} \Delta a + \frac{\tau_n}{\tau_D} \sum_{i=1}^{N} \tilde{\beta}_{a_{i-1}} \Delta a \right).
\]

(102)

This establishes (99b) and (99c). }

**Lemma A.7** For the economy of Section 4.1, under Assumptions (34a) and (34b), and with \(o(\cdot)\) being the “little o” notation,

\[
\tilde{\beta}_{a_{n-1}} \Delta a = \frac{b_n}{\sqrt{N}} + o \left( \frac{1}{\sqrt{N}} \right)
\]

(103a)

and

\[
\lambda = \frac{1}{L \sqrt{N}} + o \left( \frac{1}{\sqrt{N}} \right),
\]

(103b)

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where
\[
b_n = \frac{\tau_n}{2\tau_D + \tau_n^2} \sqrt{\frac{\tau_D}{\tau_0 \mathbb{E} \left[ \frac{\tau (\tau_D + \tau)}{(2\tau_D + \tau)^2} \right]}} for some constant \( b_n \) for each \( n \). This is equivalent to making the conjecture
\[
\tilde{\beta}_{a_{n-1}} \Delta a = \frac{b_n}{\sqrt{N}} + o \left( \frac{1}{\sqrt{N}} \right)
\]
for some constant \( b_n \) for each \( n = 1, \ldots, N \), and each \( N \). Let
\[
\bar{b}_N = \frac{1}{N} \sum_{i=1}^{N} b_i,
\]
\[
\bar{b}^2 \tau^{-1}_N = \frac{1}{N} \sum_{i=1}^{N} \frac{b_i^2}{\tau_i},
\]
\[
\bar{b} \tau^{-1}_N = \frac{1}{N} \sum_{i=1}^{N} \frac{b_i}{\tau_i},
\]
and
\[
L = \mathbb{E} \left[ \frac{\tau}{2\tau_D + \tau} \right] \sqrt{\frac{\tau_D}{\tau_0 \mathbb{E} \left[ \frac{\tau (\tau_D + \tau)}{(2\tau_D + \tau)^2} \right]}}.
\]

Moreover,
\[
\mathbb{E} [b] = \lim_{N \to \infty} \frac{1}{N} \sum_{n=1}^{N} b_n < \infty,
\]
\[
\mathbb{E} [b^2] = \lim_{N \to \infty} \frac{1}{N} \sum_{n=1}^{N} b_n^2 < \infty,
\]
\[
\lim_{N \to \infty} \sum_{n=1}^{N} \frac{b_n^2}{n^2 \tau_n} < \infty,
\]
and
\[
\lim_{N \to \infty} \sum_{n=1}^{N} \frac{b_n^4}{n^2 \tau_n^2} < \infty.
\]

**Proof.** The proof follows the conjecture-and-verify method. First, conjecture that
\[
\lim_{N \to \infty} \sqrt{N} \tilde{\beta}_{a_{n-1}} \Delta a = b_n
\]
for some constant \( b_n \) for each \( n \).
and
\[ \tau^{-1} N = \frac{1}{N} \sum_{i=1}^{N} \frac{1}{\tau_i} \quad (107d). \]

Further conjecture that the quantities in (107a), (107b), and (107c) have finite limits as \( N \to \infty \). I denote these limits respectively as \( \bar{E}[b] \), \( \bar{E}[b^2/\tau] \), and \( \bar{E}[b/\tau] \). The quantity in (107d) has a finite limit as \( N \to \infty \) by assumption (34a).

Next, by Lemma (A.6) we get that
\[ \sqrt{N} \bar{b}_N + o(1) = \frac{\bar{b}_N}{N (\bar{b}_N)^2 + o(N) + \tau_D (b^2 \tau^{-1} N + \tau_D o(1)) 2b \tau^{-1} N + \tau^{-1} N} + \frac{\tau D}{\tau_0}, \quad (108) \]
from which we obtain
\[ \sqrt{N} \lambda = \frac{\bar{b}_N + o(1)}{(\bar{b}_N)^2 + o(1) + \frac{1}{N} \tau_D (b^2 \tau^{-1} N + \frac{1}{\tau_0}) + \tau_D o(1) [2b \tau^{-1} N + \tau^{-1} N]} \quad (109) \]

Because \( \bar{b}_N, b^2 \tau^{-1} N, \bar{b} \tau^{-1} N \) and \( \tau^{-1} N \) have finite limits in \( N \), Equation (109) implies that
\[ \lim_{N \to \infty} \sqrt{N} \lambda = \frac{1}{\bar{E}[b]}, \quad (110) \]
which we can write as
\[ \lambda = \frac{1}{\bar{E}[b] \sqrt{N}} + o \left( \frac{1}{\sqrt{N}} \right). \quad (111) \]
This proves that (103b) holds with \( L = \bar{E}[b] \) as long as we can prove (103a).

Lemma A.6 implies
\[ \bar{\beta}_{a_{n-1}} \Delta a = \frac{\lambda \tau_D}{1 - \lambda \sum_{i=1}^{N} \bar{\beta}_{a_{i-1}} \Delta a} \left[ 2 + \frac{\tau_n}{\tau_D} - \delta \lambda \left( \frac{\bar{\beta}_{a_{n-1}} \Delta a}{\tau_n} \right)^2 \right] + \delta \left[ 1 + \lambda \left( 2 \bar{\beta}_{a_{n-1}} \Delta a + \frac{\tau_n}{\tau_D} \sum_{i=1}^{N} \bar{\beta}_{a_{i-1}} \Delta a \right) \right]^\tau_n \quad (112) \]

and Equations (99a) and (106) yield
\[ \frac{\lambda \tau_D}{1 - \lambda \sum_{i=1}^{N} \bar{\beta}_{a_{i-1}} \Delta a} = \frac{\sum_{i=1}^{N} \bar{\beta}_{a_{i-1}} \Delta a}{\sum_{i=1}^{N} \left( \frac{\bar{\beta}_{a_{i-1}} \Delta a}{\tau_i} \right)^2 + \frac{1}{\tau_0}} = \frac{\sqrt{N} \bar{b}_N + o(1)}{(b^2 \tau^{-1} N + \frac{1}{\tau_0}) + o(1) [2b \tau^{-1} N + \tau^{-1} N]} \quad (113) \]
Substituting (106), (111) and (113) into (112) yields

\[ \tilde{\beta}_{a_{n-1}} \Delta a = \frac{\tau_n}{\left(\sqrt{N} b_N + o(\sqrt{N})\right) + o(1)} \left( \frac{\tau_n}{\tau_D} + 2 \frac{1}{N} b_n + o\left(\frac{1}{N}\right) + o(1) \right), \]  

(114)

and multiplying through by \(\sqrt{N}\) gives

\[ \tilde{\beta}_{a_{n-1}} \Delta a \sqrt{N} = \frac{\tau_n}{\left(b_N + o(1)\right) + o(1)} \left( \frac{\tau_n}{\tau_D} + 2 \frac{1}{N} b_n + o\left(\frac{1}{N}\right) + o(1) \right). \]  

(115)

Taking the limit as \(N \to \infty\) on the right-hand side of (115) shows that

\[ \lim_{N \to \infty} \sqrt{N} \tilde{\beta}_{a_{n-1}} \Delta a = \frac{\tau_n \left( \mathbb{E} \left[ \frac{b^2}{\tau} \right] + \frac{1}{\tau_\theta} \right)}{\left(2 + \frac{\tau_n}{\tau_D}\right) \mathbb{E} \left[ b \right]} \]  

(116)

because by conjecture, \(b_N, b_2^{-1} N, b_T^{-1} N \) and \(\tilde{a}_N^{-1} N\) have finite limits as \(N \to \infty\). This confirms the conjecture (106) for some constant \(b_n\).

To prove (103a) it remains to solve for the constant \(b_n, n = 1, \ldots, N\), and to confirm that (107a), (107b), and (107c) have finite limits.

To solve for \(b_n, n = 1, \ldots, N\), we match asymptotic orders of \(N\) in (106) and (116). We obtain

\[ b_n = \frac{\tau_n \tau_D \left( \mathbb{E} \left[ \frac{b^2}{\tau} \right] + \frac{1}{\tau_\theta} \right)}{\left(2 + \frac{\tau_n}{\tau_D}\right) \mathbb{E} \left[ b \right]}. \]  

(117)

Taking averages on both sides of (117) gives

\[ \mathbb{E} \left[ b \right] = \mathbb{E} \left[ \frac{\tau}{2 + \tau} \right] \mathbb{E} \left[ \frac{b^2}{\tau} \right] \mathbb{E} \left[ b \right] \]  

(118)

and therefore

\[ \mathbb{E} \left[ b \right] = \sqrt{\mathbb{E} \left[ \frac{\tau}{2 + \tau} \right] \mathbb{E} \left[ \frac{b^2}{\tau} \right]} \mathbb{E} \left[ b \right]. \]  

(119)

Substituting this into (117) gives

\[ b_n = \frac{\tau_n \tau_D \mathbb{E} \left[ \frac{b^2}{\tau} \right] + \frac{1}{\tau_\theta}}{\left(2 + \frac{\tau_n}{\tau_D}\right) \mathbb{E} \left[ b \right]}. \]  

(120)
Squaring, dividing by $\tau_n$ and taking averages on both sides now yields

$$
\mathbb{E}\left[ \frac{b^2}{\tau} \right] = \mathbb{E}\left[ \frac{\tau}{(2\tau_D + \tau)^2} \right] \tau_D \left( \mathbb{E}\left[ \frac{b^2}{\tau} \right] + \frac{1}{\tau_n} \right),
$$

(121)

from which we obtain

$$
\mathbb{E}\left[ \frac{b^2}{\tau} \right] = \frac{\tau_D \mathbb{E}\left[ \frac{\tau}{(2\tau_D + \tau)^2} \right]}{\tau_n \mathbb{E}\left[ \frac{\tau}{(2\tau_D + \tau)^2} \right]}.
$$

(122)

Substituting this into (120) and (119) establishes (103c) and (103d).

We can now proceed with the limits of (107a), (107b), and (107c). Because $[\tau_n/(2\tau_D + \tau_n)]^2 < 1$, it follows that

$$
\frac{\tau_n}{(2\tau_D + \tau_n)^2} < \frac{1}{\tau_n},
$$

(123)

and therefore

$$
\frac{1}{N} \sum_{n=1}^{N} \frac{\tau_n}{(2\tau_D + \tau_n)^2} < \frac{1}{N} \sum_{n=1}^{N} \frac{1}{\tau_n}.
$$

(124)

Taking the limit as $N \to \infty$ and using assumption (34a) shows that

$$
\mathbb{E}\left[ \frac{\tau}{(2\tau_D + \tau)^2} \right] < \infty.
$$

(125)

Moreover, because $[(\tau_D + \tau_n)/(2\tau_D + \tau_n)] < 1$, it follows that

$$
\frac{(\tau_D + \tau_n) \tau_n}{(2\tau_D + \tau_n)^2} < \frac{\tau_n}{2\tau_D + \tau_n},
$$

(126)

and therefore

$$
\frac{1}{N} \sum_{n=1}^{N} \frac{\tau_n (\tau_D + \tau_n)}{(2\tau_D + \tau_n)^2} < \frac{1}{N} \sum_{n=1}^{N} \frac{\tau_n}{2\tau_D + \tau_n} < \frac{1}{N} \sum_{n=1}^{N} 1 = 1
$$

(127)

where the second inequality follows from that $\tau_n/(2\tau_D + \tau_n) < 1$. By taking the limit as $N \to \infty$ in (127) it now follows that

$$
\mathbb{E}\left[ \frac{\tau (\tau_D + \tau)}{(2\tau_D + \tau)^2} \right] < \mathbb{E}\left[ \frac{\tau}{2\tau_D + \tau} \right] < \infty.
$$

(128)
Equation (122), together with the bounds in (125) and (128), implies that

$$\mathbb{E}\left[\frac{b^2}{\tau}\right] < \infty.$$  \hspace{1cm} (129)

Equation (119), and the bounds in (128) and (129) further imply that

$$\mathbb{E}[b] < \infty.$$  \hspace{1cm} (130)

What is more, by (120) it follows that

$$\mathbb{E}\left[\frac{b}{\tau}\right] = \mathbb{E}\left[\frac{1}{2\tau_D + \tau}\sqrt{\frac{\tau_D}{\mathbb{E}\left[\frac{b^2}{\tau}\right] + \frac{1}{\tau}}}ight] < \mathbb{E}\left[\frac{1}{\tau}\sqrt{\frac{\tau_D}{\mathbb{E}\left[\frac{b^2}{\tau}\right] + \frac{1}{\tau}}}ight] < \infty \hspace{1cm} (131)$$

where the first inequality is because $1/ (2\tau_D + \tau_n) < 1/\tau_n$, and the second inequality is by assumption (34a), and the bounds established in (128) and (129). The bounds in (129), (130), and (131) confirm that (107a), (107b), and (107c) have finite limits.

Finally, I establish the bounds in (104b), (104c), and (104d); the bound in (104a) follows from (130).

From $[(\tau_D + \tau_n) / (2\tau_D + \tau_n)]^2 < 1$ and (103c) it follows that

$$\frac{1}{N} \sum_{n=1}^{N} b_n^2 = \frac{\tau_D}{\tau_D \mathbb{E}\left[\frac{\tau_D + \tau}{(2\tau_D + \tau)^2}\right]} \frac{1}{N} \sum_{n=1}^{N} \frac{\tau_n}{2\tau_D + \tau_n}^2 < \frac{\tau_D}{\tau_D \mathbb{E}\left[\frac{\tau_D + \tau}{(2\tau_D + \tau)^2}\right]} \sum_{n=1}^{N} \frac{1}{n^2 \tau_n},$$

and taking the limit as $N \to \infty$ on the left-hand side proves that the bound in (104b) holds. From (103c) and (123) we have

$$\sum_{n=1}^{N} \frac{1}{n^2 \tau_n} = \frac{\tau_D}{\tau_D \mathbb{E}\left[\frac{\tau_D + \tau}{(2\tau_D + \tau)^2}\right]} \sum_{n=1}^{N} \frac{1}{n^2 (2\tau_D + \tau_n)^2} < \frac{\tau_D}{\tau_D \mathbb{E}\left[\frac{\tau_D + \tau}{(2\tau_D + \tau)^2}\right]} \sum_{n=1}^{N} \frac{1}{n^2 \tau_n},$$

and taking the limit as $N \to \infty$ establishes the bound in (104c) by assumption (34b). From $[\tau_n / (2\tau_D + \tau_n)]^4 < 1$ it follows that

$$\frac{\tau_n^3}{(2\tau_D + \tau_n)^4} < \frac{1}{\tau_n},$$

(134)
and therefore by (103c) we obtain

\[
\sum_{n=1}^{N} \frac{b_n^4}{n^2 \tau_n} = \left( \frac{\tau_D}{\tau_D \mathbb{E}\left[ \frac{\tau_D}{(2\tau_D + \tau_n)} \right]} \right)^2 \sum_{n=1}^{N} \frac{\tau_n^3}{n^2 (2\tau_D + \tau_n)^4} < \left( \frac{\tau_D}{\tau_D \mathbb{E}\left[ \frac{\tau_D}{(2\tau_D + \tau_n)} \right]} \right)^2 \sum_{n=1}^{N} \frac{1}{n^2 \tau_n}. \quad (135)
\]

Taking the limit as \(N \to \infty\) establishes the bound in (104d) by assumption (34b). This concludes the proof. □

**Proof of Theorem 7.** By Lemma A.7 we have that

\[
\lim_{N \to \infty} \lambda = \lim_{N \to \infty} \left[ \frac{1}{L \sqrt{N}} + o \left( \frac{1}{\sqrt{N}} \right) \right] = 0. \quad (136)
\]

Moreover, from (5), (30a), (31a), and Lemma A.7 we obtain

\[
\lim_{N \to \infty} P = \lim_{N \to \infty} \left[ \left( \frac{1}{L} + o(1) \right) \frac{1}{N} \sum_{i=1}^{N} b_i + \frac{1}{L} o(1) + o(1) \right] D
\]

\[+ \lim_{N \to \infty} \left[ \frac{1}{L} + o(1) \right] \left( \frac{1}{N} \sum_{i=1}^{N} \varepsilon_i + \frac{1}{N} \sum_{i=1}^{N} b_i \varepsilon_i \right)
\]

\[= D + \frac{1}{L} \left( \lim_{N \to \infty} \frac{1}{N} \sum_{i=1}^{N} \varepsilon_i + \lim_{N \to \infty} \frac{1}{N} \sum_{i=1}^{N} b_i \varepsilon_i \right), \quad (137)
\]

where the second equality follows from (103c) and (103d) of Lemma (A.7). By Kolmogorov’s Law of Large Numbers,

\[
\lim_{N \to \infty} \frac{1}{N} \sum_{i=1}^{N} \varepsilon_i = 0 \quad (138)
\]

as long as

\[
\lim_{N \to \infty} \sum_{n=1}^{N} \frac{1}{n^2 \tau_n} < \infty, \quad (139)
\]

and

\[
\lim_{N \to \infty} \frac{1}{N} \sum_{i=1}^{N} b_i \varepsilon_i = 0 \quad (140)
\]

as long as

\[
\lim_{N \to \infty} \sum_{n=1}^{N} \frac{1}{n^2 \tau_n} < \infty. \quad (141)
\]

Requirement (139) is guaranteed by assumption (34b), and requirement (141) is guaranteed by
Lemma (A.7). It now follows that
\[
\lim_{N \to \infty} P = D. \tag{142}
\]

**Proof of Proposition 8.** For the economy with Itô signals we have that the rational variation is
\[
V_R = \sum_{n=1}^{N} X_t^2 = \sum_{n=1}^{N} \beta_{a_{n-1}}^2 (\Delta z_t)^2 = \sum_{n=1}^{N} \beta_{a_{n-1}}^2 \left( D^2 \Delta a^2 + 2 D \Delta a \sqrt{\tau_{a_{n-1}}^{-1} \Delta B_{a_n}} + \tau_{a_{n-1}}^{-1} \Delta B_{a_n}^2 \right)
\]
\[
= \Delta a D^2 \sum_{n=1}^{N} \beta_{a_{n-1}}^2 \Delta a + 2 \Delta a D \sum_{n=1}^{N} \beta_{a_{n-1}}^2 \sqrt{\tau_{a_{n-1}}^{-1} \Delta B_{a_n}} + \sum_{n=1}^{N} \beta_{a_{n-1}}^2 \tau_{a_{n-1}}^{-1} \Delta B_{a_n}^2. \tag{143}
\]

The expression inside the square bracket above contains three different sums, with three different types of limits as \( N \to \infty \). The sum
\[
\sum_{n=1}^{N} \beta_{a_{n-1}}^2 \Delta a
\]
converges to the Riemann integral
\[
\int_0^1 \beta^2(a) dt. \tag{145}
\]
The sum
\[
\sum_{n=1}^{N} \beta_{a_{n-1}}^2 \sqrt{\tau_{a_{n-1}}^{-1} \Delta B_{a_n}}
\]
converges to the stochastic integral
\[
\int_0^1 \frac{\beta^2(a)}{\sqrt{\tau(a)}} dB_t. \tag{147}
\]
Finally, the sum
\[
\sum_{n=1}^{N} \beta_{a_{n-1}}^2 \tau_{a_{n-1}}^{-1} \Delta B_{a_n}^2
\]
is a quadratic variation sum, so it converges to the Riemann integral
\[
\int_0^1 \frac{\beta^2(a)}{\tau(a)} da. \tag{149}
\]
Taking the limit in (143) therefore yields
\[
\lim_{N \to \infty} V_R = \left( \lim_{N \to \infty} \Delta a \right) D^2 \int_0^1 \beta^2(a) dt + 2 \left( \lim_{N \to \infty} \Delta a \right) D \int_0^1 \frac{\beta^2(a)}{\sqrt{\tau(a)}} dB_t + \int_0^1 \frac{\beta^2(a)}{\tau(a)} da
\]
\[ \int_0^1 \frac{\beta^2(a)}{\tau(a)} da. \quad (150) \]

This establishes (39).

For the economy with Kolmogorov signals, we have, by Lemma A.7, that the rational variation is

\[ V_R = \sum_{n=1}^N X_n^2 = \sum_{n=1}^N \left( \frac{b_n}{\sqrt{N}} + o \left( \frac{1}{\sqrt{N}} \right) \right)^2 (D + \varepsilon_n)^2 = \frac{1}{N} \sum_{n=1}^N [b_n + o(1)]^2 (D + \varepsilon_n)^2 \]

\[ = D^2 \left[ \frac{1}{N} \sum_{n=1}^N b_n^2 + 2o(1) \frac{1}{N} \sum_{n=1}^N b_n + o(1) \right] \]

\[ + 2D \left[ \frac{1}{N} \sum_{n=1}^N b_n^2 \varepsilon_n + 2o(1) \frac{1}{N} \sum_{n=1}^N b_n \varepsilon_n + o(1) \frac{1}{N} \sum_{n=1}^N \varepsilon_n \right] \]

\[ + \frac{1}{N} \sum_{n=1}^N b_n^2 \varepsilon_n^2 + 2o(1) \frac{1}{N} \sum_{n=1}^N b_n \varepsilon_n^2 + o(1) \frac{1}{N} \sum_{n=1}^N \varepsilon_n^2 \quad (151) \]

By (103c) of Lemma A.7, the coefficient of \( D^2 \) in (151) converges to

\[ \frac{\tau_D \E \left[ \left( \frac{\tau}{2\tau_D + \tau} \right)^2 \right]}{\tau_E \E \left[ \frac{\tau \left( \tau_D + \tau \right)}{2\tau_D + \tau} \right]^2}. \quad (152) \]

because

\[ \frac{1}{N} \sum_{n=1}^N b_n^2 \quad (153) \]

and

\[ \frac{1}{N} \sum_{n=1}^N b_n \quad (154) \]

have finite limits by (104b) and (104a) of Lemma A.7.

Consider next the coefficient of \( D \) in (151). By the Kolmogorov Law of Large Numbers

\[ \lim_{N \to \infty} \frac{1}{N} \sum_{n=1}^N \varepsilon_n = 0 \quad (155a) \]

if

\[ \lim_{N \to \infty} \sum_{n=1}^N \frac{1}{n^2 \tau_n} < \infty, \quad (155b) \]
\[ \lim_{N \to \infty} \frac{1}{N} \sum_{n=1}^{N} b_n \varepsilon_n = 0 \]  

(156a)

if

\[ \lim_{N \to \infty} \sum_{n=1}^{N} \frac{b_n^2}{n^2 \tau_n} < \infty, \]  

(156b)

and

\[ \lim_{N \to \infty} \frac{1}{N} \sum_{n=1}^{N} b_n^2 \varepsilon_n = 0 \]  

(157a)

if

\[ \lim_{N \to \infty} \sum_{n=1}^{N} \frac{b_n^4}{n^2 \tau_n} < \infty, \]  

(157b)

Condition (155b) is guaranteed by Assumption (34b). Condition (156b) is guaranteed by (104c) of Lemma A.7, and condition (157b) is guaranteed by (104d) of Lemma A.7. It follows that as \( N \to \infty \) the coefficient of \( D \) in (151) converges to zero.

Finally, consider the last three terms in (151). By the Kolmogorov Law of Large Numbers

\[ \lim_{N \to \infty} \frac{1}{N} \sum_{n=1}^{N} \varepsilon_n^2 = \mathbb{E} \left[ \frac{1}{\tau} \right] \]  

(158a)

if

\[ \lim_{N \to \infty} \sum_{n=1}^{N} \frac{1}{n^2} \text{Var}(\varepsilon_n^2) = 2 \lim_{N \to \infty} \sum_{n=1}^{N} \frac{1}{n^2 \tau_n^2} < \infty, \]  

(158b)

\[ \lim_{N \to \infty} \frac{1}{N} \sum_{n=1}^{N} b_n \varepsilon_n^2 = \mathbb{E} \left[ \frac{b}{\tau} \right] \]  

(159a)

if

\[ \lim_{N \to \infty} \sum_{n=1}^{N} \frac{1}{n^2} \text{Var}(b_n \varepsilon_n^2) = 2 \lim_{N \to \infty} \sum_{n=1}^{N} \frac{b_n^2}{n^2 \tau_n^2} < \infty, \]  

(159b)

and

\[ \lim_{N \to \infty} \frac{1}{N} \sum_{n=1}^{N} b_n^2 \varepsilon_n^2 = \mathbb{E} \left[ \frac{b^2}{\tau} \right] \]  

(160a)

if

\[ \lim_{N \to \infty} \sum_{n=1}^{N} \text{Var}(b_n^2 \varepsilon_n^2) = 2 \lim_{N \to \infty} \sum_{n=1}^{N} \frac{b_n^4}{n^2 \tau_n^2} < \infty. \]  

(160b)

Condition (158b) is guaranteed by Assumption (40a). Condition (159b) is also guaranteed by
Assumption (40a), because from
\[
\frac{1}{(2\tau_D + \tau_n)^2} < \frac{1}{\tau_n^2}
\] (161)
and Lemma A.7 we obtain
\[
\lim_{N\to\infty} \sum_{n=1}^{N} \frac{b_n^2}{n^2 \tau_n^2} < \frac{\tau_D}{\tau_n} \mathbb{E} \left[ \frac{\tau}{(2\tau_D + \tau)^2} \right] \lim_{N\to\infty} \sum_{n=1}^{N} \frac{1}{n^2 \tau_n^2} < \infty.
\] (162)

Lastly, condition (160b) is guaranteed by Assumption (40a) because from
\[
\frac{\tau_n^2}{(2\tau_D + \tau_n)^4} < \frac{1}{\tau_n^2}
\] (163)
and Lemma A.7 we obtain
\[
\lim_{N\to\infty} \sum_{n=1}^{N} \frac{b_n^2}{n^2 \tau_n^2} < \frac{\tau_D}{\tau_n} \mathbb{E} \left[ \frac{\tau}{(2\tau_D + \tau)^2} \right] \lim_{N\to\infty} \sum_{n=1}^{N} \frac{1}{n^2 \tau_n^2} < \infty.
\] (164)

It now follows that as \( N \to \infty \) the last three terms in (151) converge to
\[
\mathbb{E} \left[ \frac{b^2}{\tau} \right] = \frac{\tau_D}{\tau} \mathbb{E} \left[ \frac{\tau}{(2\tau_D + \tau)^2} \right].
\] (165)
where the equality follows from (122). This establishes (40b) and concludes the proof. \( \blacksquare \)

**Proof of Proposition 9.** For the economy with Itô signals, by Corollary A.4 we have
\[
\mathbb{E}[u(\pi_n; \Delta z_{a_n})] = \left( 1 - \lambda \sum_{i=1}^{N} \beta_{a_{i-1}} \Delta a \right) \frac{\beta_{a_{n-1}}}{2\tau_D} \Delta a + \lambda \frac{\beta_{a_{n-1}}^2}{2\tau_D} (\Delta a)^2.
\] (166)
Summing over \( n \) gives
\[
\sum_{n=1}^{N} \mathbb{E}[u(\pi_n; \Delta z_{a_n})] = \left( 1 - \lambda \sum_{i=1}^{N} \beta_{a_{i-1}} \Delta a \right) \sum_{n=1}^{N} \frac{\beta_{a_{n-1}}}{2\tau_D} \Delta a + \lambda \Delta a \sum_{n=1}^{N} \frac{\beta_{a_{n-1}}^2}{2\tau_D} \Delta a,
\] (167)
and taking the limit shows that
\[
W = \frac{1}{2\tau_D} \left( 1 - \lambda \int_{0}^{1} \beta(a) da \right) \int_{0}^{1} \beta(a) da.
\] (168)
Combining this expression with Lemma A.3 establishes (42).
For the economy with Kolmogorov signals, combining the first-order condition of agent $n$ with his utility function gives

$$u(\pi_n; \Delta z_{a_n}) = \frac{1}{2} \left( \mathbb{E} \left[ D \middle| \Delta z_{a_n} \right] - \lambda \sum_{i=1}^{N} \mathbb{E} \left[ X_i \middle| \Delta z_{a_n} \right] \right) =$$

$$\frac{1}{2} \left( 1 - \lambda \sum_{i=1 \atop i \neq n}^{N} \bar{\beta}_{a_{i-1}} \Delta a \right) \bar{\beta}_{a_{n-1}} \Delta a - \frac{1}{2} \left( \sum_{i=1 \atop i \neq n}^{N} \bar{\beta}_{a_{i-1}} \Delta a \right) \bar{\beta}_{a_{n-1}} \Delta a - \frac{1}{2} \lambda \sum_{i=1 \atop i \neq n}^{N} \bar{\beta}_{a_{i-1}} \Delta a \bar{\beta}_{a_{n-1}} \Delta a,$$

and therefore

$$\mathbb{E} [u(\pi_n; \Delta z_{a_n})] = \frac{1}{2 \tau_D} \left( 1 - \lambda \sum_{i=1 \atop i \neq n}^{N} \bar{\beta}_{a_{i-1}} \Delta a \right) \bar{\beta}_{a_{n-1}} \Delta a - \frac{1}{2 \tau_D} \lambda \left( \bar{\beta}_{a_{n-1}} \Delta a \right)^2. \quad (169)$$

Summing over $n$ gives

$$\sum_{n=1}^{N} \mathbb{E} [u(\pi_n; \Delta z_{a_n})] = \frac{1}{2} \left( 1 - \lambda \sum_{i=1}^{N} \bar{\beta}_{a_{i-1}} \Delta a \right) \bar{\beta}_{a_{n-1}} \Delta a + \frac{1}{2 \tau_D} \lambda \sum_{n=1}^{N} \left( \bar{\beta}_{a_{n-1}} \Delta a \right)^2$$

$$= \frac{1}{2} \left( 1 - \lambda \sum_{i=1}^{N} \bar{\beta}_{a_{i-1}} \Delta a \right) \bar{\beta}_{a_{n-1}} \Delta a + \frac{1}{2 \tau_D} \lambda \sum_{n=1}^{N} \left( \bar{\beta}_{a_{n-1}} \Delta a \right)^2 = \frac{1}{2} \left( \sum_{i=1}^{N} \left( \bar{\beta}_{a_{i-1}} \Delta a \right)^2 \tau_n + \tau_D \lambda \sum_{n=1}^{N} \left( \bar{\beta}_{a_{n-1}} \Delta a \right)^2 \right), \quad (171)$$

where the second equality follows from (99a) of Lemma A.6. By Lemma A.7 we obtain

$$\sum_{n=1}^{N} \mathbb{E} [u(\pi_n; \Delta z_{a_n})] = \frac{1}{2} \left[ \frac{1}{L \sqrt{N}} + o \left( \frac{1}{\sqrt{N}} \right) \right]$$

$$\left[ \bar{\beta}^2 \tau^{-1} N + \frac{1}{\tau_D} + \bar{\beta}^2 N + \left( \frac{2 \bar{b} \tau^{-1} N + 2 \bar{b} + \tau^{-1} N \right) \sigma(1) + o(1) \right]. \quad (172)$$

Taking the limit as $N \to \infty$ we get, by Lemma A.6 and due to Assumptions (34a) and (34b), that

$$W = \lim_{N \to \infty} \sum_{n=1}^{N} \mathbb{E} [u(\pi_n; \Delta z_{a_n})] = \lim_{N \to \infty} \left[ \frac{1}{L \sqrt{N}} + o \left( \frac{1}{\sqrt{N}} \right) \right] \left( \mathbb{E} \left[ \bar{\beta}^2 \tau^{-1} \right] + \frac{1}{2 \tau_D} \mathbb{E} \left[ \bar{\beta}^2 \right] \right) = 0. \quad (173)$$